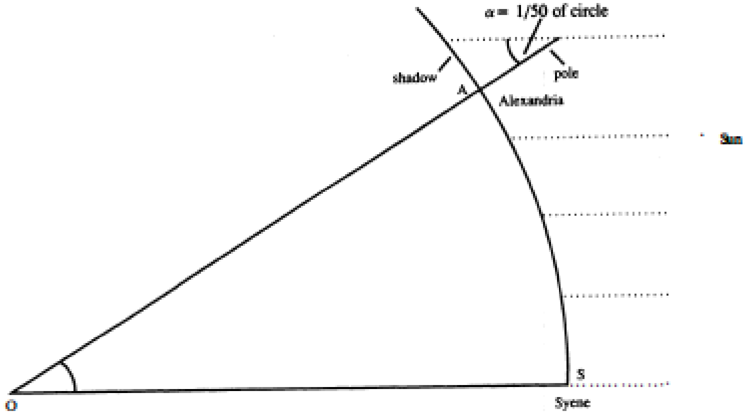
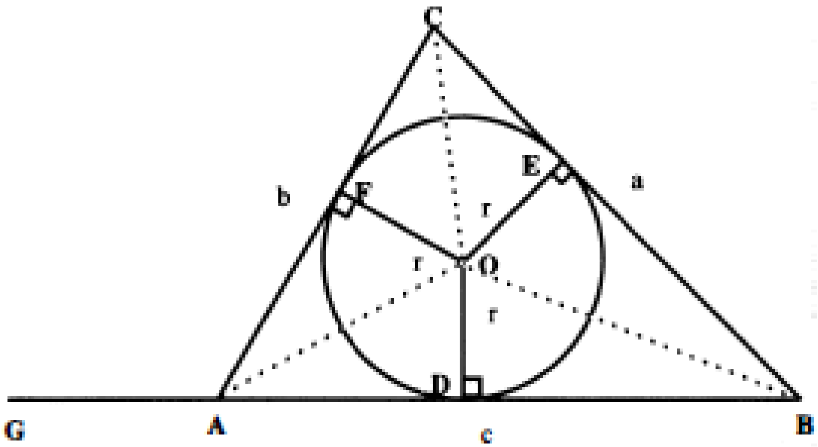
Greek trigonometry: Eratosthenes and Heron

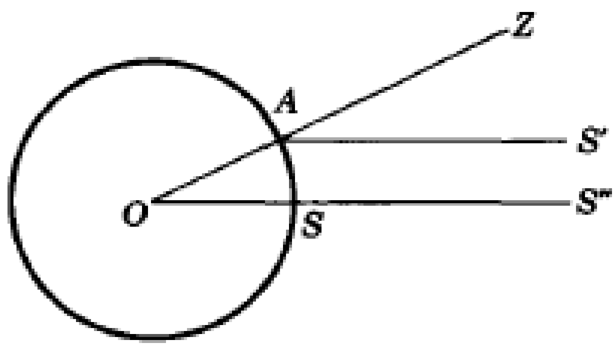


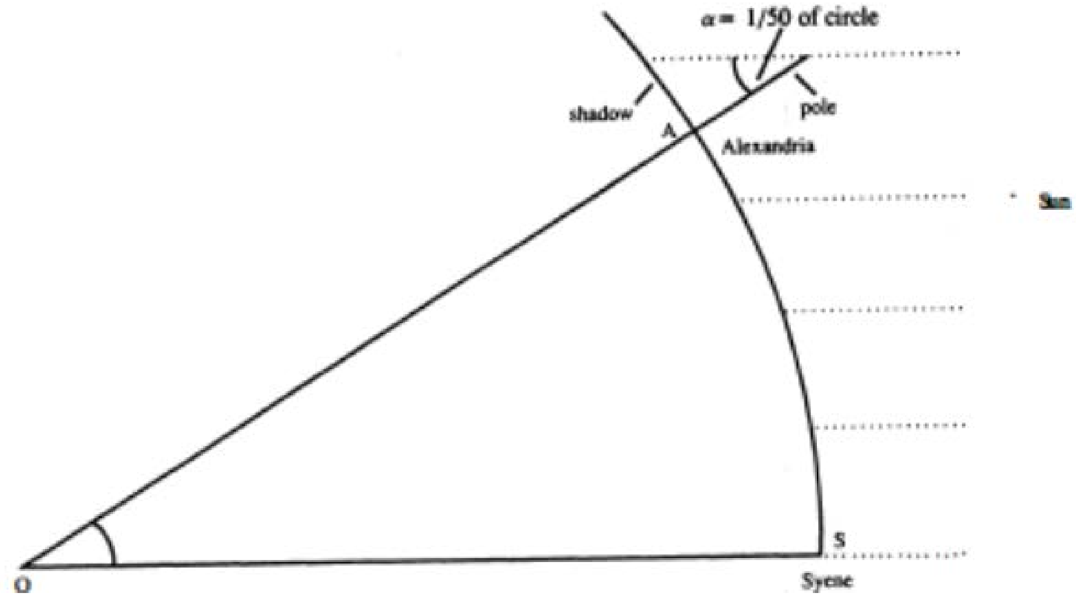


The circumference of the earth

- Eratosthenes was a scholar and mathematician at the Library of Alexandria and was the first person to approximate the circumference of the earth.
- He did this by using simple trigonometry and similar triangles.
- It was known that in the Egyptian town of Syene, located south of Alexandria, the sun stood directly overhead at noon on the first day of summer so that a vertical pole cast no shadow at that time.
- But a vertical pole did cast a shadow at high noon at Alexandria.

The basic idea

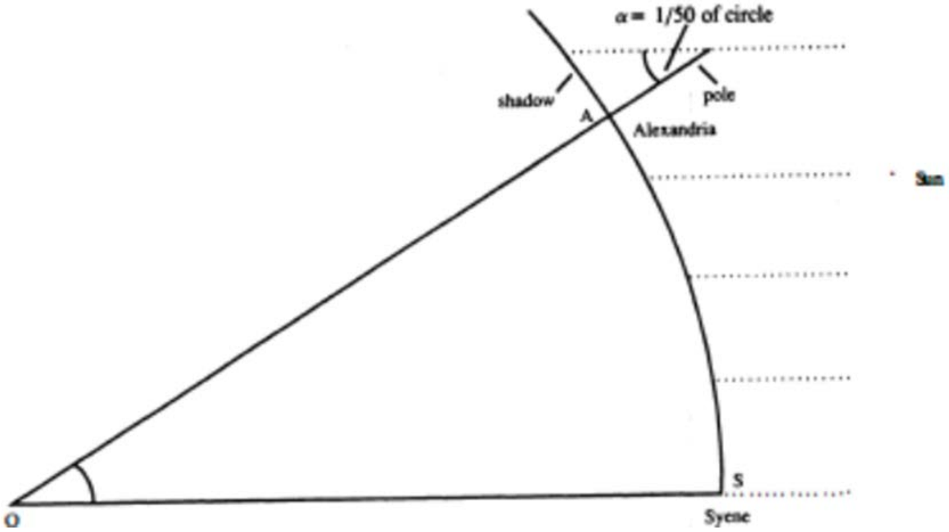




- The event can be visualised by the diagrams above.
- The angle subtended at the center of the earth is proportional to the distance between Syene and Alexandria.

The calculation

- By knowing the length of the pole, and measuring the length of the shadow, Eratosthenes calculated the circumference of the globe to be about 24,466 miles which is very close to 24,860 accepted today.

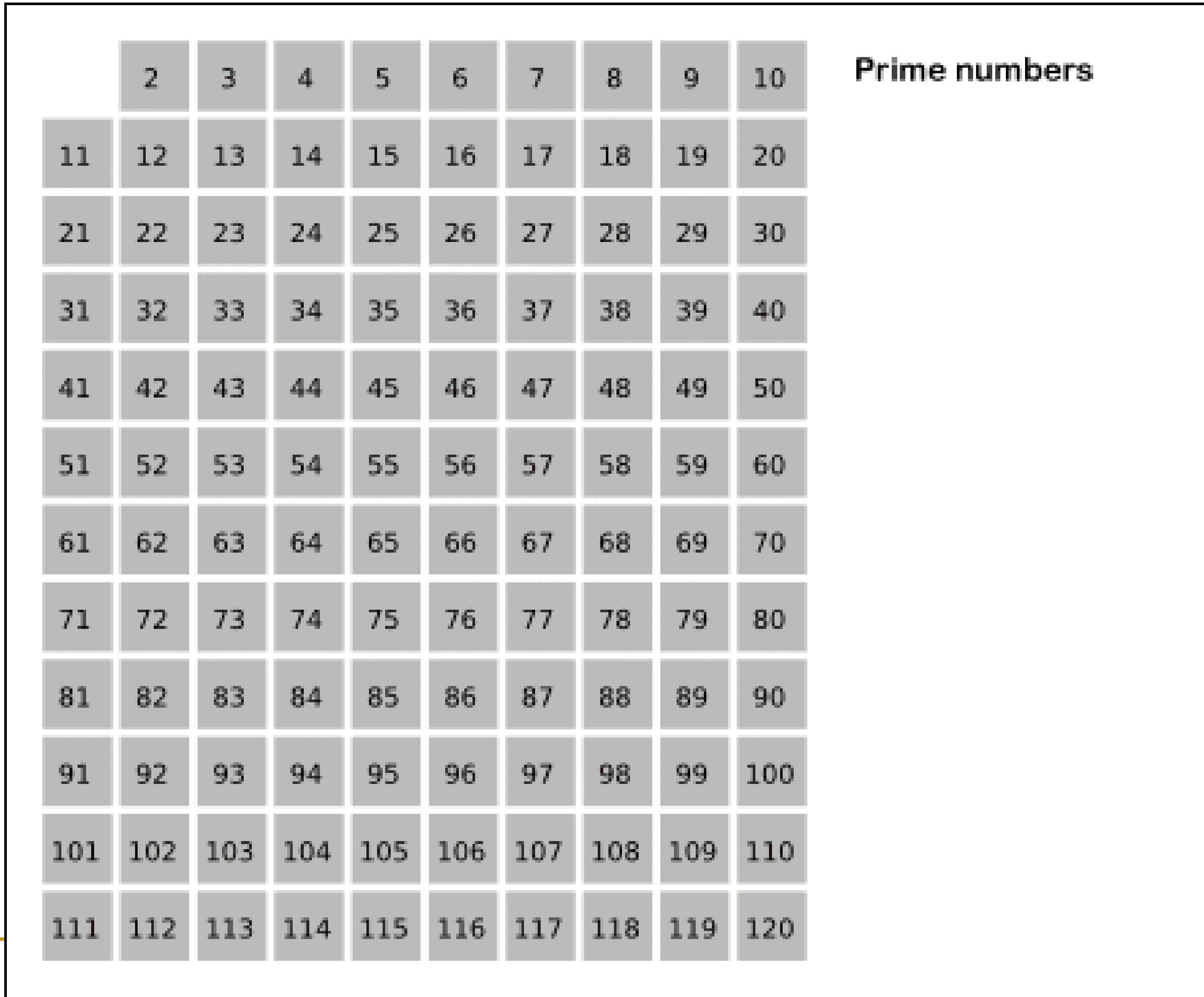


The sieve of Eratosthenes

- Eratosthenes also devised a method of determining all prime numbers below a given bound X.
- The idea is to write down all the numbers from 1 to X and then to begin with 2, and delete all even numbers from 4 onwards, then multiples of 3 from 6 onwards and so forth.

2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, 11, ~~12~~, 13, ~~14~~, ~~15~~, ~~16~~, 17, ~~18~~, 19,
~~20~~, ~~21~~, ~~22~~, 23, ~~24~~, ~~25~~, ~~26~~, ~~27~~, ~~28~~, 29, ~~30~~, 31, ~~32~~, ~~33~~, ~~34~~, ~~35~~,
~~36~~, 37, ~~38~~, ~~39~~, ~~40~~, 41, ~~42~~, 43, ~~44~~, ~~45~~, ~~46~~, 47, ~~48~~, ~~49~~, ~~50~~, ~~51~~,
~~52~~, 53, ~~54~~, ~~55~~, ~~56~~, ~~57~~, ~~58~~, 59, ~~60~~, 61, ~~62~~, ~~63~~, ~~64~~, ~~65~~, ~~66~~, 67,
~~68~~, ~~69~~, ~~70~~, 71, ~~72~~, 73, ~~74~~, ~~75~~, ~~76~~, ~~77~~, ~~78~~, 79, ~~80~~, ~~81~~, ~~82~~, 83,
~~84~~, ~~85~~, ~~86~~, ~~87~~, ~~88~~, 89, ~~90~~, ~~91~~, ~~92~~, ~~93~~, ~~94~~, ~~95~~, ~~96~~, 97, ~~98~~, ~~99~~

Animation of the sieve of Eratosthenes



Legendre's reformulation

- In 1808, Legendre gave a more analytic formulation of Eratosthenes sieve.
- He recognized the sieve as a larger idea embodied in the inclusion-exclusion principle.
- To this end, he used the Mobius function μ defined on the natural numbers as follows: $\mu(1)=1$; $\mu(n)=0$ if n is divisible by the square of a prime number and if n is a product r distinct primes, $\mu(n)=(-1)^r$.

Legendre's formula

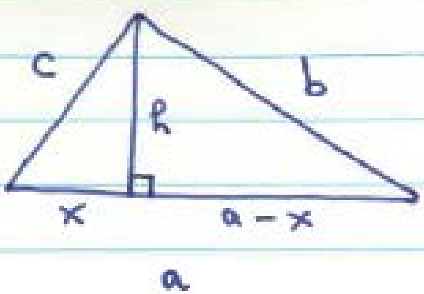
- Let $\pi(X)$ denote the number of primes up to X.
- Then n is a prime between $\sqrt{X}$ and X, if and only if it is not divisible by a prime p less than $\sqrt{X}$.
- Thus, $\pi(X) - \pi(\sqrt{X}) + 1 = \sum_{d|P} \mu(d)[X/d]$, where [t] denotes the greatest integer function and P denotes the product of the primes less than $\sqrt{X}$.
- The proof relies on the basic property of the Mobius function: $\Sigma_{d|n}\mu(d)=0$ unless n=1, in which case, the sum is 1.

Heron's formula

- Heron derived a formula for the area of a triangle using only the length of the sides.
- Given a triangle with side lengths a, b and c, let s denote the semi-perimeter. That is, $2s=a+b+c$.
- Heron showed that the area is $\sqrt{s(s-a)(s-b)(s-c)}$.
- He did this in a complicated manner which can be described as geometric rather than algebraic.

A proof of Heron's formula

- We will discuss later Heron's proof, but indicate here how the formula can be derived using the Pythagorean theorem.



$$\text{Area} = A = \tfrac{1}{2} a h$$

$$\therefore\ A^2 = \tfrac{1}{4} a^2 h^2.$$

$$\text{Pythagoras} \Rightarrow h^2 = c^2 - x^2$$
$$= b^2 - (a-x)^2$$

We can solve for x:

$$c^2 = b^2 - a^2 + 2ax \Rightarrow x = \frac{a^2+c^2-b^2}{2a}$$

Therefore

$$A^2 = \frac{1}{4} a^2 (c-x)(c+x) = \frac{1}{4} a^2 \left(c - \frac{a^2+c^2-b^2}{2a}\right)\left(c + \frac{a^2+c^2-b^2}{2a}\right)$$

Final simplification

- This expression is easily simplified:

$$
\begin{aligned}
16A^2 &= (2ac - a^2 - c^2 + b^2)(2ac + a^2 + c^2 - b^2) \\
&= (b^2 - (a-c)^2)((a+c)^2 - b^2) \\
&= (b - (a-c))(b + (a-c))(a+c-b)(a+c+b) \\
&= 2s(2s-2a)(2s-2b)(2s-2c) \\
\Rightarrow \quad A^2 &= s(s-a)(s-b)(s-c).
\end{aligned}
$$

A formula for the radius of the inscribed circle in a triangle

- Heron's formula allows us to derive a nice formula for the radius of the inscribed circle.
- Area=½(a+b+c)r.=rs=√s(s-a)(s-b)(s-c).

r= √(s-a)(s-b)(s-c)/√s.