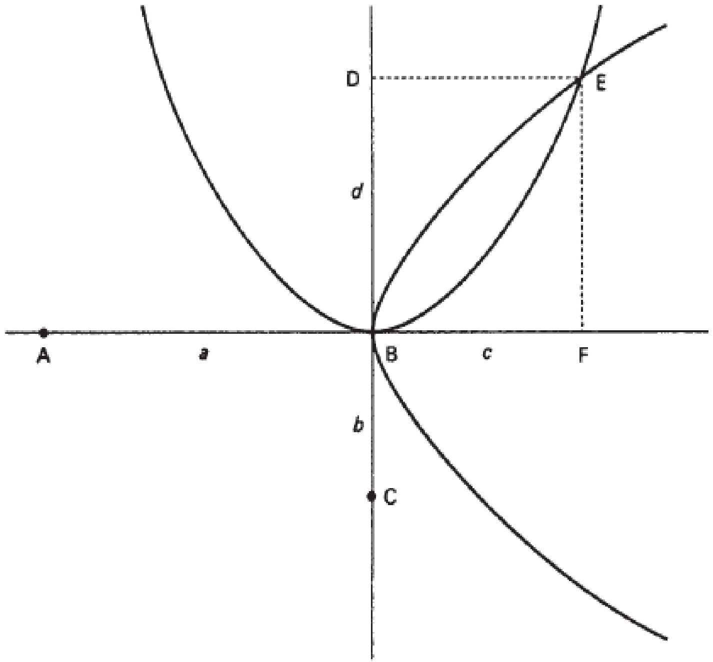
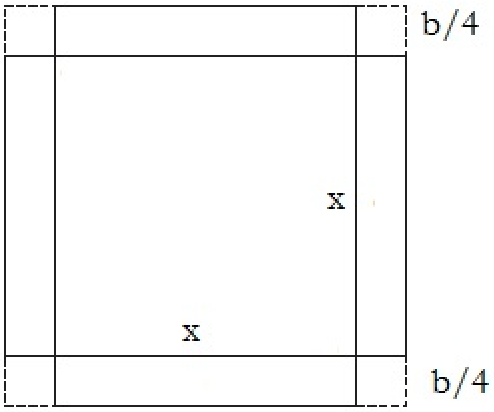
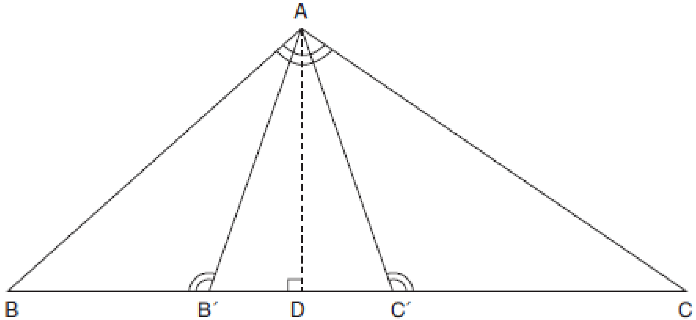
Al-Khwarizmi and Arabic mathematics

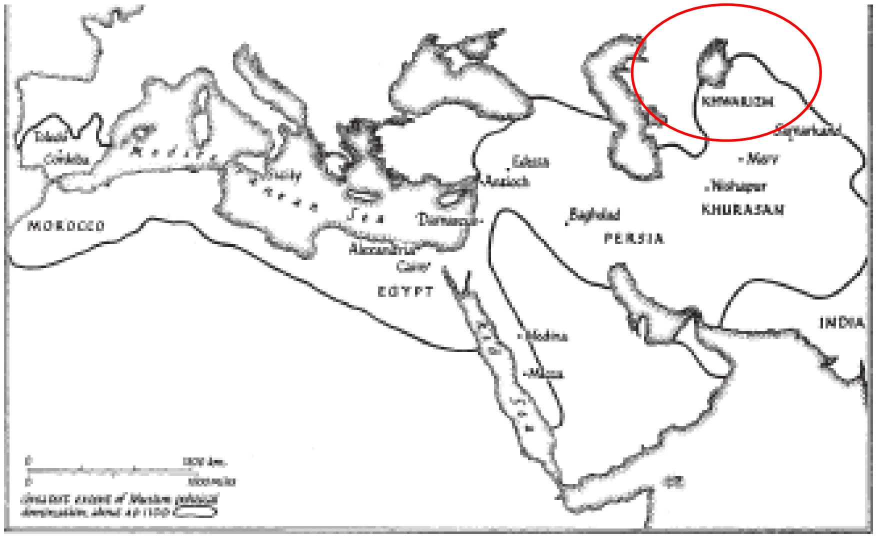






Baghdad and the House of Wisdom

- Prophet Mohammed was born in Mecca in 570 CE.
- He is considered as the founder of Islam and with its rise, Baghdad became the center of learning.
- The "House of Wisdom" rivalled the ancient Library of Alexandria. There many Arabic scholars and mathematicians began to translate the ancient Greek texts and the Hindu texts.
- The most famous was the work of Al-Khwarizmi who translated the works of Brahmagupta dealing with the decimal system.



The birth of algebra

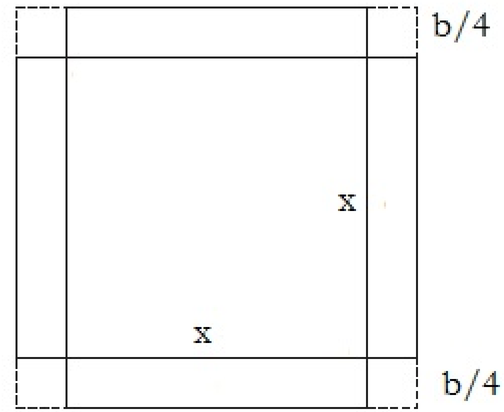
- In translating works of Diophantus and Brahmagupta, Al-Khwarizmi gave a systematic account of arithmetic.
- In fact, the modern word “algorithm” comes from the name Al-Khwarizmi.
- But his most famous work is “Al-jabr wa’l muqabalah” from which the modern word “algebra” is derived.
- The word “al-jabr” refers to moving negative quantities to the other side of an equation so as to make them positive. The word “muqabalah” means to cancel like terms on both sides of the equation.

Quadratic equations

- An important part of Al-Khwarizmi's Algebra is dedicated to the study of six cases of the roots of quadratic equations.
- These cases are:
- 1. $x^2=bx$; 2. $x^2=c$; 3. $x^2+bx+c=0$;
- 4. $x^2=bx+c$; 5. $x^2+bx=c$; 6. $x^2+c=bx$.
- The problem lay in trying to visualize these equations in order to find the roots.

The method of completing the squares

- Let us consider the method to solve $x^2 + bx = c$.
- Construct first a square with area x^2.
- Construct on each of the four sides a rectangle of width $b/4$ so that their total area is bx.
- If we add the four corner squares of area $b^2/16$ we get a big square whose area is $(x+b/2)^2$.



Thus, $(x+b/2)^2 = c+b^2/4$, from which the solution is easily found.

The problem of the 17 camels

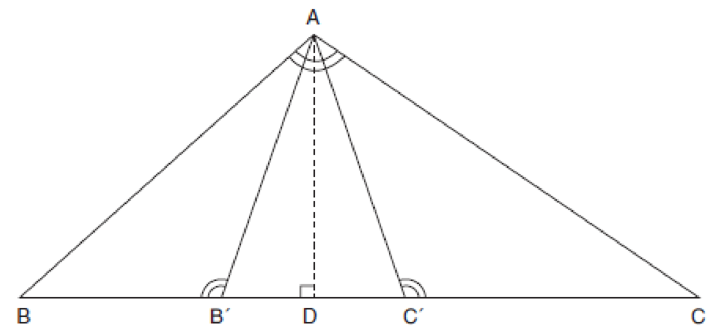
- The second part of Al-Khwarizmi's Algebra deals with "inheritance problems" and how to divide property.
- Here is one which seems to be a 1400 year old joke: a man died leaving 17 camels. He stipulated that ½ of them should go to his eldest son, 1/3 to his second son and 1/9 to the third son. How should the camels be divided?
- A wise man came and said, "Not to worry, I add my camel to make 18 and now make the division." So the eldest son got 9, the second got 6 and the third got 2; the wise man then took his camel!

The work of Abd al-Hamid ibn-Turk

- In the work of Al-Khwarizmi, we find a rejection of roots which are negative as well as the cases where the discriminant of the quadratic is negative.
- A work by Abd al-Hamid ibn Turk which may have been earlier recognizes the case of negative discriminant and states there are no solutions to the equation in such cases.
- It is quite possible that like Euclid's Elements, these two works were compiling general knowledge about such quadratic equations current at that time.

Thabit ibn-Qurra: Generalization of the Pythagorean theorem

- Thabit ibn-Qurra (826-901) proved the following generalization of the Pythagorean theorem.



$$AC^2 + AB^2 = BC(BB' + CC').$$

Draw AB' and AC' so that the angle at B' and C' equals the angle at A of the original triangle. Then:

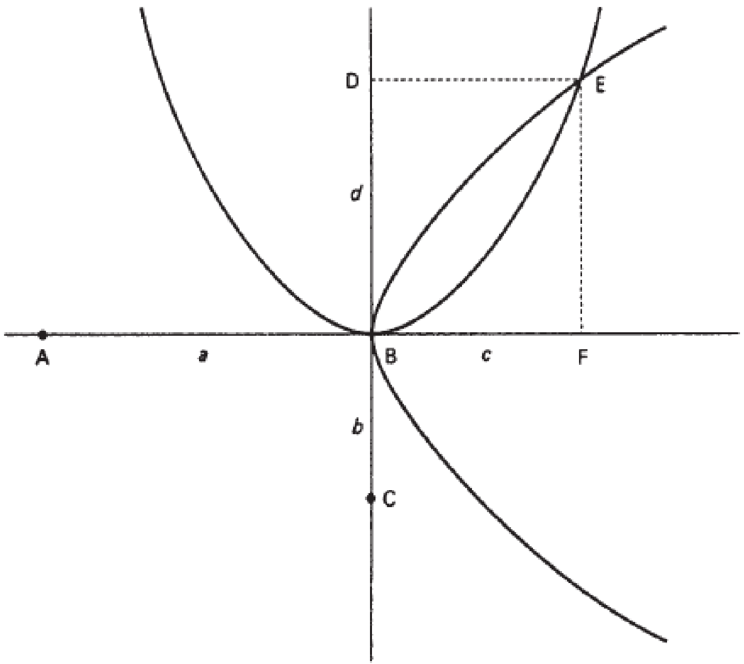
The proof uses similar triangles:
AB:BC:CA = B'B:BA:AB'=C'A:AC:CC'
so that AB/BC=B'B/BA and BC/CA=AC/CC' from which we get $(AB)^2 = (BC)(BB')$ and $(AC)^2 = (BC)(CC')$ and the result follows by addition.

Thabit's work on amicable numbers

- Two numbers m and n are called *amicable* if each is equal to the sum of the **proper** divisors of the other.
- In mathematical notation, let $\sigma(n)=\sigma(m)=m+n$, where $\sigma(n)$ is the sum of the divisors of n.
- Thabit showed that if p, q and r are distinct prime numbers of the form $p=3x2^{n-1}-1$, $q=3x2^{n}-1$, $r=9x2^{2n-1}-1$ with $n>1$, then M and N are amicable where $M=2^{n}pq$ and $N=2^{n}r$. (Exercise)
- This raises the question of the infinitude of amicable pairs. The conjecture is that there are infinitely man.

Omar Khayyam's solution of cubic equations

- Omar Khayyam (1048-1126 CE) gave methods to solve certain cubic equations. We illustrate his method in the case $x^3=a$.
- The method is based on a simple observation.
- Let a,b,c,d be such that $b/c=c/d=d/a$.
- Then $(b/c)^2= (c/d)(d/a)= c/a$.
- Therefore $c^3=b^2a$. If $b=1$, c gives the cube root of a. We can therefore find c as a solution of two quadratic equations: $c^2=d$ and $d^2=ac$.
- In other words, the solution is the intersection of two parabolas.



Khayyam's methods for other cubic equations

- In the table below, we list how Khayyam used conic sections to solve cubic equations.

TABLE 11.1: SOME OF OMAR KHAYYAM'S SOLUTIONS OF CUBIC EQUATIONS

Type ($a > 0$, $c > 0$)	*Method*
1. $x^3 = c$	Intersection of two parabolas
2. $x^3 + ax = c$	Intersection of circle and parabola
3. $x^3 \pm c = ax$	Intersection of hyperbola and parabola
4. $x^3 = ax + c$	Intersection of two hyperbolas

Note: In each case one positive root was found.

Source: The Crest of the Peacock, by George Joseph, p. 330.

Who was Omar Khayyam? He is well-known today as a poet and his famous for his work called The Rubaiyat of Omar Khayyam. He was a poet and a mathematician, a very rare combination. Here is a sample verse:

I need a jug of wine and a book of poetry,
Half a loaf for a bite to eat,
Then you and I, seated in a deserted spot,
Will have more wealth than a Sultan's realm.

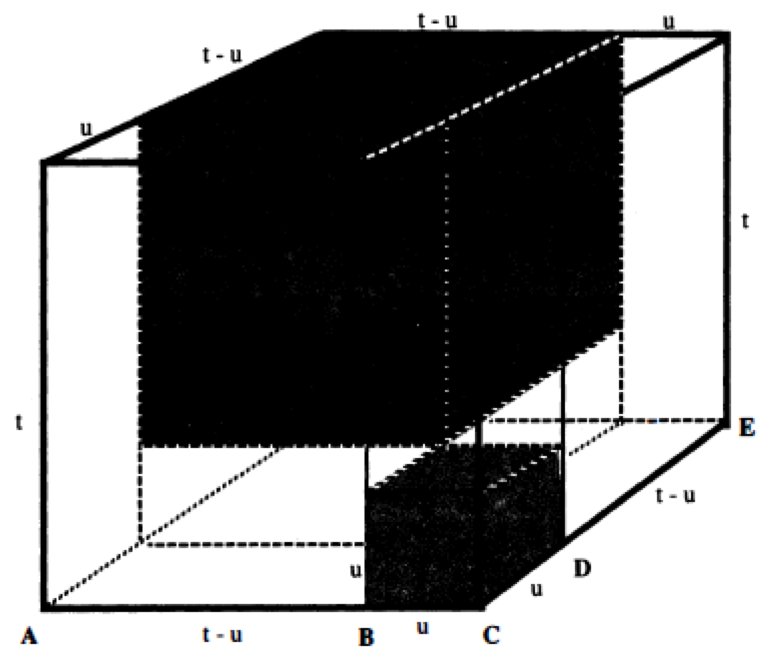
Cardano's method of solving the cubic

- Cardano (1501-1576) undoubtedly built upon Khayyam's works when he wrote his famous Ars Magna (or "Great Art") in 1545.
- In it, he outlines a method to treat the specific case of $x^3 + 6x = 20$. One can re-write his method in modern notation and describe a general solution as follows.
- We note first that any general cubic can be reduced to the study of $x^3 + mx = n$.

"Completing the cube"

- Cardano imagined a large cube having side AC of length t. AC is divided at B into BC of length u and AB of length t-u. We get:

- a small cube in the lower front corner, with volume u^3
- a larger cube in the upper back corner, with volume $(t-u)^3$
- two upright slabs, one facing front along *AB* and the other facing to the right along *DE*, each with dimensions $t-u$ by u by t (the length of the side of the big cube) and thus each with volume $tu(t-u)$
- a tall block in the upper front corner, standing upon the small cube, with volume $u^2(t-u)$
- a flat block in the lower back corner, beneath the larger cube, with volume $u(t-u)^2$



Clearly the large cube's volume, t^3, equals the sum of these six component volumes. That is,

$$t^3 = u^3 + (t-u)^3 + 2tu(t-u) + u^2(t-u) + u(t-u)^2$$

Simplification

This could have been derived purely algebraically without cubes and slabs!

- Moving u^3 to the other side, we get:

$$(t-u)^3 + [2tu(t-u) + u^2(t-u) + u(t-u)^2] = t^3 - u^3$$

and factoring the common $(t-u)$ from the bracketed expression gives

$$(t-u)^3 + (t-u)[2tu + u^2 + u(t-u)] = t^3 - u^3 \qquad \text{or simply}$$

$$(t-u)^3 + 3tu(t-u) = t^3 - u^3 \qquad (*)$$

In $(*)$ we have arrived at an equation reminiscent of the original cubic $x^3 + mx = n$. That is, if we let $t - u = x$, then $(*)$ becomes $x^3 + 3tux = t^3 - u^3$, and this instantly suggests that we set

$$3tu = m \qquad \text{and} \qquad t^3 - u^3 = n$$

If we now can determine the quantities t and u in terms of m and n from the original cubic, then $x = t - u$ will yield the solution we seek.

Reduction to the quadratic case

- As Khayyam suggested in his book, one can reduce solving the cubic to the study of intersections of conics. That is precisely what we have here.

To begin, consider his two conditions on t and u, namely

$$3tu = m \qquad \text{and} \qquad t^3 - u^3 = n$$

From the former, we see that $u = m/3t$, and substituting this into the latter yields

$$t^3 - \frac{m^3}{27t^3} = n$$

Multiply both sides by t^3 and rearrange terms to get the equation

$$t^6 - nt^3 - \frac{m^3}{27} = 0$$

$$(t^3)^2 - n(t^3) - \frac{m^3}{27} = 0$$

The final answer

- The quadratic is easily solved using the familiar formula.

$$t^3 = \frac{n \pm \sqrt{n^2 + \frac{4m^3}{27}}}{2}$$

$$= \frac{n}{2} \pm \frac{1}{2}\sqrt{n^2 + \frac{4m^3}{27}} = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}$$

Then, using only the positive square root, we have

$$t = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

Now, we also know that $u^3 = t^3 - n$, and so we conclude that

$$u^3 = \frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}} - n \quad \text{or}$$

$$u = \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

At last, we have the algebraic version of Cardano's rule for solving the depressed cubic $x^3 + mx = n$, namely

$$x = t - u$$

$$= \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

Q.E.D.

Summary of Arabic mathematics

- With Al-Khwarizmi, we have the birth of modern algebra and the theory of algorithms.
- The work on quadratic equations and cubic equations studied by Omar Khayyam paved the way for the general solution of the cubic in 16th century Europe by Cardano.
- There are also notable contributions to trigonometry. In particular, the Arab mathematicians introduced the tangent and cotangent functions that we use today. They also derived the addition formulas for the trigonometric functions.