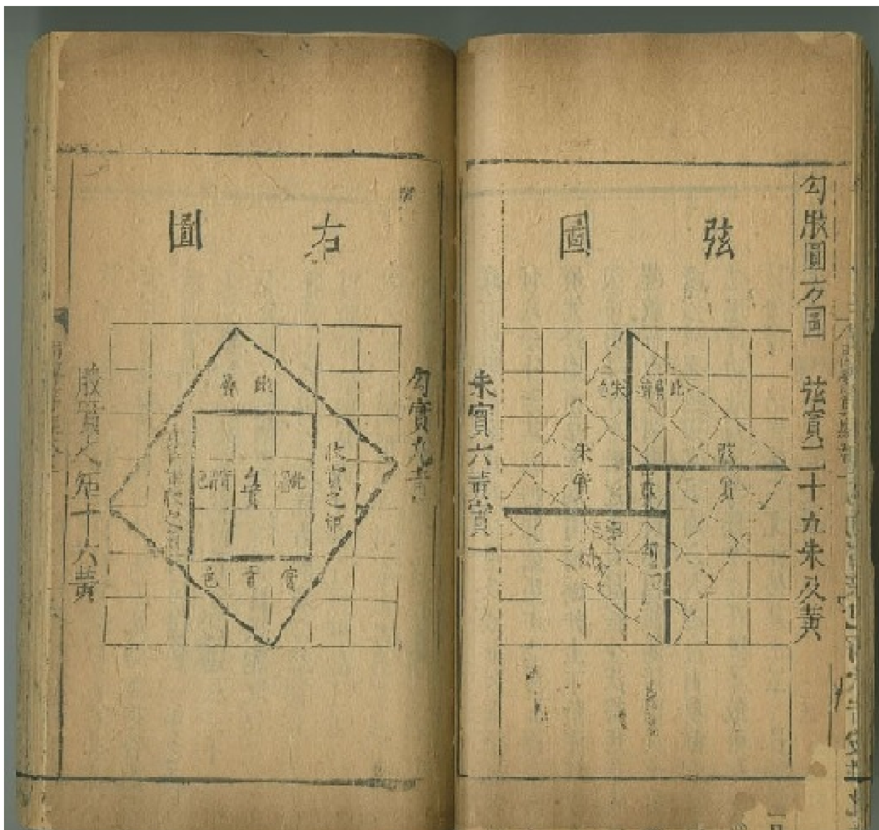
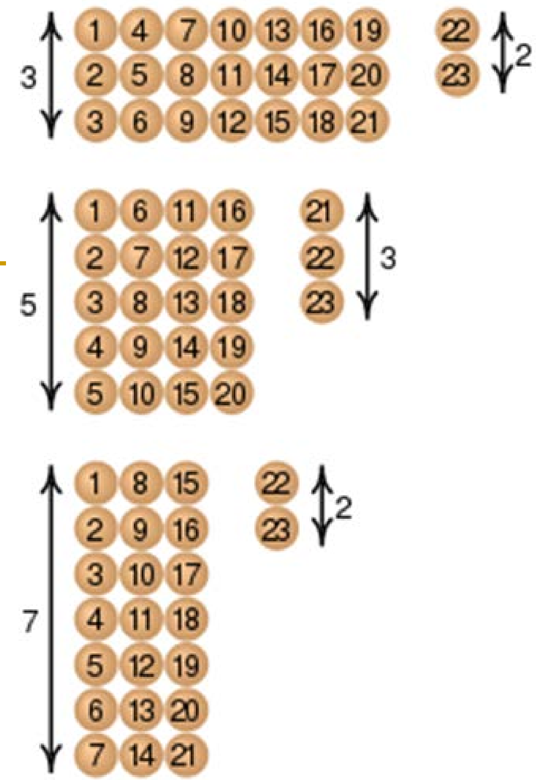
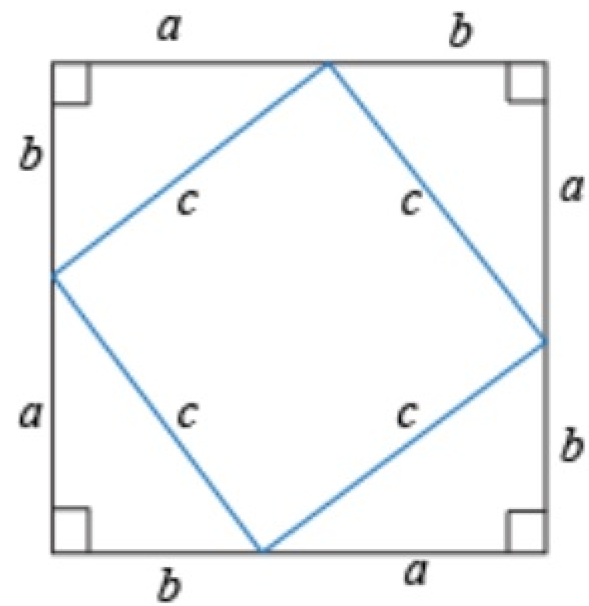
Mathematics in Ancient China





Chinese proof of the Pythagorean theorem

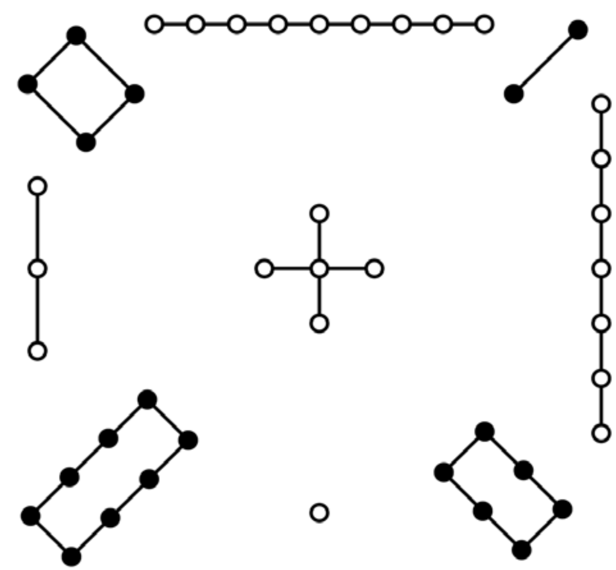
- The Chinese mathematicians seemed to have known about the Pythagorean theorem centuries before Pythagoras.
- The "Chou Pei Suan Ching" is considered the oldest mathematical work known and a conservative date puts it around 300 BCE. Some scholars place it even earlier.
- Here is a pictorial demonstration of the Pythagorean theorem.



"Nine Chapters" and Magic Squares

- A slightly later work called "Chiu-chang suan-shu" or "Nine chapters" is a collection of about 246 problems dealing with surveying and is dated between 300 BCE and 200 CE.
- There is a discussion of magic squares in this work.
- These are n x n matrix configurations containing all the numbers from 1 to n^2 whose rows, columns and diagonals add up to the same number.
- Here is an example: Lo Shu square dated 650 BCE.

4	9	2
3	5	7
8	1	6



Source: Crest of the Peacock by George Joseph

TABLE 6.1: MAJOR CHINESE MATHEMATICAL SOURCES UP TO THE SEVENTEENTH CENTURY

Title	*Author*	*Date*	*Notable subjects covered*
Zhou Bi Suan Jing (The Mathematical Classic of the Gnomon and the Circular Paths of Heaven)	Unknown	c. 500–200 BC	Pythagorean theorem; simple rules of fractions and arithmetic operations
Suanshu Shu (A Book on Arithmetic)	Unknown	300–150 BC	Operations with fractions; areas of rectangular fields; fair taxes
**Jiu Zhang Suan Shu* (Nine Chapters on Mathematical Arts)	Unknown	300 BC–AD 200	Root extraction; ratios (including the rule of three and the rule of false position); solution of simultaneous equations; areas and volumes of various geometrical figures and solids; right-angled triangles

Ta Tai Li Chi (Records of Rites Compiled by Tai the Elder)	Unknown	AD 80	Magic square order of 3
Commentary on *Jiu Zhang*	Chang Heng	130	π = square root of 10
Shu Shu Chi Yi (Manual on the Traditions of the Mathematical Arts)	Xu Yue	c. 200	Theory of large numbers; magic squares; first mention of the abacus
Commentary on *Zhou Bi*	Zhao Zhujing	c. 200–300	Solution of quadradic equations of the type $x^2 + ax = b^2$
Hai Dao Suan Jing (Sea Island Mathematical Manual)	Liu Hui	263	Extentions of problems in geometry and algebra from the *Nine Chapters*
Sun Zu Suan Jing (Master Sun's Mathematical Manual)	Sun Zu	400	A problem in indeterminate analysis; square root extraction; operations with rod numerals

Title	*Author*	*Date*	*Notable subjects covered*
Sui Shu (Official History of the Sui Dynasty)	Zu Chongzhi	450	Evaluation of π; method of finite differences
Ji Gu Suan Jing (Continuation of Ancient Mathematics)	Wang Xiaotong	625	Solution of third-degree equations; practical problems for engineers, architects, and surveyors
**Suan Jing Shi Shu* (The Ten Mathematical Manuals)	Li Chungfeng	656	An encyclopedia of mathematical classics of the past
Meng Xi Bi Tan (Dream Pool Essays)	Shen Kuo	1086	Summation of series by piling up a number of kegs in a space shaped like a dissected pyramid
**Shu Shu Jiu Zhang* (Nine Sections of Mathematics)	Qin Jiushao	1247	Numerical solutions of equations of high degree; indeterminate analysis
Ce Yuan Hai Jing (The Sea Mirror of the Circle Measurements)	Li Ye	1248	Solutions of high-degree equations; applications of the Pythagorean theorem to practical problems; use of a diagonal line across a digit to indicate a minus quantity

**Xiang Jie Jiu Zhang Suan Fa Zuan Lei* (Detailed Analysis of the Nine Chapters)	Yang Hui	1261	Arithmetic progressions; decimal fractions; quadratic equations with negative coefficients of *x*
Yuan Shi (Official History of the Yuan Dynasty)	Guo Shoujing	1280	Foundations of spherical trigonometry; cubic interpolation formula; biquadratic equations
**Si Yuan Yu Jian* (The Precious Mirror of the Four Elements)	Zhu Shijie	1303	Pascal's triangle; solutions of simultaneous equations with five unknowns by matrix methods
Suan Fa Dong Zong (A Systematic Treatise on Arithmetic)	Cheng Tai We	1593	Magic squares; introduction to abacus
Ji He Yuan Pen (Elements of Geometry)	Ricci and Xu	1607	Six books of Euclid's *Elements* translated into Chinese

Magic squares, matrices and the Chinese remainder theorem

- There is an inextricable link between the discovery of magic squares and the concept of a matrix.
- Matrix theory as we use it today was developed by Hermann Grassmann in his book *Ausdehnungslehre* in 1844.
- But it is clear that the ancient Chinese mathematicians were developing this idea through the centuries.
- Matrices are related to solving simultaneous equations. The Chinese remainder theorem deals with simultaneous congruences.

Column reduction of matrices

- In the Nine Chapters, we find discussion of solving systems of linear equations via the method of column reduction (or equivalently row reduction) usually attributed to Gauss in the 19th century.
- Here is an example from the Nine Chapters.

$$3x + 2y + z = 39$$
$$2x + 3y + z = 34$$
$$x + 2y + 3z = 26$$

by performing column operations on the matrix

1	2	3
2	3	2
3	1	1
26	34	39

to reduce it to

0	0	3
0	5	2
36	1	1
99	24	39

The second form represented the equations $36z = 99$, $5y + z = 24$, and $3x + 2y + z = 39$, from which the values of z, y, and x are successively found with ease.

How to construct a magic square?

- It is not difficult to show that the Lo-Shu magic square of order 3 is essentially unique.
- Let us first consider the problem of constructing a magic square of order n.
- We have to arrange the numbers from 1 to n^2 in an n x n matrix so that each row, each column and each diagonal add up to the same number.
- Since the sum of all the numbers is $n^2(n^2+1)/2$ and there are n rows, each row must sum to $n(n^2+1)/2$.
- Two questions arise: existence and uniqueness.

Uniqueness of the Lo-Shu square

- Using basic linear algebra, we can show that the Lo-Shu square is essentially unique.
- Each row, column and diagonal has to add up to 15.

$$\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & I \end{vmatrix}$$

A+B+C=15, D+E+F=15, G+H+I=15,

A+D+G-15, B+E+H=15, C+F+I=15,

A+E+I-15, and G+E+C=15

Eliminating seven of the unknowns, we arrive at the matrix-

$$\begin{vmatrix} A & B & 15-A-B \\ 20-2A-B & 5 & 2A+B-10 \\ A+B-5 & 10-B & 10-A \end{vmatrix}$$

How to choose A and B?

$$\begin{vmatrix} A & B & 15-A-B \\ 20-2A-B & 5 & 2A+B-10 \\ A+B-5 & 10-B & 10-A \end{vmatrix}$$

- We have to choose A and B such that all the entries are positive, distinct and less than 9.
- We see immediately that $6 \leq A+B \leq 14$.
- The only solutions are (A,B)=(4,3), (2,9), (6,7) and (8,1) and they produce the squares:

These are all seen to be essentially the same because they are all rotations of the Lo Shu square.

$$\begin{vmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{vmatrix}, \begin{vmatrix} 6 & 7 & 2 \\ 1 & 5 & 9 \\ 8 & 3 & 4 \end{vmatrix} \quad and \quad \begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix}$$

What about magic squares of higher order?

- One can take this 3 x 3 square and use it to construct a magic square of order 6 as follows.
- If we add 15+3k, with k=0,9, 18 and 27, we get:

$$\begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix}, \begin{vmatrix} 17 & 10 & 15 \\ 12 & 14 & 16 \\ 13 & 18 & 11 \end{vmatrix}, \begin{vmatrix} 26 & 19 & 24 \\ 21 & 23 & 25 \\ 22 & 27 & 20 \end{vmatrix}, \begin{vmatrix} 35 & 28 & 33 \\ 30 & 32 & 34 \\ 31 & 36 & 29 \end{vmatrix}$$

$$\begin{vmatrix} 8 & 1 & 6 & 26 & 19 & 24 \\ 3 & 5 & 7 & 21 & 23 & 25 \\ 4 & 9 & 2 & 22 & 27 & 20 \\ 35 & 28 & 33 & 17 & 10 & 15 \\ 30 & 32 & 34 & 12 & 14 & 16 \\ 31 & 36 & 29 & 13 & 18 & 11 \end{vmatrix}$$

We see all the numbers from 1 to 36 occur. We may try to paste together these to form a magic square of order 6.

But the row sums are not equal to 111 .

Modification of the matrix

- The first three rows add up to 84 and the last three to 138, so there is a deficit of 27. We modify our matrix accordingly.

$$\begin{vmatrix} 8 & 1 & 6 & 26 & 19 & 24 \\ 3 & 5 & 7 & 21 & 23 & 25 \\ 4 & 9 & 2 & 22 & 27 & 20 \\ 35 & 28 & 33 & 17 & 10 & 15 \\ 30 & 32 & 34 & 12 & 14 & 16 \\ 31 & 36 & 29 & 13 & 18 & 11 \end{vmatrix}$$

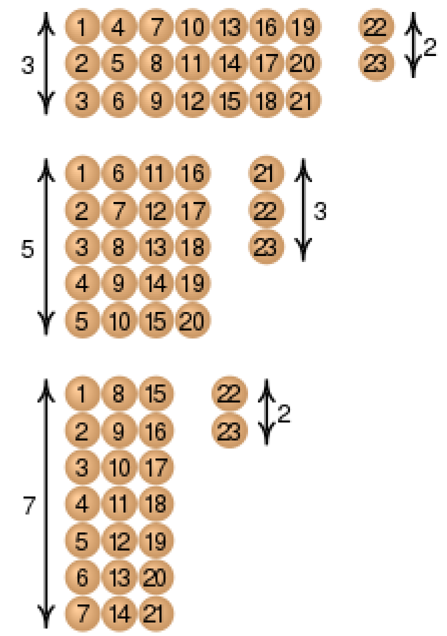
$8 \leftrightarrow 35$ $\quad$ $5 \leftrightarrow 32$ $\quad$ $4 \leftrightarrow 31$

$$\begin{vmatrix} 35 & 1 & 6 & 26 & 19 & 24 \\ 3 & 32 & 7 & 21 & 23 & 25 \\ 31 & 9 & 2 & 22 & 27 & 20 \\ 8 & 28 & 33 & 17 & 10 & 15 \\ 30 & 5 & 34 & 12 & 14 & 16 \\ 4 & 36 & 29 & 13 & 18 & 11 \end{vmatrix}$$

This is a valid 6 x 6 magic square.

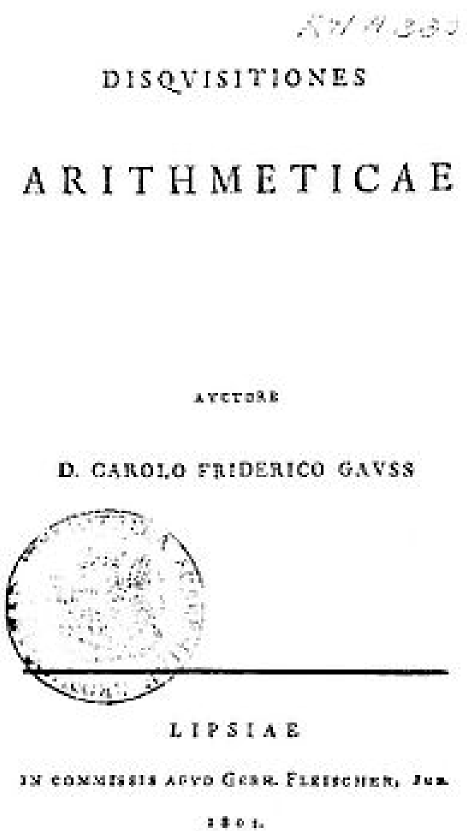
The Chinese remainder theorem

- In the 3rd century, Sun-Tzu posed the following problem: find a number x which when divided by 3 leaves a remainder of 2, when divided by 5 leaves a remainder of 3, and when divided by 7 leaves a remainder of 2.
- He gave the answer 23. But it is clear that if we add any multiple of 105, it will still work.
- In fact, these are all the solutions in positive integers.



Sun-Tzu, Aryabhata and Brahmagupta

- Sun-Tzu gave no formal proof of his method. This was later done independently by Aryabhata in the 6th century and Brahmagupta in the 7th century.
- But it is clear from these writings that he had a general method which we now call the Chinese remainder theorem.
- Today, we state it as follows: if $n_1, \ldots, n_k$ are coprime natural numbers and $a_1, \ldots, a_k$ are arbitrary, then the congruence $x \equiv a_1 \pmod{n_1}, \ldots, x \equiv a_k \pmod{n_k}$ has a unique solution.
- This version appears in Gauss's Disquisitiones written in 1801.



Proof of the Chinese remainder theorem

- There are several ways to prove the Chinese remainder theorem.
- The simplest is by an inductive argument, we can reduce to k=2.
- Thus, given two coprime numbers M and N, we have to show we can solve $x \equiv a \pmod{M}$ and $x \equiv b \pmod{N}$.
- Since M and N are coprime, there are integers u and v such that $Mu + Nv = 1$.
- Put $e=Mu$ and $f=Nv$. Then $e=1 \pmod{N}$ and $0 \pmod{M}$, $f=1 \pmod{M}$ and $0 \pmod{N}$. Then $x=af + be$ works.
- Uniqueness is clear for if x and y are distinct solutions mod MN, then $x=y \pmod{MN}$.