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# Generalization of a theorem of Hurwitz

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**Abstract.** This paper is an exposition of several classical results formulated and unified using more modern terminology. We generalize a classical theorem of Hurwitz and prove the following: let

$$G_k(z) = {\sum_{m,n}}' \frac{1}{(mz+n)^k}$$

be the Eisenstein series of weight $k$ attached to the full modular group. Let $z$ be a CM point in the upper half-plane. Then there is a transcendental number $\Omega_z$ such that

$$G_{2k}(z) = \Omega_z^{2k} \cdot \text{(an algebraic number)}.$$

Moreover, $\Omega_z$ can be viewed as a fundamental period of a CM elliptic curve defined over the field of algebraic numbers. More generally, given any modular form $f$ of weight $k$ for the full modular group, and with algebraic Fourier coefficients, we prove that $f(z)\pi^k/\Omega_z^k$ is algebraic for any CM point $z$ lying in the upper half-plane. We also prove that for any automorphism $\sigma$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $(f(z)\pi^k/\Omega_z^k)^\sigma = f^\sigma(z)\pi^k/\Omega_z^k$.

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## 1. Introduction

In his epochal paper [15], Ramanujan introduces the power series

$$\Phi_{r,s}(x) = \sum_{n=1}^{\infty} \sigma_{r,s}(n) x^n,$$

where

$$\sigma_{r,s}(n) = \sum_{de=n} d^r e^s,$$

is the generalized divisor function. He isolates the functions

$$P := 1 - 24\Phi_{0,1}(x), \quad Q := 1 + 240\Phi_{0,3}(x), \quad R := 1 - 504\Phi_{0,5}(x),$$

and proves that $\Phi_{r,s}(x)$ is a polynomial in $P$, $Q$, $R$ with rational coefficients whenever $r + s$ is an odd positive integer. It is still a mystey if a similar result holds when $r + s$ is an even positive integer.

The purpose of this paper is to re-examine a classical result of Hurwitz and to relate it to the functions $P$, $Q$, $R$.

In 1898, Hurwitz [8] proved that

$$\sum_{(m,n)\neq(0,0)} \frac{1}{(mi+n)^{4k}} = \Omega^{4k} \cdot \text{(a rational number)}, \qquad i = \sqrt{-1} \tag{1}$$

where

$$\Omega = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}} = \int_1^{\infty} \frac{dx}{\sqrt{x^3 - x}}.$$

Since this integral allows us to identify $\Omega$ as a fundamental period of the elliptic curve $y^2 = x^3 - x$, we can apply a famous 1937 theorem of Schneider [17] to deduce that $\Omega$ is transcendental.

There are several ways to view this elegant theorem of Hurwitz. One way is to view it as the Gaussian analogue of the celebrated theorem of Euler giving the explicit values of the Riemann zeta function at positive even integers in terms of integer powers of $\pi$ and the (rational) Bernoulli numbers. From this perspective, one is led to ask the question if a similar result can be obtained for the sums

$$\sum_{\alpha\neq 0} \alpha^{-k}, \qquad k \geq 4$$

where the sum is over the non-zero elements of the ring of integers of an imaginary quadratic field (Hurwitz's theorem corresponding to the ring of Gaussian integers). Another perspective is to view it as a special value of a certain Hecke $L$-series, as done in Damerell [3] and Harder-Schappacher [7].

Viewing the rational numbers appearing in Hurwitz's theorem as generalizations of the classical Bernoulli numbers, one is led to study the arithmetic and divisibility properties of these numbers, as done in Coates-Wiles [2].

In this paper, we take yet another perspective which is not unrelated to the two viewpoints mentioned above.

The classical Eisenstein series for the full modular group is defined to be

$$G_k(z) = \sum_{m,n}{}' \frac{1}{(mz+n)^k} \tag{2}$$

where $\sum'$ means that $(m,n) \neq (0,0)$ is a modular form of weight $k$ for the full modular group when $k \geq 4$. Thus, what Hurwitz proved is that

$$G_{4k}(i) = r\Omega^{4k}, \qquad \text{where } r \in \mathbb{Q}. \tag{3}$$

Let us observe that

$$S := \sum_{m,n}{}' \frac{1}{(mi+n)^{4k+2}} = 0$$

since

$$S = \sum_{m,n}{}' \frac{1}{i^{4k+2}(m-ni)^{4k+2}} = -S.$$

Using very basic notions of the theory of modular forms, we will give another proof of (3), which is much simpler and more conceptual than that of Hurwitz. Other than $i = \sqrt{-1}$, another interesting CM point is $\rho = \frac{1+\sqrt{-3}}{2}$ which leads to analogous results and this we discuss at the end of the paper.

We must add that at the time Hurwitz wrote his paper, the theory of modular forms was not fully developed. Our proof will use some basic facts from the elementary theory of modular forms that can be found in Serre's graceful monograph [19]. At the same time, we will show that $G_k(z)$ is transcendental whenever $z$ is a CM point in the upper half-plane, that is, whenever, $\mathbb{Q}(z)$ is an imaginary quadratic field. Our proof will also show an intimate connection with the celebrated Chowla-Selberg formula.

More generally, we prove the following.

**Theorem 1.** *Let $f(z)$ be a modular form of weight $k$ for the full modular group. Suppose that $f$ has algebraic Fourier coefficients. For each CM point $z$ in the upper half-plane, there is a transcendental number $\Omega_z$ which is algebraically independent with $\pi$ such that $f(z)(\Omega_z/\pi)^{-k}$ is algebraic. In particular, $f(z)$ is an algebraic multiple of the $k$-th power of a transcendental number. Also, for any automorphism $\sigma$ of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, $(f(z)\pi^k/\Omega_z^k)^\sigma = f^\sigma(z)\pi^k/\Omega_z^k$.*

As a corollary, we deduce

**Theorem 2.** *Let $G_{2k}(z)$ be the Eisenstein series of weight $k$ on the full modular group and let $z$ be a CM point in the upper half-plane. Then there is a transcendental number $\Omega_z$ such that*

$$G_{2k}(z) = \Omega_z^{2k} \cdot \text{(an algebraic number)}.$$

*Moreover, $\Omega_z$ can be viewed as a fundamental period of a CM elliptic curve defined over the field of algebraic numbers.*

The CM elliptic curve can be defined over the Hilbert class field of $\mathbb{Q}(z)$. However, in the case $z = i$ or $z = \rho$, the algebraic number of the theorem is really a rational number since in these cases, it is easily checked the curve is defined over the rationals.

It is possible that the algebraic number of the theorem is zero. However, a theorem of Rankin and Swinnerton-Dyer [16] shows that any zero of the Eisenstein series $G_{2k}(z)$ lies on the unit circle $|z| = 1$. In addition, Kohnen [9] has shown that apart from $z = i$ and $z = \rho$, all the other zeros of $G_{2k}(z)$ are transcendental. This work was recently extended by Gun [4] to other modular forms. Thus, if $z$ is a CM point unequal to $i$ or $\rho$, then $G_{2k}(z)$ is necessarily transcendental.

Our theorem as stated here, seems new. However, the necessary ingredients needed to prove it are scattered throughout the literature and the main purpose of this paper is to pull together these disparate strands of thought and present them in a unified way from the viewpoint of Hurwitz's theorem. In this context, we set

$$E_{2k}(z) = G_{2k}(z)/2\zeta(2k)$$

where $\zeta(s)$ denotes the Riemann zeta function. Then, the constant term of $E_{2k}(z)$ is 1 and the Fourier expansion has rational coefficients.

Apart from the constant terms, both $E_4$ and $E_6$ coincide with Ramanujan's $Q$ and $R$ respectively. One can also define $E_2$ appropriately and this is (again apart from the constant term), Ramanujan's $P$. His result that any $\Phi_{r,s}(x)$ is a polynomial in $P$, $Q$, $R$ is a precursor of the general result that the algebra of quasi-modular forms for the full modular group is generated by $E_2$, $E_4$ and $E_6$.

The transcendence of $E_{2k}(z)$ when $z$ is a CM point is nascent in a more general theorem proved by Gun, Murty and Rath [5]. If $j$ denotes the modular invariant, and $\alpha$ lies in the upper half-plane such that $j(\alpha)$ is algebraic, then they showed that for any modular form $f$ for the full modular group, with algebraic coefficients, the values $f(\alpha)$ and $e^{2\pi i\alpha}$ are algebraically independent, provided that $\alpha$ is not a zero of $f$. (See Theorem 4 in [5].) In particular, $f(\alpha)$ is transcendental. In [5], a pivotal role is played by an application of Nesterenko's theorem (see [12] as well as Chapters 1 and 3 of [13]) stating that for any $z$ in the upper half-plane, at least three of the four numbers

$$e^{2\pi iz}, E_2(z), E_4(z), E_6(z)$$

are algebraically independent. The same setting is true in our context. If $j(\alpha)$ is algebraic, then as $j(z) = E_4(z)^3/\Delta(z)$, with $\Delta(z)$ being Ramanujan's cusp form, we deduce that $E_4(\alpha)$ and $E_6(\alpha)$ are algebraically dependent. Unravelling this fact, and applying Nesterenko's theorem gives us the desired transcendence result.

The goal of our paper is to isolate and highlight this particular application in the context of Eisenstein series and its relation to the classical theorem of Hurwitz. As such, our result can be viewed as a generalization of Hurwitz's theorem. We also make prominent the interpretation of the special value of the Eisenstein series as a period of a CM elliptic curve, something which was not explicitly done in [5].

Much of this work can be generalized to higher levels. This has now been done in a forthcoming paper of Hamieh and Murty [6].

## 2. Preliminary lemmas

Now, let us recall the definition of a modular function, and state the valence formula which will be useful in our proof. (See [19], Chapter VII for details.) A meromorphic function $f$ on the upper half plane $\mathcal{H}$ is called weakly modular of weight $2k$ if

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \tag{4}$$

for all $z \in \mathcal{H}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G := \mathrm{SL}_2(\mathbb{Z})$. Since $f(z+1) = f(z)$, $f$ can be expressed as a function $\tilde{f}$ of $q = e^{2\pi iz}$ which is meromorphic in the punctured disk $0 < |q| < 1$. If $\tilde{f}$ extends to a meromorphic function at the origin $q = 0$ then we say $f$ is a modular function of weight $2k$.

For $p \in \mathcal{H}$ let $v_p(f)$ be the order of $f$ at $p$ and $v_\infty(f)$ be the order of $\tilde{f}$ at $q = 0$. Note that $v_p(f) = v_{g(p)}(f)$ for $g \in G$ and that $v_p(f)$ only depends on the image of $p$ in $\mathcal{H}/G$. If the order of the stabilizer of the point $p$ is denoted by $e_p$ then one can show that $e_p = 2$ (respectively $e_p = 3$) if $p$ is congruent modulo $G$ to $i$ (respectively to $\frac{1+\sqrt{3}i}{2}$) and $e_p = 1$ otherwise.

**Lemma 3 (The valence formula).** *Let f be a modular function of weight $2k$, not identically zero. One has:*

$$v_\infty(f) + \sum_{p\in\mathcal{H}/G} \frac{1}{e_p} v_p(f) = \frac{k}{12},$$

*or equivalently*

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum_{p\in\mathcal{H}/G}^{*} v_p(f) = \frac{k}{12}$$

*where* $\sum^*$ *means a summation over the points of* $\mathcal{H}/G$ *distinct from the classes of* $i$ *and* $\rho = \frac{1+\sqrt{-3}}{2}$.

*Proof.* See [19], Theorem 3 of Chapter VII. □

As a corollary of the valence formula, we easily deduce that $E_6(i) = 0$ and $E_4(\rho) = 0$ and these are the only zeros of $E_6$ and $E_4$ respectively.

A modular function $f$ of weight $2k$ is called a modular form if it is holomorphic everywhere i.e. $\widetilde{f}$ is holomorphic in the disk $|q| < 1$.

**Lemma 4.** *For* $k \geqslant 1$, $G_{2k}$, *the Eisenstein series of weight* $2k$, *can be expressed as a rational linear combination of monomials* $G_4^\alpha G_6^\beta$, *where* $\alpha$ *and* $\beta$ *are integers with* $2\alpha + 3\beta = k$.

*Proof.* For $k \geqslant 4$, $G_{2k}$ is given by Theorem 1.13 of [1] as

$$(2k+1)(k-3)(2k-1)G_{2k} = 3\sum_{r=2}^{k-2}(2r-1)(2k-2r-1)G_{2r}G_{2k-2r}, \tag{5}$$

which is easily established using the second order differential equation satisfied by the Weierstrass $\wp$-function. This enables us to express $G_{2k}$ as

$$G_{2k} = \sum_{\alpha,\beta} c_{\alpha\beta} G_4^\alpha G_6^\beta, \qquad \text{with } c_{\alpha\beta} \in \mathbb{Q}, \tag{6}$$

by applying (5) repeatedly via induction. We also find by induction that the sum is over positive integers $\alpha, \beta$ satisfying $4\alpha + 6\beta = 2k$. □

**Remark.** In general, the space $M_k$ of modular forms of weight $2k$ has a basis consisting of monomials $G_4^\alpha G_6^\beta$ over $\mathbb{C}$, where $\alpha, \beta$ nonnegative integers and $2\alpha + 3\beta = k$. This can be found for example in Chapter VII, Corollary 2 of [19].

**Lemma 5.** *For any imaginary quadratic field with discriminant* $-D$ *and character* $\chi_D$, *the numbers* $\pi, e^{\pi\sqrt{D}}, \prod_{a=1}^{D-1}\Gamma(a/D)^{\chi_D(a)}$ *are algebraically independent.*

This is Corollary 3.2 of [13].

## 3. Periods of elliptic curves

Let

$$E: \qquad y^2 = 4x^3 - g_2x - g_3$$

be an elliptic curve. It is well-known that if $e_1, e_2, e_3$ are the roots of the cubic on the right hand side of this equation, then the fundamental periods $\omega_1, \omega_2$ of $E$ are given by (see for example, p. 65 of [14])

$$\frac{\omega_1}{2} = \int_{e_2}^{e_3} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}, \quad \frac{\omega_2}{2} = \int_{e_1}^{e_3} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}.$$

Schneider [17] has shown that if $g_2, g_3$ are algebraic, then the periods $\omega_1, \omega_2$ are transcendental. If the elliptic curve has periods $\omega_1, \omega_2$, then one can arrange (by changing sign) that $z = \omega_2/\omega_1$ lies in the upper half-plane.

The Eisenstein series of weight $2k$ associated to a lattice $L$ is defined to be

$$G_{2k}(L) := \sum_{\omega \in L'} \frac{1}{\omega^{2k}}, \tag{7}$$

where $L'$ denotes the set of non-zero periods.

Let $L_z = [\omega_1, \omega_2]$ be the lattice associated to $z = \omega_2/\omega_1 \in \mathcal{H}$. Then the corresponding $g_2(z) = g_2(L_z)$ and $g_3(z) = g_3(L_z)$ are related to the Eisenstein series $G_4(L_z), G_6(L_z)$ via the formulas

$$g_2(z) = 60G_4(L_z), \quad g_3(z) = 140G_6(L_z). \tag{8}$$

(See [10] or [14] for details.)

Here is the promised elementary proof of the theorem of Hurwitz.

**Theorem 6.** *$G_{4k}(i) = r\Omega^{4k}$, where $r \in \mathbb{Q}$.*

*Proof.* As $G_{4k} = \sum_{2\alpha+3\beta=2k} c_{\alpha\beta} G_4^{\alpha} G_6^{\beta}$ with $c_{\alpha\beta} \in \mathbb{Q}$ (by Lemma 4) and $G_6(i) = 0$ (by Lemma 3), we conclude that

$$G_{4k}(i) = c_{k0} G_4(i)^k, \tag{9}$$

since all other terms vanish.

Notice that $\Omega$ in (3) is a period of the elliptic curve

$$y^2 = 4x^3 - 4x.$$

This curve corresponds to the point $z = i$ in the standard fundamental domain and has $j$-invariant 1728. The period is given by

$$\Omega = \int_1^{\infty} \frac{dx}{\sqrt{x^3 - x}}.$$

Comparing the definitions of (2) and (7), one can check easily that

$$G_4(z) = \omega_1^4 G_4(L_z), \qquad \text{for } z = \frac{\omega_2}{\omega_1} \in \mathcal{H}.$$

On the other hand, we know from (8) that $4 = 60G_4(L_i)$ so that

$$G_4(i) = \frac{1}{15}\omega_1^4.$$

A simple change of variable transforms the integral to

$$\Omega = \frac{1}{2}\int_0^1 u^{-3/4}(1-u)^{-1/2}du,$$

which is easily recognized to be a beta integral. Again, by expressing $\omega_1, \omega_2$ and $\omega_3$ as beta integrals, we get $\omega_1 = -\Omega,\ \omega_2 = -i\Omega$ and $\omega_3 = -(1+i)\Omega$ (so that $\frac{\omega_2}{\omega_1} \in \mathcal{H}$ and $\omega_1 + \omega_2 = \omega_3$). Hence, $\omega_1^4 = \Omega^4$.

We therefore deduce that

$$G_4(i) = \frac{1}{15}\omega_1^4 = \frac{1}{15}\Omega^4.$$

This, together with (9), completes the proof. □

**Remark.** If we recall the evaluation of the beta function given by

$$B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\operatorname{Re} x, \operatorname{Re}(y) > 0$, we can evaluate

$$\Omega = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}}.$$

## 4. The general CM case, the Chowla-Selberg formula and proof of the main theorem

We review chapters 16 and 17 of [14] here. For a CM point $z$, let $E_z$ be an elliptic curve $y^2 = 4x^3 - g_2x - g_3$ over $\mathbb{Q}$ with complex multiplication by $z = \omega_2/\omega_1$ where $\omega_i$'s are its fundamental periods. One can choose a model for $E_z$ so that $g_2$ and $g_3$ are algebraic. Then $\Omega_z := \omega_1$ is given by the Chowla-Selberg formula:

$$\Omega_z = \alpha_z\sqrt{\pi} \prod_{0<a<d_z,(a,d_z)=1} \Gamma(a/d_z)^{\frac{w_z\,\chi_z(a)}{4h_z}} \tag{10}$$

where $\alpha_z$ is an algebraic number, $w_z$ is the number of roots of unity in $K_z := \mathbb{Q}(z)$, $-d_z$ is the discriminant of $K_z$, $\chi_z$ is the quadratic character mod $d_z$ determined by $K_z$ and $h_z$ is the class number of $K_z$. For $z = \omega_2/\omega_1$ lying in the upper half-plane and from the fact that (see [14] Corollary 17.9)

$$E_4(z) = \frac{3}{4}\left(\frac{\Omega_z}{\pi}\right)^4 g_2, \quad E_6(z) = \frac{27}{8}\left(\frac{\Omega_z}{\pi}\right)^6 g_3, \tag{11}$$

where $E_{2k}(z)$ is the normalized Eisenstein series given by $G_{2k}(z)/2\zeta(2k)$, we see that $G_{2k}(z)$ is equal to ${\Omega_z}^{2k}$ up to some algebraic multiple.

More explicitly, by Lemma 4,

$$G_{2k}(z) = \sum_{4\alpha+6\beta=2k} c_{\alpha\beta} G_4(z)^\alpha G_6(z)^\beta$$

so that for suitable rational numbers $c^*_{\alpha\beta}$,

$$\begin{aligned} G_{2k}(z) &= \sum_{4\alpha+6\beta=2k} c^*_{\alpha\beta}(\Omega_z^4 g_2)^\alpha(\Omega_z^6 g_3)^\beta \\ &= {\Omega_z}^{2k} \sum_{4\alpha+6\beta=2k} c^*_{\alpha\beta} g_2^\alpha g_3^\beta \\ &= \delta_0 {\Omega_z}^{2k} \quad \text{for some } \delta_0 \in \overline{\mathbb{Q}}. \end{aligned} \tag{12}$$

From the Chowla-Selberg formula (10) (see also [11]) we deduce that

$${\Omega_z}^{2k} = \alpha_z^{2k}\pi^k \left( \prod_{0<a<d_z,(a,d_z)=1} \Gamma(a/d_z)^{\frac{w_z\,\chi_z(a)}{4h_z}} \right)^{2k},$$

we derive that

$$G_{2k}(z) = \delta\pi^k \prod_{0<a<d_z,(a,d_z)=1} \Gamma(a/d_z)^{\frac{w_z k\,\chi_z(a)}{2h_z}}$$

where $\delta$ is an algebraic number. This completes the proof of our main theorem. □

Hurwitz's result is easily derived from this by using the functional equation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

We now give another example that is similar to Hurwitz's theorem mentioned in our introduction. Let us consider the case when $\rho = \frac{1+\sqrt{-3}}{2}$.

By the valence formula, one can check that $G_4(\rho) = 0$. Also by Lemma 4 $G_{6k+r}(\rho) = 0$ if $0 < r < 6$. Since $G_{6k}(\rho) = c_{0k}G_6(\rho)^k$, it is equal to

$$\pi^{3k} \prod_{0<a<3,(a,3)=1} \Gamma(a/3)^{\frac{18k}{2}\chi_z(a)} = \left(\frac{\pi\,\Gamma(1/3)^3}{\Gamma(2/3)^3}\right)^{3k} \tag{13}$$

up to some algebraic multiple.

By Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, we get $\frac{1}{\Gamma(2/3)} = \frac{\sqrt{3}\,\Gamma(1/3)}{2\pi}$. Now (13) gives that $G_{6k}(\rho)$ is

$$\left(\frac{\Gamma(1/3)^3}{\pi}\right)^{6k} \tag{14}$$

up to some nonzero algebraic multiple.

Since this is (up to an algebraic multiple) the power of a period of an elliptic curve with algebraic coefficients, Schneider's theorem (see [17]) implies that it is transcendental.

However, it is possible to deduce this using a result of Čudnovskiĭ. We state this as:

**Proposition 7.** *$E_{6k}(\rho)$ is transcendental where $\rho = \frac{1+\sqrt{-3}}{2}$.*

*Proof.* By Theorem 14 of [20](due to G. V. Čudnovskiĭ), the two numbers

$$\Gamma\left(\frac{1}{3}\right) \quad \text{and} \quad \pi$$

are algebraically independent. This, together with (14), implies the result. □

An elliptic curve $E$ over $\mathbb{Q}$ whose $j$-invariant is not 0 nor 1728 has the Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ where $g_2$ and $g_3$ are nonzero rational numbers. If $E$ has complex multiplication and $g_2,\ g_3 > 0$ or $g_2,\ g_3 < 0$, then the recursion relation (5) for Eisenstein series shows that the above $G_{2k}(z)$ is a nonzero number for all $k$ since $G_{2k}$ is expressible as a polynomial in $g_2 = 60G_4$ and $g_3 = 140G_6$ with positive rational coefficients.

As remarked earlier, by the result of Kohnen [9] $G_{2k}(z)$ is non-zero when $z$ is a CM point unequal to $i$ or $\rho$ and so in these cases, it is the $2k$-th power of a transcendental number.

## 5. Proof of the main theorem

Given any modular form $f$ of weight $k$ with algebraic Fourier coefficients, we can write it as

$$f(z) = \sum_{4a+6b=k} c_{ab}E_4^aE_6^b$$

with $c_{ab}$ algebraic. This is evident since the collection of elements $E_4^a E_6^b$ with $4a + 6b = k$ is a basis (with rational Fourier coefficients) for the space of modular forms of weight $k$ for the full modular group. Applying the Galois automorphism to $f$ simply leads to applying the Galois automorphism to the $c_{ab}$'s. Thus, inserting formula (11), we see that

$$f(z) = (\Omega_z/\pi)^k(\text{an algebraic number}),$$

whose action under Galois is fully determined. Applying Lemma 5, completes the proof.

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