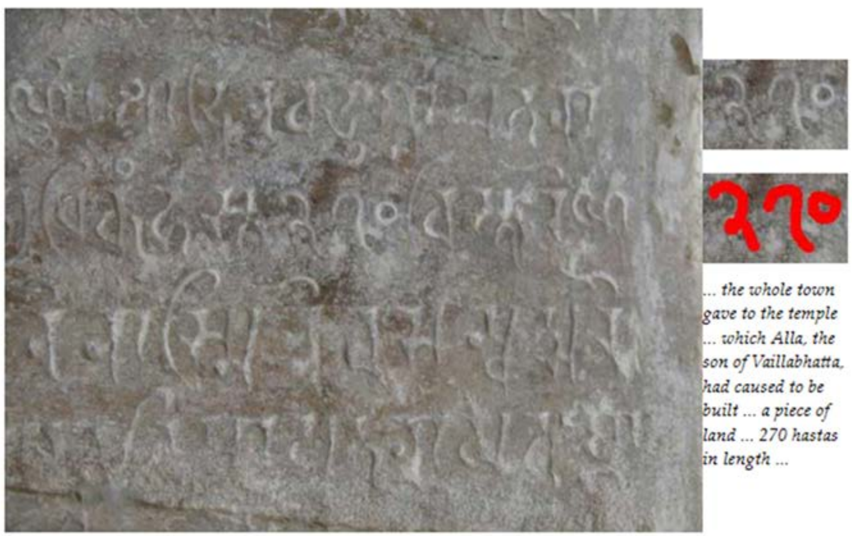
Mathematics in Ancient India



Magic squares in India

Chautisa Yantra: Parshvanath Jain temple in Khajuraho, India (10th century)

Each row, column and diagonal adds up to 34 as it should.

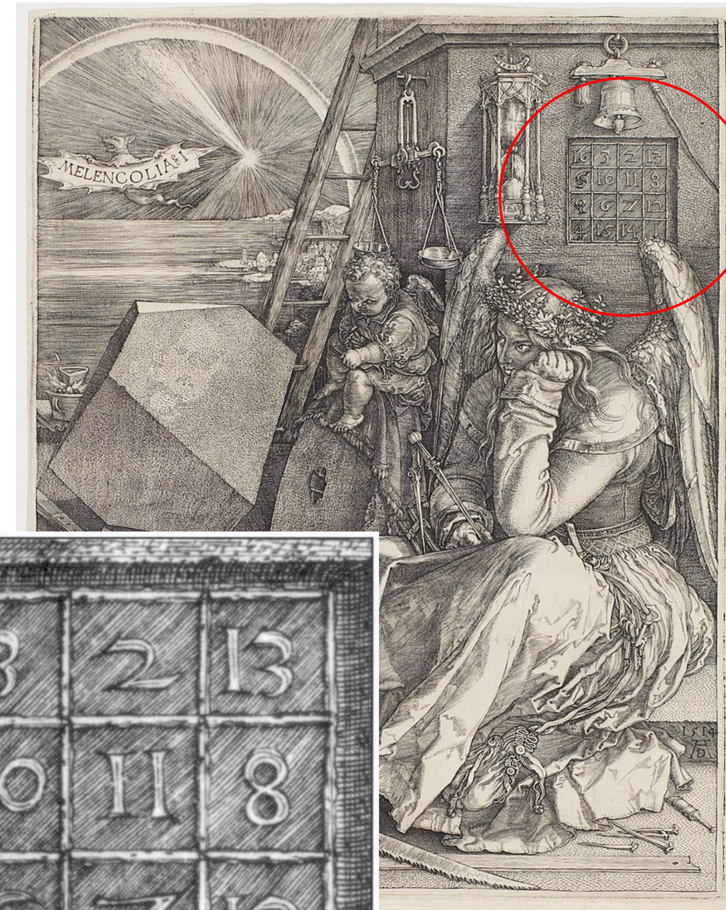
7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

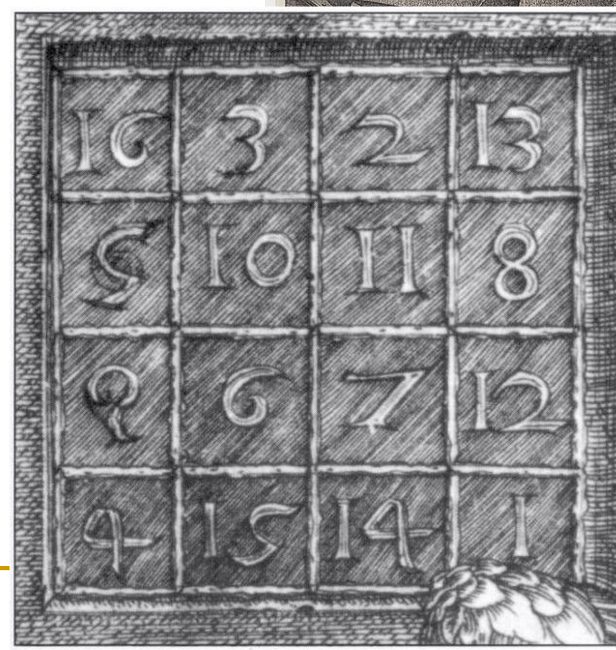
Mathematics and Leonardo da Vinci

- In the Renaissance period, we see interest in the relation between art and mathematics.
- Echoing Plato who wrote "Let no one ignorant of geometry enter this Academy", Leonardo da Vinci wrote "Let no one who is not a mathematician read my works."

Magic squares in the Renaissance

- The interest in mathematics is evident in the work of Albrecht Durer's "Melancholia".
- We can see the presence a variation of the 4 x 4 magic square of India in his painting.





Existence of magic squares of order n

- We will show how to place all the numbers from 1 to n^2 in an n x n grid so as to form a magic square.
- Recall:

?	?	?	$\cdots$	?	S
?	?	?	$\cdots$	?	S
?	?	?	$\cdots$	?	S
?	?	?	$\cdots$	?	$\vdots$
?	?	?	$\cdots$	?	S
					$n \cdot S$

So

$$\begin{aligned} n \cdot S &= 1 + 2 + 3 + \cdots + n^2 \\ &= \frac{n^2(n^2+1)}{2} \\ S &= \frac{n(n^2+1)}{2} \end{aligned}$$

Lehmer's algorithm (1929)

The numbers 1 through n^2 are arranged as follows. Choose a position (p, q) in the square M so that 1 is the entry m_{pq}. Pick numbers α and β ($\alpha < n$, $\beta < n$), called "steps," to determine the desired position for the number 2. Then 2 is placed as entry $m_{p+\alpha,q+\beta}$, 3 is in $m_{p+2\alpha,q+2\beta}$, and so forth through n, putting each number k in the position $m_{p+(k-1)\alpha,q+(k-1)\beta}$, where $p+(k-1)\alpha$ and $q+(k-1)\beta$ are reduced modulo n. To keep the number $n+1$ from being in the same place as 1, introduce a "break step" (a, b) so that $n+1$ becomes entry $m_{p+a,q+b}$, and so forth. In general, a number x is placed in entry m_{ij} where

$$i \equiv p + \alpha(x-1) + [\![\tfrac{x-1}{n}]\!]\, a \pmod n \quad \text{and} \quad j \equiv q + \beta(x-1) + [\![\tfrac{x-1}{n}]\!]\, b \pmod n.$$

(The symbol $[\![y]\!]$ represents the greatest integer function.) The square formed by this method is magic if and only if $a, b, \alpha, \beta,$ and $(\alpha b - \beta a)$ are each relatively prime to n

Why does this work?

Theorem. The Lehmer algorithm produces a magic square of order n if α, β, a, b and

$$\alpha b - a\beta$$

are coprime to n.

Proof. The number x is to be placed in the (i, j) position where

$$i \equiv p + \alpha(x-1) + a\left[\frac{x-1}{n}\right] \pmod{n}$$

$$j \equiv q + \beta(x-1) + b\left[\frac{x-1}{n}\right] \pmod{n}$$

and the reduced residues in $[1, n]$ are chosen.

The map is one-to-one

We first show the map $x \mapsto (i, j)$
from the set
$$1 \le x \le n^2$$
to $[1, n] \times [1, n]$ is one-one and onto.

Since both sets have the same size,
it suffices to show the map is
one-one.

Inverting a matrix mod n

This reduces to congruence matrix equation

$$\begin{bmatrix} \alpha & a \\ \beta & b \end{bmatrix} \begin{bmatrix} x-y \\ \left[\frac{x-1}{n}\right]-\left[\frac{y-1}{n}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{n}.$$

This means we need to do linear algebra (mod n).

Since

$$\det \begin{bmatrix} \alpha & a \\ \beta & b \end{bmatrix} = \alpha b - a\beta$$

is coprime to n, the matrix is invertible (mod n). Thus

$$x \equiv y \pmod{n}$$

and

$$\left[\frac{x-1}{n}\right] \equiv \left[\frac{y-1}{n}\right] \pmod{n}$$

Thus the map is one-one.

Checking row, column & diagonal sums

We fix a row i_0 and add the entries in that row. The entries are those x satisfying the congruence

$$p + \alpha(x-1) + a\left[\frac{x-1}{n}\right] \equiv i_0 \pmod{n}.$$

Write $x-1 = qn + r$ so that

$0 \le r \le n-1$, and $0 \le q \le n-1$.

Thus $\left[\frac{x-1}{n}\right] = q$. We have

$$p + \alpha(qn+r) + aq \equiv i_0 \pmod{n}$$

$$\Rightarrow (\alpha n + a)q + \alpha r \equiv i_0 - p \pmod{n}.$$

Choosing any r with $0 \le r \le n-1$ gives a unique q, since

$\alpha n + a$ is coprime to n.

Therefore, the x's that appear in row i_0 have the form

$$x = (qn+r) + 1 \qquad 0 \le q,r \le n-1$$

Final tally

As r runs through $0 \le r \le n-1$ we get a unique q in the range

$$0 \le q \le n-1.$$

The sum is

$$n\left(\frac{1}{2}n(n-1)\right) + \frac{1}{2}n(n-1) + n$$

$$= \frac{1}{2}n(n^2+1), \quad \text{as desired.}$$

- The verification on the columns, and diagonals is similar and left as an exercise.

Source: Crest of the Peacock by George Joseph

Table 8.1: Chronology of Indian History and Mathematics

Period	*Main historical events*	*Mathematics*	*Notable mathematicians*
3000–1500 BC	The Indus Valley civilization (script undeciphered) covering 1–2 million square km; main urban centers Harappa, Lothal, and Mohenjo-Daro	Weights, artistic designs, "Indus scale"; brick technology probably influenced the construction of Vedic altars in the next period	
1500–500 BC	The coming of the Aryans; the formation of Hindu civilization; the emergence of the *Code of Manu*; the recording of the Vedas and *Upanishads*	*Vedangas* and *Sulbasutras*; problems in astronomy, arithmetical operations, Vedic geometry	Baudhayana, Apastamba, Katyayana

500–200	The establishment of Indian states; the rise of Buddhism and Jainism; contacts with Persia maintained; the Mauryan empire, culminating in the reign of Asoka, who spread Buddhism abroad	Vedic mathematics continues during the earlier years but declines with ending of ritual sacrifices; beginnings of Jaina mathematics: number theory, permutations and combinations, the binomial theorem; astronomy
200 BC–AD 400	Triple division: Kushan dynasty (North), Pandyas (South), Bactrian-Persian (Punjab); pervading influence of Buddhism in art and sculpture	Jaina mathematics: rules of mathematical operations, decimal-place notation, first use of 0; algebra including simple, simultaneous, and quadratic equations; square roots; details of how to represent unknown quantities and negative signs

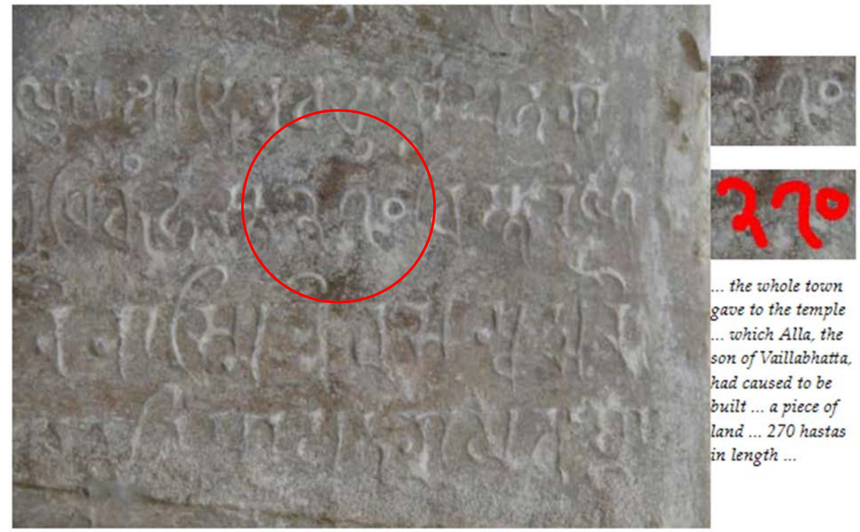
400–1200	Imperial Guptas reaching their height in the reign of Harsha (606–647); flowering of Indian civilization as shown in science, philosophy, medicine, logic, grammar, and literature	The Classical period of Indian mathematics; important works: the Bakshali Manuscript, *Aryabhatiya*, *Panca-siddhantika*, *Aryabhatiya Bhasya*, *Maha Bhaskariya*, *Brahma Shputa-siddhanta*, *Patiganita*, *Ganita Sara Samgraha*, *Ganitilaka*, *Lilavati*, *Bijaganita*	Aryabhata I, Varahamihira, Bhaskara I, Brahmagupta, Sridhara, Mahavira, Bhaskara II (also known as Bhaskaracharya)
1200–1600	Early Muslim dynasties; birth of Sikhism; the Hindu kingdom of Vijaynagar in the South	Decline of mathematics and learning in the North; the rise of the Kerala school of astronomy and mathematics; work on infinite series and analysis	Narayana, Madhava, Nilakantha

The Sulvasutras

- The word “sulva” means rope. The Sulvasutras are “rules of the chord” or the art of measurement.
- In Apastamba’s Sulvasutras, we find the first mention of the Pythagorean theorem.
- He also discusses how the construct a square whose area is equal to a given rectangle.
- We also find here references on how to approximate $\sqrt{2}$.

The decimal system and the use of zero

- The earliest known evidence for the use of the modern decimal system dates back to the Vedas and are dated around 1500 BCE.
- But there are inscriptions in Indian temples dating to 800 CE. But it is clear from Brahmagupta's work, the system was already in place around 600 CE in India.
- The best known is the famous Vishnu temple in Gwalior, Madhya Pradesh where the inscription clearly shows the number 270.



Laplace on the value of the decimal system

- The adoption of the decimal system in Europe was late in coming.
- At first, the sexagesimal system of the Babylonians gained prominence that Copernicus was using it.
- The Arab world adopted it in the 10^{th} century.
- Laplace is quoted as saying "It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of the achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity"

Pierre-Simon Laplace
(1749-1827)

From magic squares to square roots

- But what we see as a recurrent theme in Indian mathematics is the study of square roots.
- In attempts to get better approximations, they were led to study a special Diophantine equation, now called the Brahmagupta-Pell equation $x^2 - Dy^2=1$.
- We will take up the work of Brahmagupta and others in the next lecture.