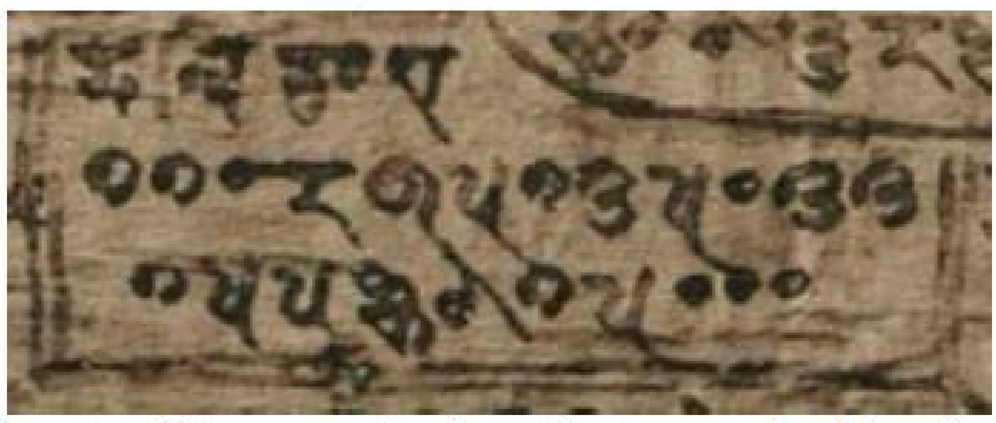
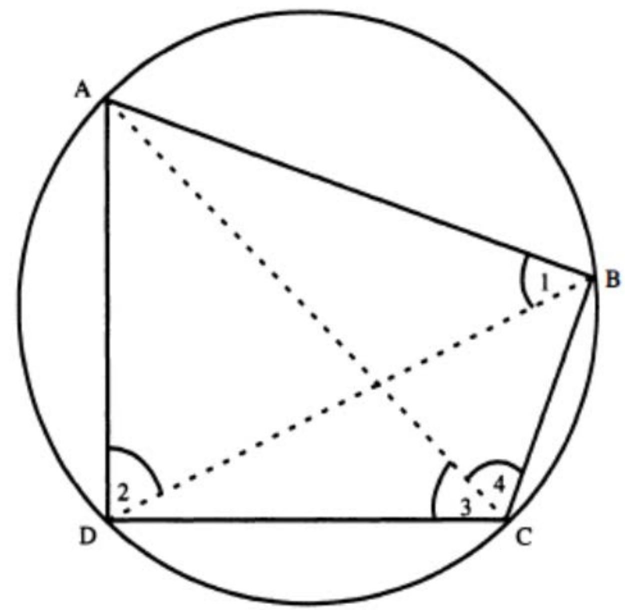
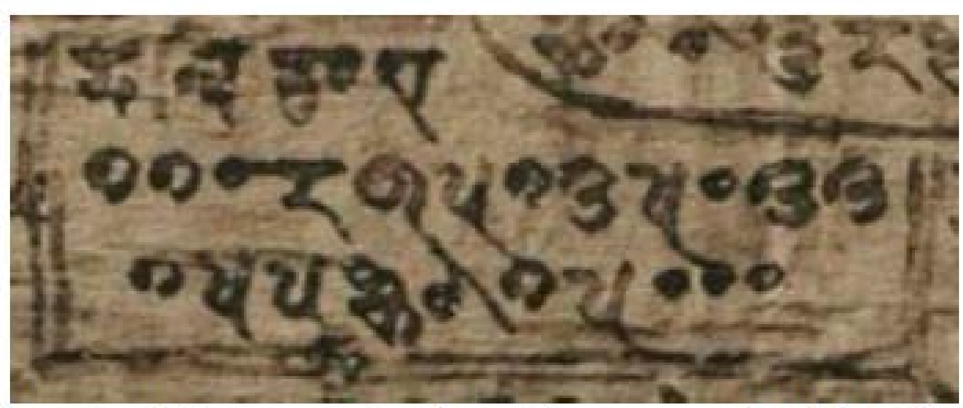
Brahmagupta and Bhaskara





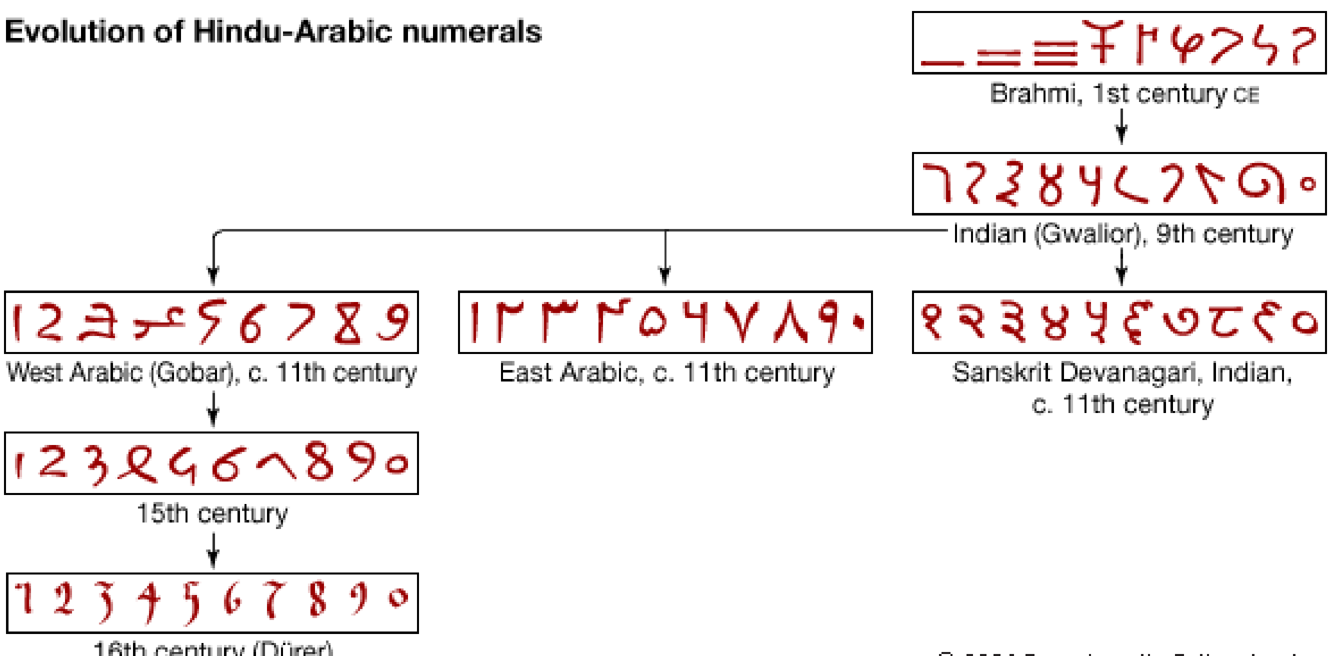
The Bakshali manuscript

- The oldest written record of the use of the decimal number system and in particular, the use of zero as a number dates back to the Bakshali manuscript discovered in northern India in 1881.
- In 2014, using carbon dating, it has been certified that the decimal system was in vogue as far back as 300 CE.



The evolution of the decimal system

- The flow chart below indicates how the original Brahmi script for the numbers morphed over time to its present form.



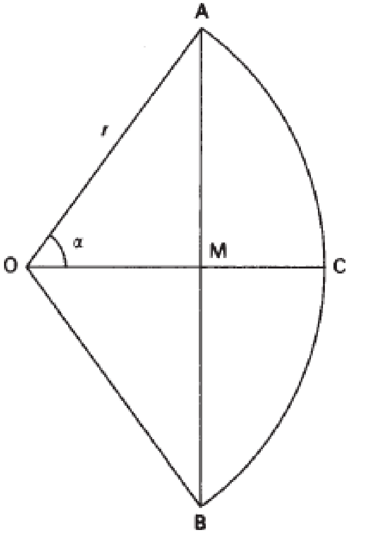
Aryabhata and the “pulverizer” method

- In the year 499 CE, Aryabhata wrote his book titled “Aryabhatiya” dealing with astronomy and mathematics.
- In it, he describes the “division algorithm” or the Euclidean algorithm labelled as the “kuttaka” method, which translates as the “pulverizer”.
- Thus, he was able to solve are now called linear Diophantine equations.
- There is an extensive section on how to work with decimal numbers.

The birth of trigonometry

- Aryabhata's book seems to contain the first occurrence of sine tables which he needed to calculate distance from the earth to various planets.
- The word he used for the sine is "jyardha" which means "half-chord" and meaning is clear from the diagram.

The word "jyardha" became abbreviated to "jya" and cosine was referred to as "kojya". When the arab mathematicians translated Indian works, they changed it to "jiba" which morped into "jyb". When Latin translators encountered this word, they took it as "jaib" which meant "opening a women's garment around the neck" and so they translated it as "sinus" which meant "bosom" or "curve". And that is the origin of the word "sine"!



Proof by Bhaskara (12th century) of the Pythagorean theorem

Here is Bhaskara's short proof of the Pythagorean theorem. Consider the right angled triangle ABC with the right angle at C along with the perpendicular CD to AB (see Figure 1). The triangles ABC, ACD and CBD are similar.

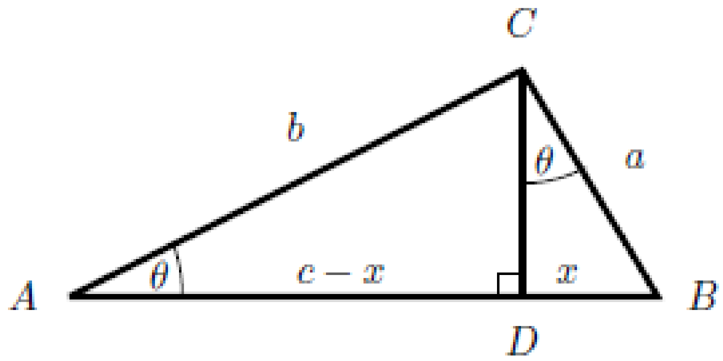


Fig. 1: Bhaskaracharya's proof of the Pythagorean theorem

Thus, comparing the smaller triangles to the big triangle ABC, we get

$$\cos\theta = \frac{c-x}{b} = \frac{b}{c},$$

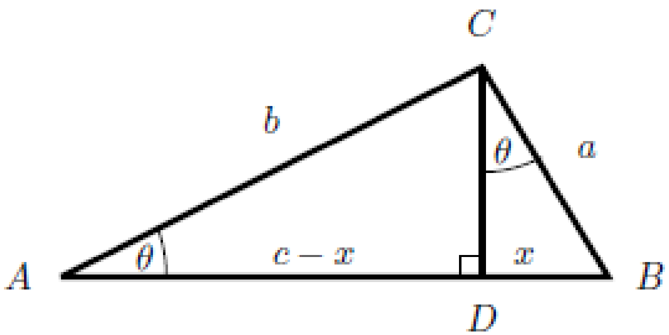
so that $b^2 = c^2 - cx$. Similarly,

$$\sin\theta = \frac{x}{a} = \frac{a}{c},$$

so that $a^2 = cx$. Putting these two equations together gives us the familiar Pythagorean theorem:

$$c^2 = a^2 + b^2,$$

The cosine law from Bhaskara’s proof



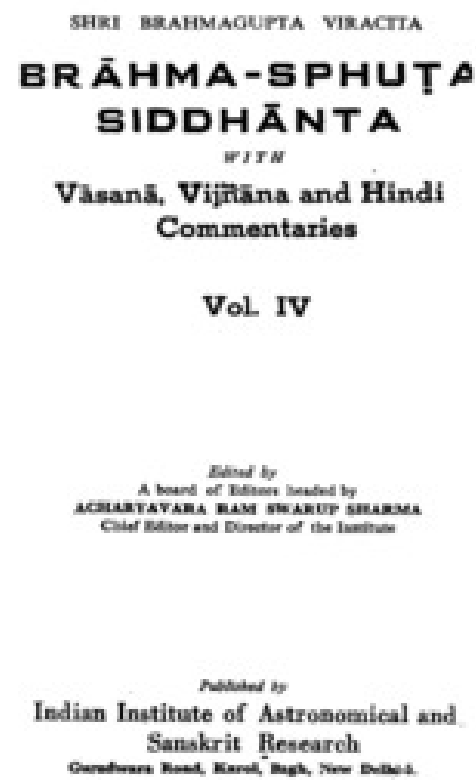
Area= ½ bc sin A

The sine law is also deduced from this!

- We can deduce the cosine law from Bhaskara’s proof as follows.
- We have $a^2 - x^2 = b^2 - (c-x)^2$ from which we easily deduce $b^2 = a^2 + c^2 - 2cx$.
- But $x/a = \cos B$ which gives the cosine law.

Brahma Sphuta Siddhanta

- Brahmagupta lived around 600 CE, in India.
- He is famous for his work titled Brahma Sphuta Siddhanta as well as his detailed study of the equation $x^2 - Ny^2 = 1$ which we now call the Brahmagupta-Pell equation because John Pell wrote a letter to Euler in the 18th century asking how to solve it.



SHRI BRAHMAGUPTA VIRACITA

BRĀHMA-SPHUṬA SIDDHĀNTA

WITH

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Brahmagupta identity

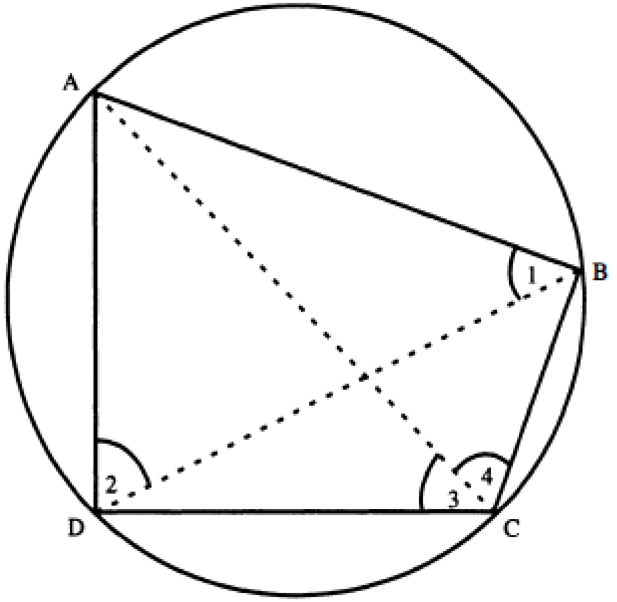
- His work contains a discussion of the Euclidean algorithm for the gcd of two numbers, which he called "the pulverizer method" since the remainder upon division by a given number "pulverizes" the number into a smaller parts by repeated application.
- All of these seem to be independent discoveries and after a discussion of what can be called linear Diophantine equations, he studies the equation x^2-Ny^2=1.
- He proved what is now called the Brahmagupta identity:

$$(x_1^2 - Ny_1^2)(x_2^2 - Ny_2^2) = (x_1x_2 + Ny_1y_2)^2 - N(x_1y_2 + x_2y_1)^2$$

The Bhavana and Chakravala

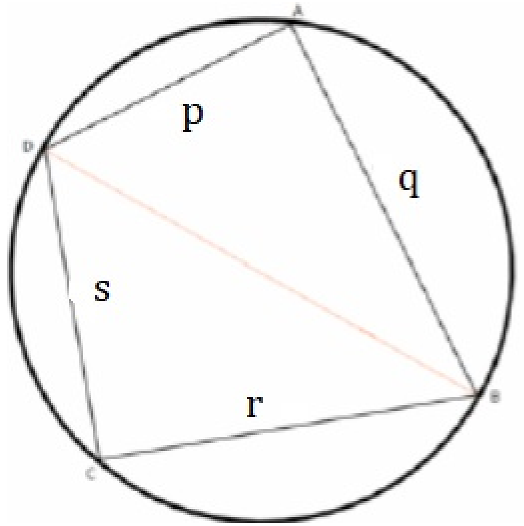
- He realized that this identity can be used to generate all the solutions by knowing one solution of the equation. He called this the Bhavana method. He called the algorithm Chakravala. In Sanskrit, "chakra" means "wheel" and so he knew that the solutions form a "cyclic group" though he didn't state it in these terms.
- He had no complete proof of his claim and it had to wait until Bhaskara gave a complete description of the method in the year 1150.
- Though Bhaskara gave a complete description of the algorithm, he didn't prove that it generates all solutions. This was done by Lagrange in the 18^{th} century.

Cyclic quadrilaterals



- In his book, Brahmagupta devotes a chapter to the study of cyclic quadrilaterals.
- One of his theorems is that the opposite angles of a cyclic quadrilateral add up to 180 degrees, which also appears in Euclid.
- The proof is easily seen from the picture.

Brahmagupta’s cyclic quadrilateral



- In his work, he describes how to calculate the area of a cyclic quadrilateral, thus generalizing Heron’s formula.
- The area K is: $= \frac{1}{2}pq\sin A + \frac{1}{2}rs\sin C.$

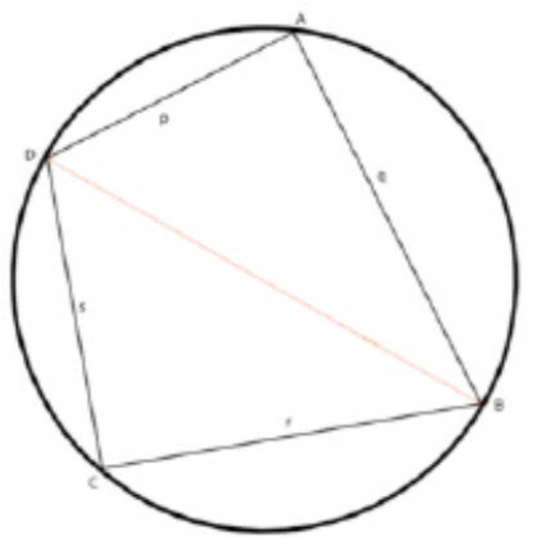
But since $ABCD$ is a cyclic quadrilateral, $\angle DAB = 180^\circ - \angle DCB$. Hence $\sin A = \sin C$. Therefore,

$$K = \frac{1}{2}pq\sin A + \frac{1}{2}rs\sin A$$

$$K^2 = \frac{1}{4}(pq + rs)^2 \sin^2 A$$

Brahmagupta’s generalization of Heron’s formula

- Simplifying:



$$4K^2 = (pq+rs)^2(1-\cos^2 A) = (pq+rs)^2 - (pq+rs)^2\cos^2 A.$$

Solving for common side DB, in $\triangle ADB$ and $\triangle BDC$, the law of cosines gives

$$p^2+q^2-2pq\cos A = r^2+s^2-2rs\cos C.$$

Substituting $\cos C = -\cos A$ (since angles A and C are supplementary) and rearranging, we have

$$2(pq+rs)\cos A = p^2+q^2-r^2-s^2.$$

Substituting this in the equation for the area,

$$4K^2 = (pq+rs)^2 - \frac{1}{4}(p^2+q^2-r^2-s^2)^2$$

$$16K^2 = 4(pq+rs)^2 - (p^2+q^2-r^2-s^2)^2.$$

The right-hand side is of the form $a^2-b^2=(a-b)(a+b)$ and hence can be written as

$$[2(pq+rs)-p^2-q^2+r^2+s^2][2(pq+rs)+p^2+q^2-r^2-s^2]$$

which, upon rearranging the terms in the square brackets, yields

$$=[(r+s)^2-(p-q)^2][(p+q)^2-(r-s)^2]$$

$$=(q+r+s-p)(p+r+s-q)(p+q+s-r)(p+q+r-s).$$

Introducing the semiperimeter $S=\frac{p+q+r+s}{2}$,

$$16K^2 = 16(S-p)(S-q)(S-r)(S-s).$$

Taking the square root, we get

$$K=\sqrt{(S-p)(S-q)(S-r)(S-s)}.$$

The treatise on zero

- But the remarkable aspect of Brahmagupta's work is his sections on arithmetic operations especially with using zero as a number, and not just as a place holder signifying "nothing".
- He described how to work with both positive and negative numbers, and that if anything is multiplied by zero, one gets zero.
- He considered even division by zero and stated incorrectly that $0/0=0$, which in modern treatments is not defined.