Mathematics of the Renaissance

Regiomantus and Nicolas Chuquet

- A singular figure who can be identified as the "Renaissance Man" is Regiomantus (1436-1476) who recognized the importance of the ancient works and began to translate them and with the invention of the printing press, publish his works.
- In France, an 1884 manuscript by Nicolas Chuquet called *Triparty en la science des nombres* explained how to work with the Hindu-Arabic numerals.
- In Italy, Luca Pacioli's *Summa* also had the same influence in the growth of mathematics.

Cardano and Ferrari's work on cubic and quartic equations

- We have already discussed Cardano's solution of the cubic equation.
- The quartic can also be solved and this is due to Lodovico Ferrari(1540), a student of Cardano.
- The fact that there is no general formula for fifth degree polynomials and higher is a famous theorem of Abel and Ruffini proved in 1824. It needs Galois theory and in particular the concept of a group to which we will come later.

Jerome Cardan

But who really solved the cubic?

- Although Al-Khwarizmi solved special cases of the cubic, he developed no general theory. In 1494, Luca Pacioli (1445-1509) wrote *Summa de Arithmetica* in which he posed the problem using for the first time a name for the "unknown" which he called "cosa" or "thing".
- Scipione del Ferro (1465-1520) of the University of Bologna (the oldest university in Europe), discovered a formula that solved the "depressed cubic" which we now know is essentially the full solution.
- The intellectual climate of that day was competitive and mathematicians would do scholarly battle, so del Ferro never published or disclosed his method but would use special cases of the cubic as challenges to defeat his opponents.
- It was only on his deathbed that he disclosed the method to Antonio Fior (1506- ?) who it is written was a "mediocre mathematician" and he used it to challenge his colleague Niccolo Fontana (1499-1557).
- Apparently, when Fontana was a child in 1512, a French soldier slashed the face of young Niccolo that disfigured him and he could no longer speak clearly. Tartaglia – the stammerer – became his nickname.

Tartaglia and the "depressed" cubic

- Tartaglia, or Fontana took up Fior's challenge and furiously worked on the cubic. On the night of February 13, 1535, he finally found the answer.
- Fior had to provide 30 lavish banquets as the reward, but Tartaglia, in a gesture of magnanimity, relieved him of this commitment.
- But then Gerolamo Cardano (1501-1576) enters the scene. We know much about his life since he wrote his Autobiography in 1575. Oysten Ore has written a modern biography of Cardano from which we have taken the details below.
- Cardano figured out how to reduce the general cubic to the "depressed cubic" but didn't know how to complete it. He heard that Tartaglia had solved this and wanted to know how to do it. But Tartaglia refused saying that he will write a book about it. Finally, on March 25, 1539, Tartaglia relented and visited Cardano and revealed the secret after Cardano swore on the Bible that he would never divulge this knowledge!
- Ludovico Ferrari (1522-1565) became Cardano's student and seeing him intelligent, Cardano shared his secret with him.

The del Ferro papers

So the credit should go to del Ferro, Tartaglia and Cardano!

- In 1543, Cardano and Ferrari travelled to Bologna and they inspected the posthumous papers of Scipione del Ferro.
- There, the found, in del Ferro's own hand writing, the solution of the depressed cubic.
- Since del Ferro is the first to have solved the depressed cubic and not Tartaglia, Cardano reasoned that he is now free to publish since his oath was to Tartaglia on the assumption that he had been the first to solve the depressed cubic!
- So in 1545, he wrote *Ars Magna*, or the "Great Art". In the preface he writes:

> Scipio Ferro of Bologna well-nigh thirty years ago discovered this rule and handed it on to Antonio Maria Fior of Venice, whose contest with Niccolo Tartaglia of Brescia gave Niccolo occasion to discover it. He gave it to me in response to my entreaties, though withholding the demonstration. Armed with this assistance, I sought out its demonstration in [various] forms. This was very difficult.

Reducing the general quartic to a "depressed" quartic

Let

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

be the general quartic equation we want to solve.

Dividing by a_4, provides the equivalent equation $x^4 + bx^3 + cx^2 + dx + e = 0$, with $b = \frac{a_3}{a_4}$, $c = \frac{a_2}{a_4}$, $d = \frac{a_1}{a_4}$, and $e = \frac{a_0}{a_4}$. Substituting $y - \frac{b}{4}$ for x gives, after regrouping the terms, the equation $y^4 + py^2 + qy + r = 0$, where

$$p = \frac{8c - 3b^2}{8} = \frac{8a_2a_4 - 3a_3{}^2}{8a_4{}^2}$$

$$q = \frac{b^3 - 4bc + 8d}{8} = \frac{a_3{}^3 - 4a_2a_3a_4 + 8a_1a_4{}^2}{8a_4{}^3}$$

$$r = \frac{-3b^4 + 256e - 64bd + 16b^2c}{256} = \frac{-3a_3{}^4 + 256a_0a_4{}^3 - 64a_1a_3a_4{}^2 + 16a_2a_3{}^2a_4}{256a_4{}^4}.$$

If y_0 is a root of this depressed quartic, then $y_0 - \frac{b}{4}$ (that is $y_0 - \frac{a_3}{4a_4}$) is a root of the original quartic and every root of the original quartic can be obtained by this process.

Ferrari's solution of the quartic

$$y^4 + py^2 + qy + r = 0.$$

- Completing the square, we see:

$$\left(y^2 + \frac{p}{2}\right)^2 = -qy - r + \frac{p^2}{4}.$$

Then, we introduce a variable m into the factor on the left-hand side by adding $2y^2m + pm + m^2$ to both sides. After regrouping the coefficients of the power of y in the right-hand side, this gives the equation

$$\left(y^2 + \frac{p}{2} + m\right)^2 = 2my^2 - qy + m^2 + mp + \frac{p^2}{4} - r,$$

which is equivalent to the original equation, whichever value is given to m.

As the value of m may be arbitrarily chosen, we will choose it in order to complete the square in the right-hand side. This implies that the discriminant in y of this quadratic equation is zero, that is m is a root of the equation

$$(-q)^2 - 4(2m)\left(m^2 + pm + \frac{p^2}{4} - r\right) = 0,$$

which may be rewritten as

$$8m^3 + 8pm^2 + (2p^2 - 8r)m - q^2 = 0. \qquad \text{(1a)}$$

The final step

$$8m^3 + 8pm^2 + (2p^2 - 8r)m - q^2 = 0.$$

- This is called the resolvent cubic, and it can be solved by Cardano's method. When m is the root of this equation, our quartic equation becomes:

$$\left(y^2 + \frac{p}{2} + m\right)^2 = \left(y\sqrt{2m} - \frac{q}{2\sqrt{2m}}\right)^2.$$

This equation is of the form $M^2 = N^2$, which can be rearranged as $M^2 - N^2 = 0$ or $(M + N)(M - N) = 0$. Therefore, equation (*1*) may be rewritten as

$$\left(y^2 + \frac{p}{2} + m + \sqrt{2m}y - \frac{q}{2\sqrt{2m}}\right)\left(y^2 + \frac{p}{2} + m - \sqrt{2m}y + \frac{q}{2\sqrt{2m}}\right) = 0.$$

This equation is easily solved by applying to each factor the quadratic formula. Solving them we may write the four roots as

$$y = \frac{\pm_1\sqrt{2m} \pm_2 \sqrt{-\left(2p + 2m \pm_1 \frac{\sqrt{2}q}{\sqrt{m}}\right)}}{2},$$

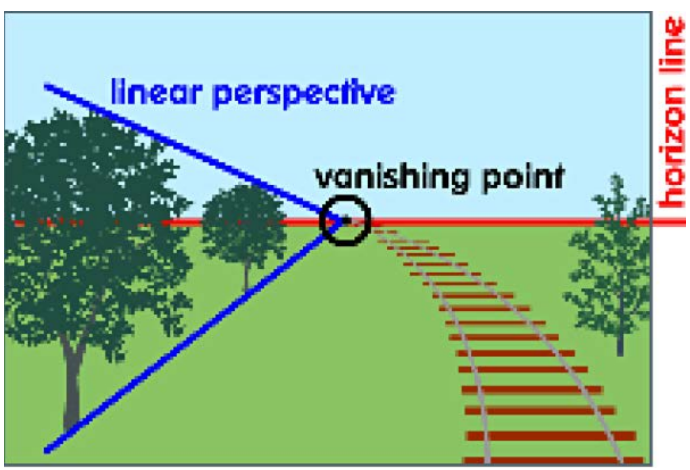
where $\pm_1$ and $\pm_2$ denote either $+$ or $-$. As the two occurrences of $\pm_1$ must denote the same sign, this leaves four possibilities, one for each root.

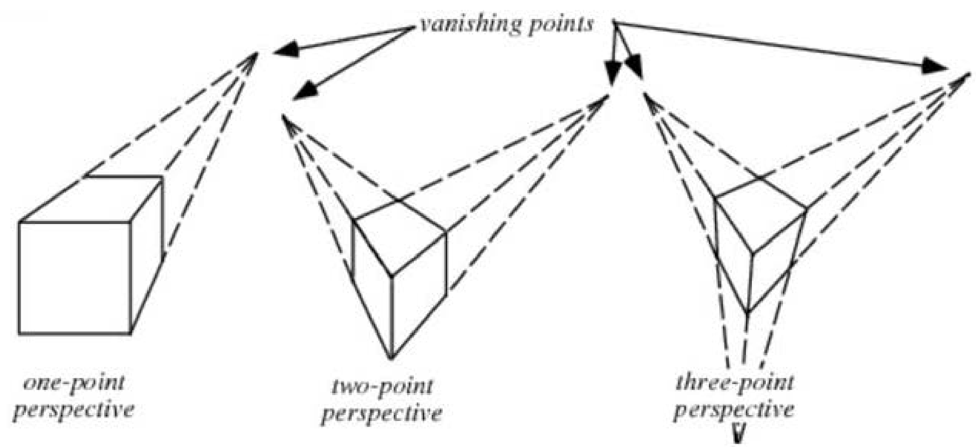
Mathematics of perspective and art

- In the Renaissance period, several artists, including Leonardo, were becoming interested in projective geometry so as to introduce three dimensionality into the two-dimensional canvas.
- Paintings before this period looked “flat”. Here are some examples.

The vanishing point

- Filipo Brunelleschi (1377-1446) described the four rules of perspective in art. They are:
- 1. The horizon appears as a horizontal line.
- 2. Straight lines in space should appear as straight lines in the painting.
- 3. Parallel lines in space meet at a **vanishing point** on the canvas.
- 4. Lines parallel to the picture frame appear as parallel lines and do not have a vanishing point.

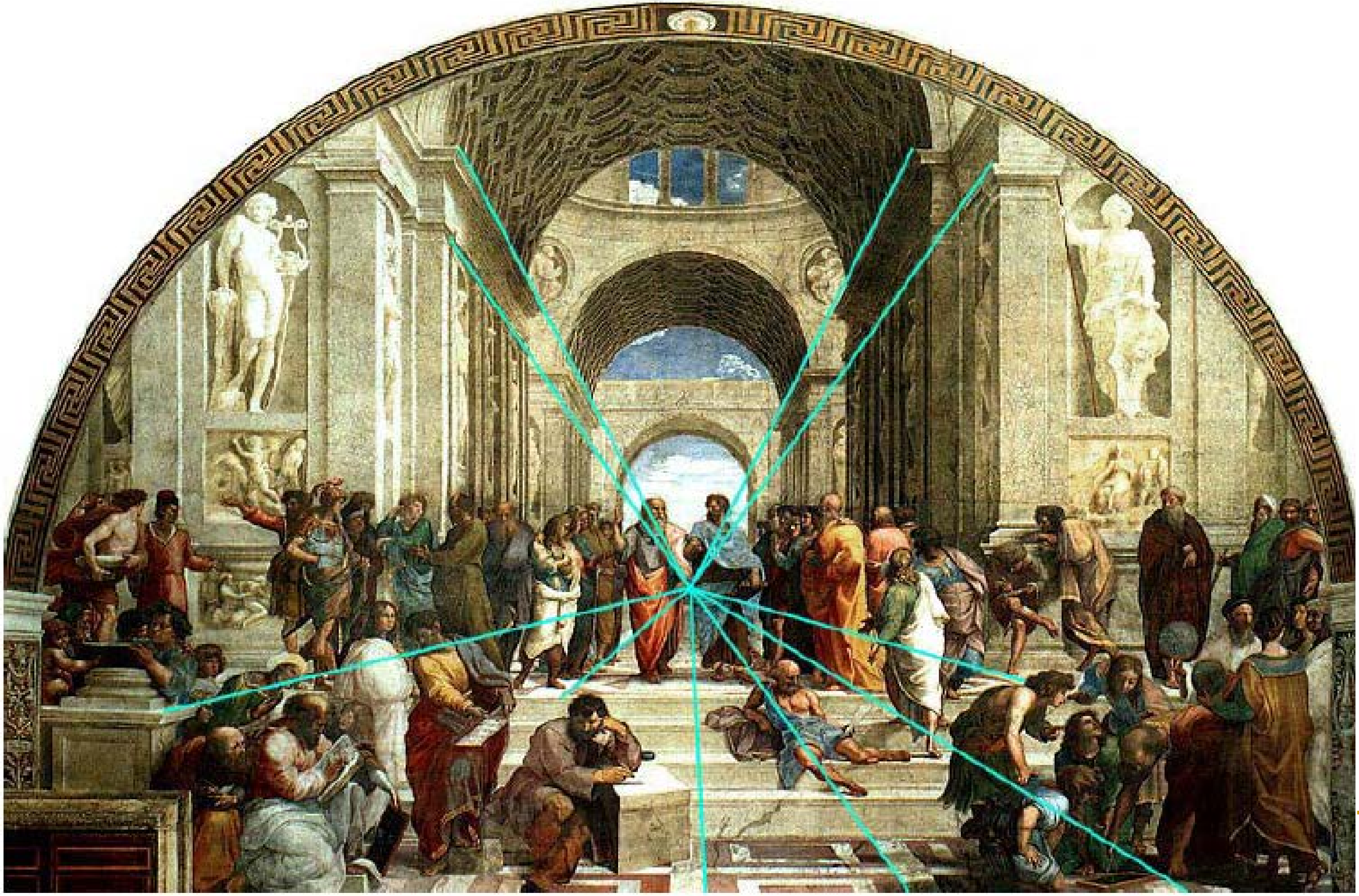




Examples of perspective art

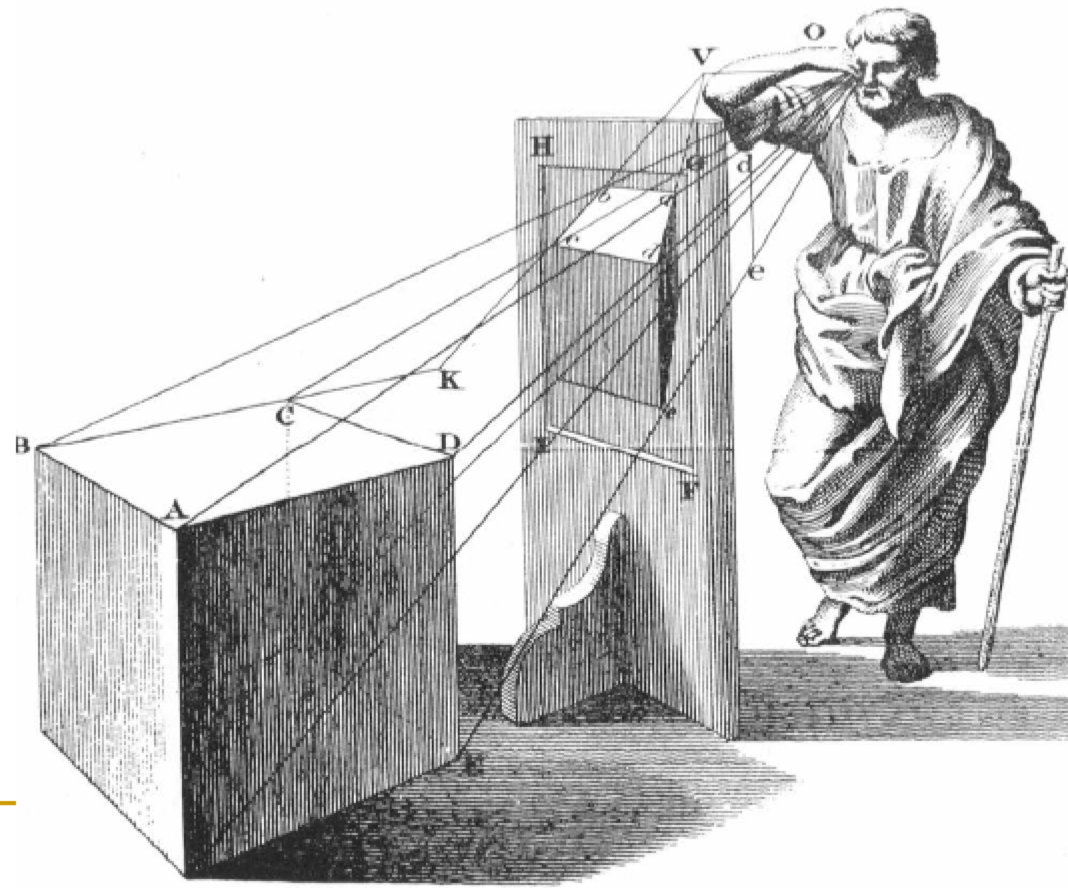
- Here are some famous examples of perspective art:
- The School of Athens by Raphael.

The "vanishing point" in the School of Athens painting



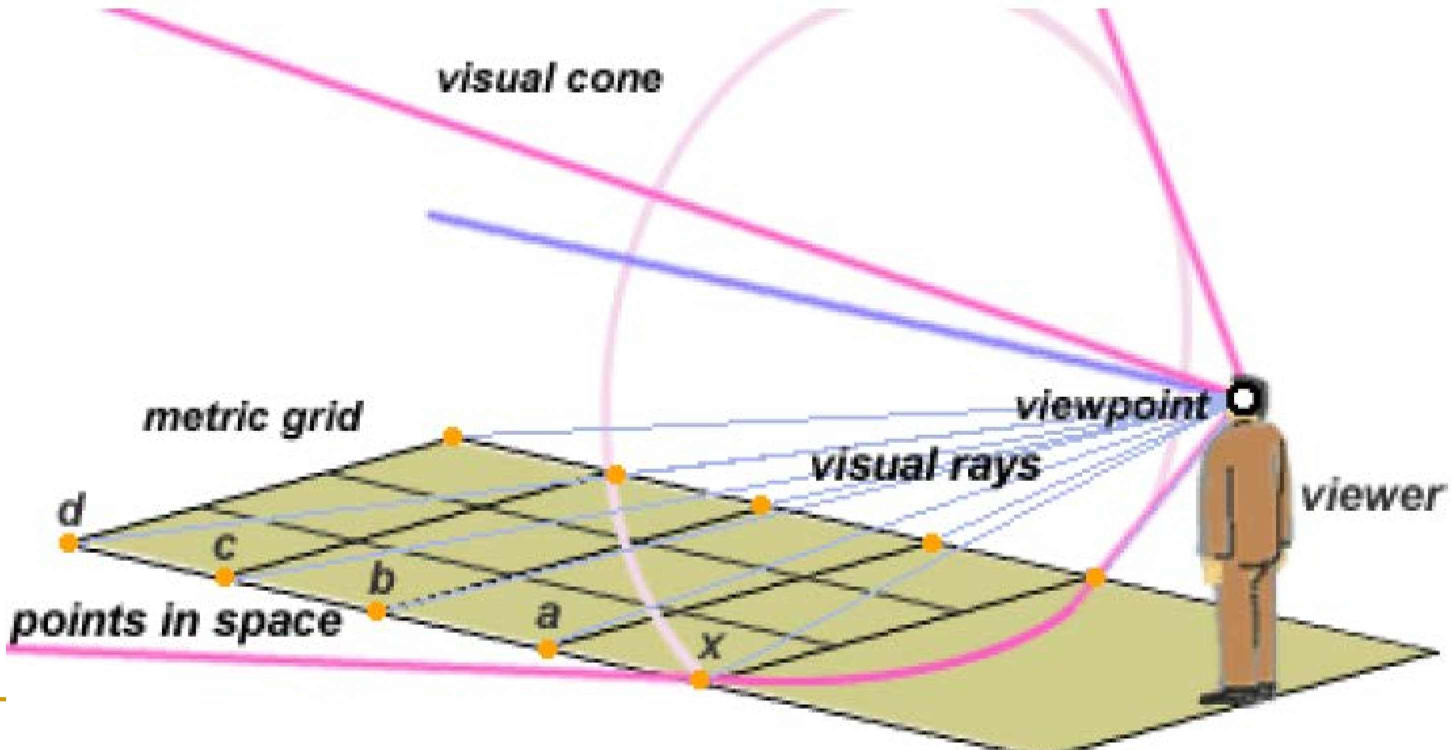
The geometry of light rays

- The essential idea in perspective art is to understand the geometry of light rays with respect to the eye of the observer.



The visual cone

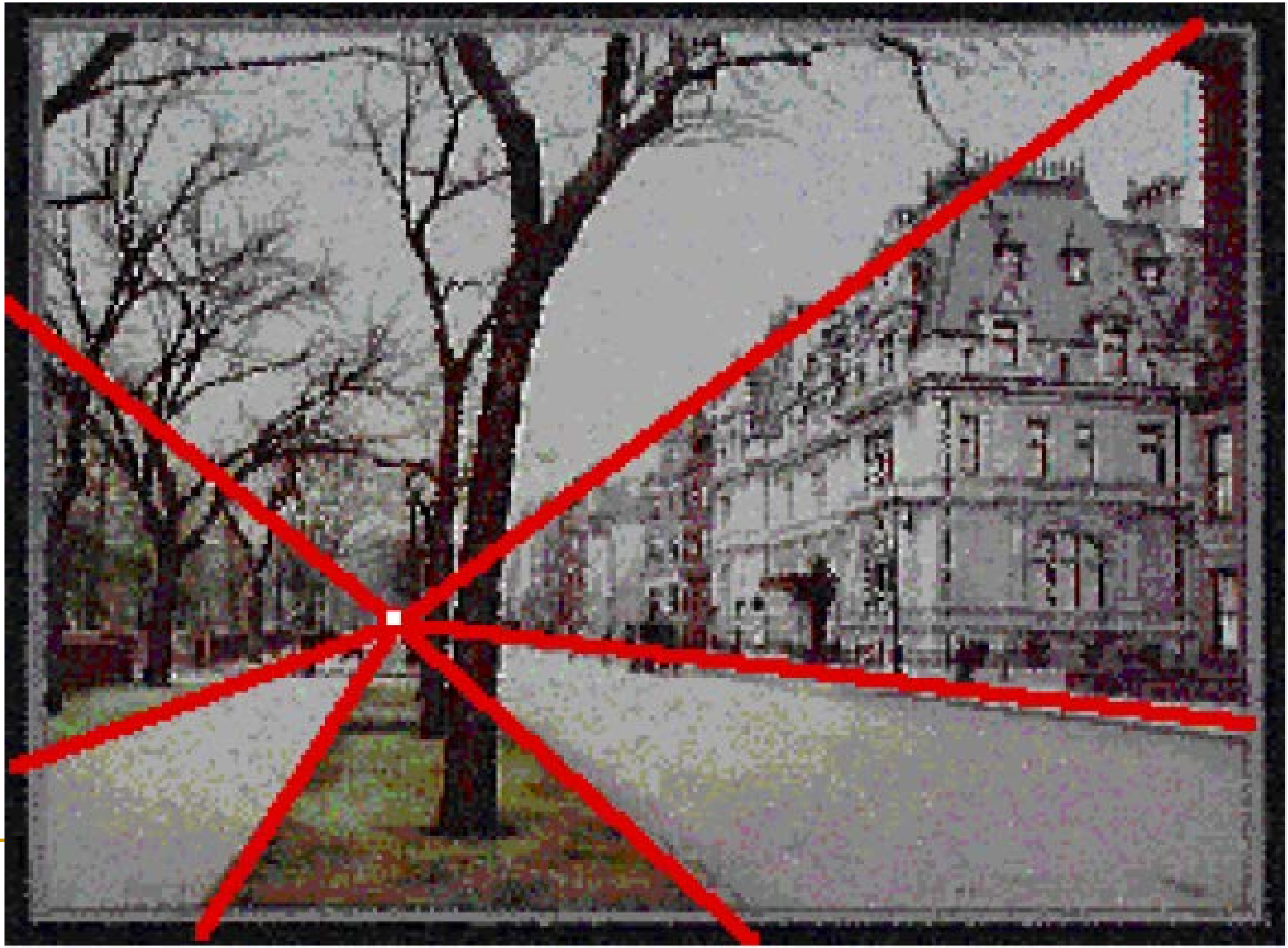
- The eye of the observer determines a visual cone in the three dimensional grid of observed space.



The Renaissance painters were aware of the visual cone

- Many texts on perspective art written in the Renaissance period reveal that the artists were aware of these facts.
- Here is a sketch of Durer.

Further examples



Da Vinci's "The Last Supper"

Mathematical contributions of the Renaissance period

- The Renaissance period was a period of blossoming knowledge.
- The printing press facilitated the spread of knowledge as well as the translations of Greek and Arabic works.
- It was also the time when some of our modern symbology was introduced.
- For example, the + and – sign along with the = sign were used in the works of Robert Recorde (1557).

Margarita philosophica (1503) by Gregor Reisch