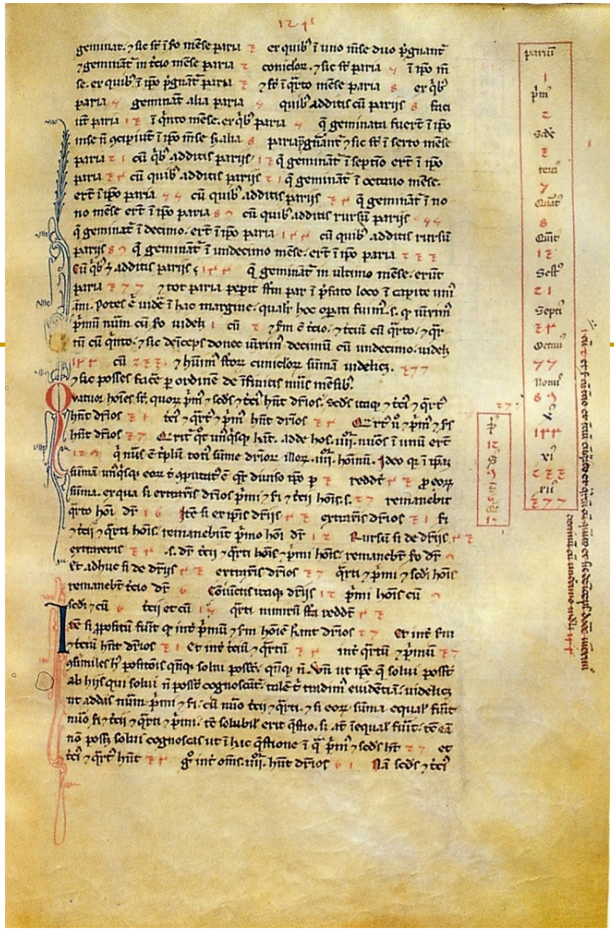
Fibonacci and Mathematics in the Middle Ages

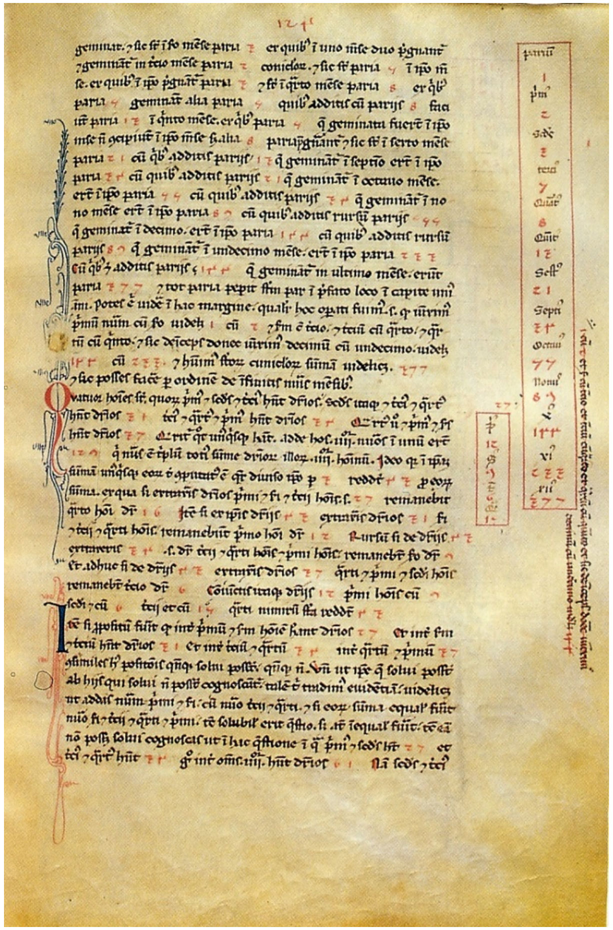


The decimal system enters Europe

- The twelfth century marks a turning point for the development of mathematics in Europe.
- After the burning of the Library of Alexandria, mathematics flourished unimpeded in India, China and the Middle East with the center of learning being Baghdad.
- However, there were some notable Italian traders who were traveling back and forth from North Africa and Italy who realized that the Arabs were using the Hindu number system for their computations which was speeding up transactions.
- Prominent among these was Fibonacci.

Fibonacci and *Liber abaci*

- There were several traders who were mathematically minded that recognized the importance of the decimal system. But it was Fibonacci (1180-1250) and his work *Liber Abaci* that was influential.
- This work was largely a translation of Arabic texts and described how to work with decimals. The word "zero" comes from the Arabic "zephirum" from which the English words "cipher" and "zero" are derived.
- It is unclear if Fibonacci knew the significance of his sequence.



A page from Liber Abaci(1202)

The Fibonacci sequence

- One of the famous problems contained in Fibonacci's book concerns the multiplication of rabbits.
- How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which in turn produces a new pair from the second month onwards.
- Let F_n denote the number of pairs in the n-th month, we see that $F_1=1$, $F_2=1$, $F_n=F_{n-1} + F_{n-2}$.
- One can show either by induction or otherwise that $F_n = (\varphi^n - \psi^n)/\sqrt{5}$ where $\varphi= (\sqrt{5}+1)/2$ is called the Golden ratio and $\psi=1-\varphi$.

The ubiquity of the Fibonacci sequence

- Yellow chamomile head showing the arrangement in 21 (blue) and 13 (aqua) spirals. Such arrangements involving consecutive Fibonacci numbers appear in a wide variety of plants,

From Scott and Marketos, "On the origin of the Fibonacci sequence"

- They write that there is a large overlap between *Liber Abaci* and Al-Khwarizmi's book *Al-jabr wa'muqabala*, written 3 centuries earlier.
- They add: "In Fibonacci's day, one witnesses a "prehistory" to science rather than science itself. Scientific fact coexists with misinformation, superstition and religious beliefs. Activities related to algebra, alchemy and astrology all represent forms of "magic" to the majority of the population at that time and face suspicion and resistance."

They add:

- "In retrospect, when considering the advance Muslim scholars had over Europeans in Fibonacci's time, it must be realized that a significant part of Fibonacci's results are unavoidably efforts in translation. These translations were "filtered" by the church authorities and consequently the "translators" had to refrain from a close association with Muslims or Muslim thought, which could pose a danger to themselves and their works. Often, results had to be "disguised". Nonetheless, with this understanding, Fibonacci's work provided an invaluable service in bringing significant mathematical contributions from the Muslim world to Christian Europe."

Bees were the inspiration for the Fibonacci sequence

Fig. 5. Barberini Exultet Roll: The Praise of The Bees. Biblioteca Vaticana (Vatican City), Cod. Barb. Lat. 592.

How to solve linear recurrences

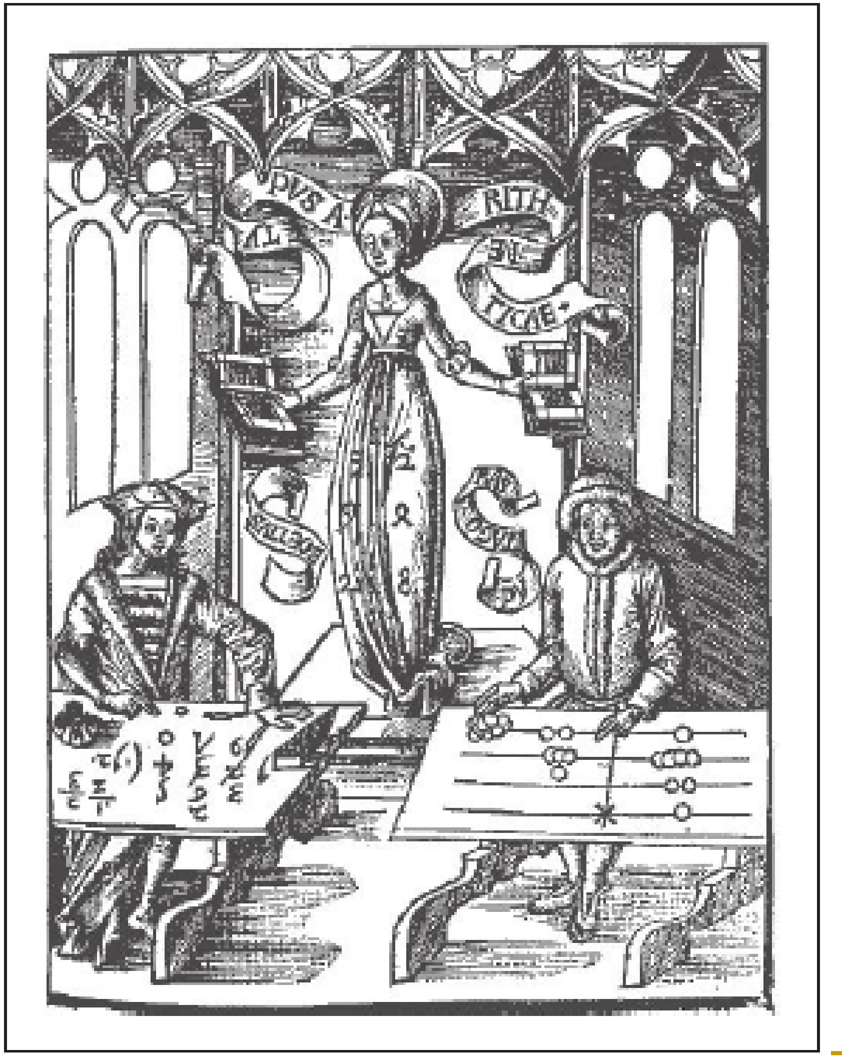
$$\begin{aligned}
s(x) &= \sum_{k=0}^{\infty} F_k x^k \\
&= F_0 + F_1 x + \sum_{k=2}^{\infty} (F_{k-1} + F_{k-2})\, x^k \\
&= x + \sum_{k=2}^{\infty} F_{k-1} x^k + \sum_{k=2}^{\infty} F_{k-2} x^k \\
&= x + x \sum_{k=0}^{\infty} F_k x^k + x^2 \sum_{k=0}^{\infty} F_k x^k \\
&= x + x s(x) + x^2 s(x).
\end{aligned}$$

The explicit formula can now be determined by the partial fraction method.

$$s(x) = \frac{x}{1 - x - x^2}$$

Mathematics in the Middle Ages

- By 1500, the importance of mathematics and especially the use of the decimal system was widely recognized that we can see it in paintings of the time.



A woodcut from Gregor Reisch, *Margarita Philosophica* (Freiburg, 1503). Arithmetic is instructing the algorist and the abacist, here inaccurately represented by Boethius and Pythagoras.

Nicole Oresme and infinite series

- In 1360, Nicole Oresme wrote a treatise titled *De Proportionibus proportionum* in which he discussed the importance of exponents and described the laws of exponents: $x^m x^n = x^{m+n}$ and $(x^m)^n = x^{mn}$.
- He also tried to describe fractional powers and even an attempt at defining $x^{\sqrt{2}}$.
- But his serious contribution seems to the proof of the divergence of the harmonic series done by ingenious grouping.
- The idea later appears in the work of Cauchy's condensation test for convergence.

Oresme's proof of the divergence of the harmonic series

- Oresme used a simple grouping idea. Let: $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad n = 1, 2, 3, \ldots.$

Nicole Oresme's proof dates back to about 1350. While the proof seems to have disappeared until after the Middle Ages, it has certainly made up for lost time.

PROOF: Consider the subsequence $\{H_{2^k}\}_{k=0}^{\infty}$.

$$\begin{aligned}
H_1 &= 1 = 1 + 0\left(\frac{1}{2}\right), \\
H_2 &= 1 + \frac{1}{2} = 1 + 1\left(\frac{1}{2}\right), \\
H_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + 2\left(\frac{1}{2}\right), \\
H_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + 3\left(\frac{1}{2}\right).
\end{aligned}$$

In general,

$$H_{2^k} \geq 1 + k\left(\frac{1}{2}\right).$$

Since the subsequence $\{H_{2^k}\}$ is unbounded, the sequence $\{H_n\}$ diverges. ∎

Cauchy condensation test

- Given a decreasing sequence of positive real numbers f(n),

, the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if

the "condensed" series $\sum_{n=0}^{\infty} 2^n f(2^n)$ converges.

$$\begin{aligned}
\sum_{n=1}^{\infty} f(n) &= f(1) + f(2)+f(3) + f(4)+f(5)+f(6)+f(7) + \cdots \\
&= f(1) + \big(f(2)+f(3)\big) + \big(f(4)+f(5)+f(6)+f(7)\big) + \cdots \\
&\leq f(1) + \big(f(2)+f(2)\big) + \big(f(4)+f(4)+f(4)+f(4)\big) + \cdots \\
&= f(1) + 2f(2) + 4f(4) + \cdots = \sum_{n=0}^{\infty} 2^n f(2^n)
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^n f(2^n) &= f(1) + \big(f(2)+f(2)\big) + \big(f(4)+f(4)+f(4)+f(4)\big) + \cdots \\
&= \big(f(1)+f(2)\big) + \big(f(2)+f(4)+f(4)+f(4)\big) + \cdots \\
&\leq \big(f(1)+f(1)\big) + \big(f(2)+f(2)+f(3)+f(3)\big) + \cdots = 2\sum_{n=1}^{\infty} f(n)
\end{aligned}$$

The decline of medieval learning

- Just when it looked like Europe was to rise out of the dark ages and into the light of mathematical thought, we see a decline in the 15^{th} century.
- Diseases such as the "Black Death" decimated more than half of the population.
- Then England and France were at war for a hundred years!
- This was followed by the War of the Roses!
- Europe had to wait another 100 years before it would awake into the period called the Renaissance.