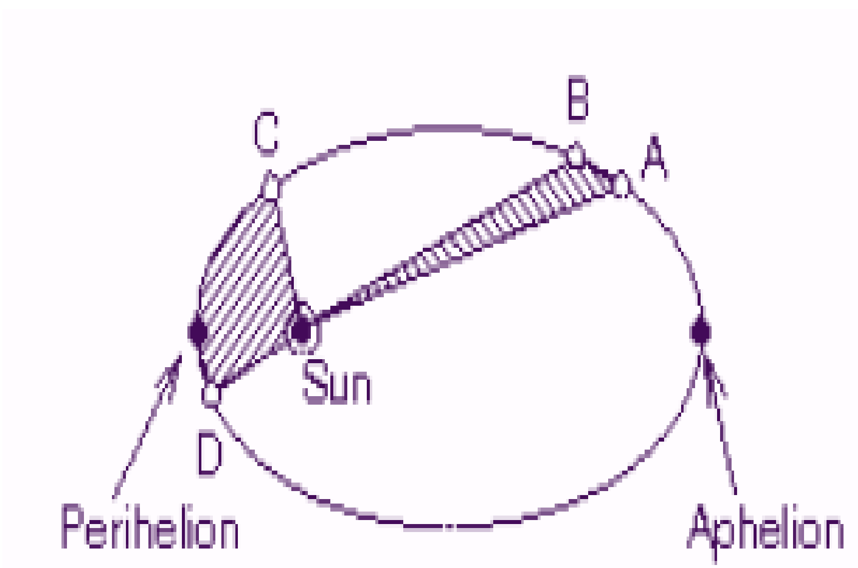
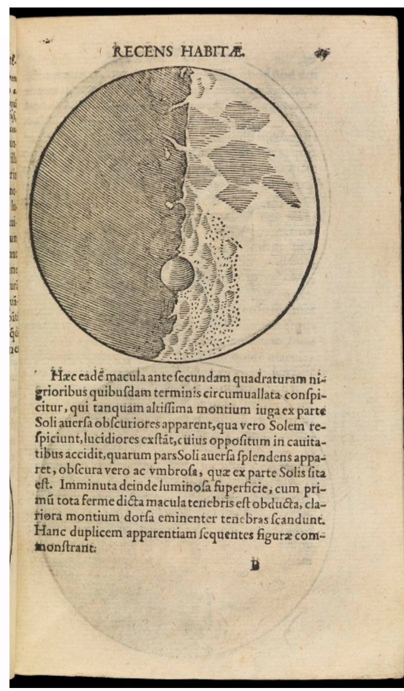
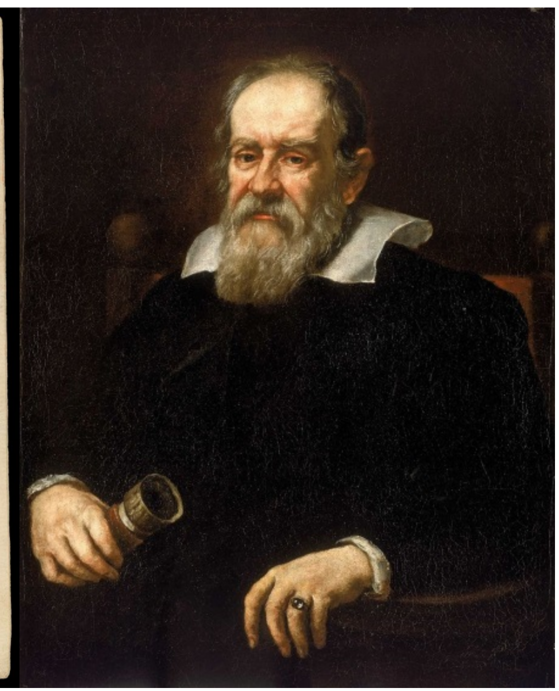
Viète, Kepler and Galileo: 16th century mathematics





RECENS HABITÆ.

Hæc eadẽ macula ante ſecundam quadraturam nigrioribus quibuſdam terminis circumuallata conſpicitur, qui tanquam altiſſima montium iuga ex parte Soli auerſa obſcuriores apparent, qua vero Solem reſpiciunt, lucidiores exſtãt, cuius oppoſitum in cauitatibus accidit, quarum pars Soli auerſa ſplendens apparet, obſcura vero ac vmbroſa, quæ ex parte Solis ſita eſt. Imminuta deinde luminoſa ſuperficie, cum primũ tota ferme dicta macula tenebris eſt obducta, clariora montium dorſa eminenter tenebras ſcandunt. Hanc duplicem apparentiam ſequentes figuræ commonſtrant.



François Viète

- François Viète (1540-1603) was not a mathematician but rather a member of parliament who had studied law. In his leisure time, he studied mathematics and most notably advocated the use of the decimal system.
- He was aware of the importance of notation and in his study of cubic equations, noticed a relationship between roots and the coefficients.
- But because he was fixated on "positive" roots, he ignored negative roots and non-real roots and thus could not make precise his intuition.

The work of Albert Girard

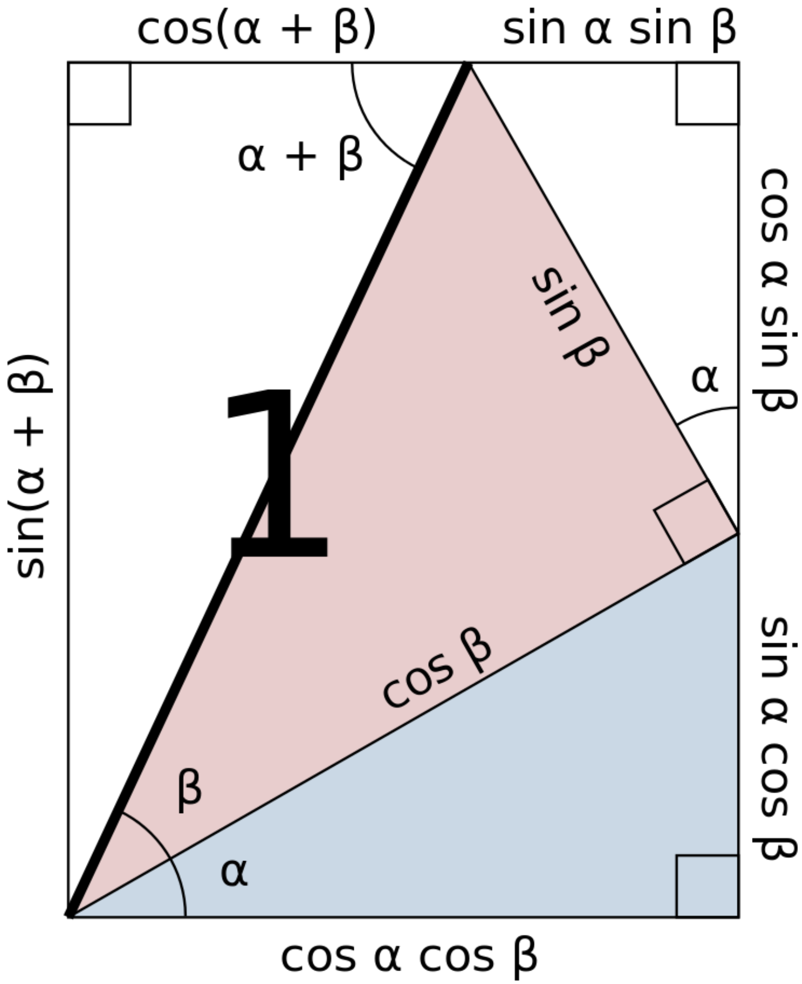
- It was Albert Girard (1590-1633) who, as late as 1629, realized that all roots are to be taken so as to note that the coefficients are symmetric functions of the roots.
- Thus, if $x^3 + Ax^2 + Bx + C = (x-r)(X-s)(X-t)$, then $C = -rst$, $B = rs + rt + st$, and $A = -(r+s+t)$.
- This simple realization was long in coming. It needed mathematicians to realize the importance of complex numbers.
- It was also through these works that much of our modern notation was evolving, such as the use of + and – signs, as well as the symbol = for equality, and > and < signs to compare numbers.

Horner’s method

- Viète is credited with discovering a method for finding roots of quadratic equations that is today called Horner’s method since Horner published a paper in the 19th century giving a formal description.
- However, the method goes back at least 500 years earlier, where it was being used by the Chinese mathematicians. Sometimes, the method is also called Newton’s method, since Newton discussed some special cases.
- The method is quite simple to illustrate. Suppose we want to solve $x^2 + 7x = 60750$. We begin with an approximation: $x_1=200$.
- Now write $x= 200 + x_2$. This gives a new quadratic: $x_2^2+407x_2= 19350$. which leads to a new approximation $x_2=40$.
- Putting $x_2=40+x_3$ gives finally $x=243$ as a root.

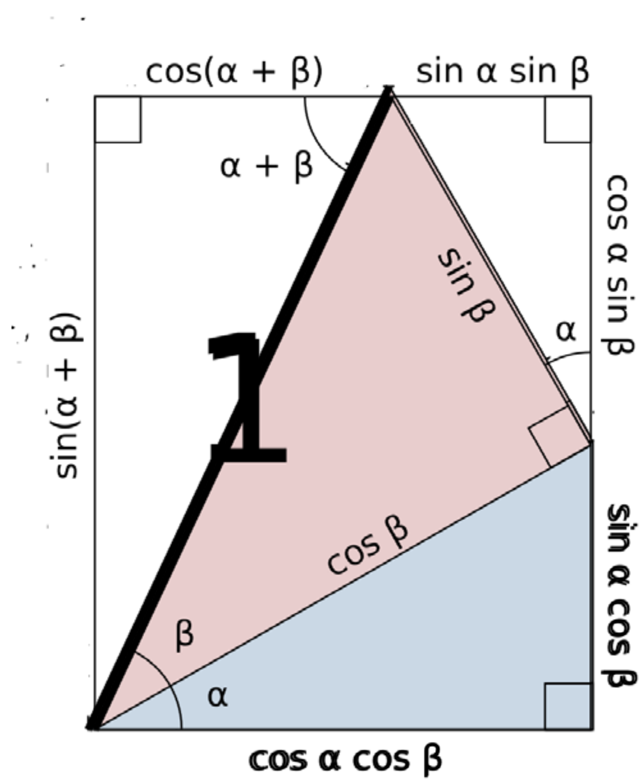
Viète's derivation of addition formulas for the trigonometric functions

- Viète used arrangements of triangles to deduce the familiar addition formulas for the sine and cosine functions.
- This remarkable picture demonstrates at once the addition formulas of sine and cosine and matches in its elegance the simplicity of the Chinese proof of the Pythagorean theorem.



The proof in detail

- We begin with the basic quadrilateral with the angles given with the length of the hypotenuse of the top right angle triangle being 1..
- We then "complete" the rectangle and the addition formulas for the trig functions are now obvious.



Viète's formula for π

- Using his addition formulas, Viète derived some interesting formulas for π.
- He began with the double angle formula:

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

Iterating this and using the familiar limit (sin x)/x →1 as x→0, he deduced

$$\sin x = 2^n \sin \frac{x}{2^n} \left(\prod_{i=1}^{n} \cos \frac{x}{2^i} \right) \qquad \frac{\sin x}{x} = \cos \frac{x}{2} \cdot \cos \frac{x}{4} \cdot \cos \frac{x}{8} \cdots$$

Substituting $x = \frac{\pi}{2}$ in this formula yields:

$$\frac{2}{\pi} = \cos \frac{\pi}{4} \cdot \cos \frac{\pi}{8} \cdot \cos \frac{\pi}{16} \cdots \qquad \frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

Deriving Viète's formula for π

- Using:

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$$

we see that for x=π/2, we have cos π/4=1/√2 and recursively, we find Viete's formula for π:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

John Napier and the logarithm

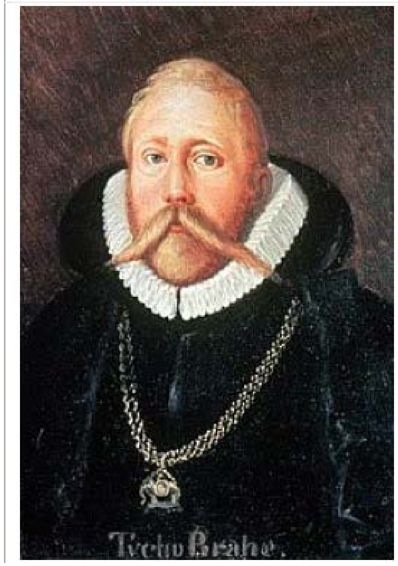
- John Napier (1550-1617) was a Scottish mathematician who realized the importance of working with the new decimal system.
- Since multiplication of huge numbers could take a long time, he discovered the logarithm function around the year 1594.
- He produced extensive tables and even coined the word "logarithm" from the Greek word "logos" meaning "ratio" and "arithmos" meaning number.
- The idea was simple enough. To find ab, we compute the sum log a + log b and then "exponentiate".
- But his logarithms were not to the base 10 and nor to the base *e* which we use today as the "natural logarithm" but a slight variation of the two.

Napier therefore chose to use $1 - 10^{-7}$ (or .9999999) as his given number. Now the terms in the progression of increasing powers are indeed close together—too close, in fact. To achieve a balance and to avoid decimals Napier multiplied each power by 10^7. That is, if $\boxed{N = 10^7(1 - 1/10^7)^L}$, then L is Napier's "logarithm" of the number N. Thus his logarithm of 10^7 is 0, his logarithm of $10^7(1 - 1/10^7) = 9999999$ is 1, and so on. If his numbers and his logarithms were to be divided by 10^7, one would have virtually a system of logarithms to the base $1/e$, for $(1 - 1/10^7)^{10^7}$ is close to $\lim_{n\to\infty} (1 - 1/n)^n = 1/e$. It must be remembered, however, that Napier had no concept of a base for a system of logarithms, for his definition was different from ours.

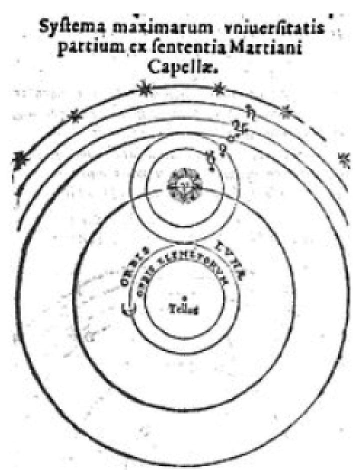
Tycho Brahe and the birth of astronomy

- Tycho Brahe (1546-1601) was a Danish nobleman and astronomer noted for his meticulous astronomical observations and tabulations, before the discovery of the telescope!
- He proposed a "geo-helio-centric" model for the solar system: the sun and the moon revolve around the earth and the other planets revolve around the sun.
- He hired Johannes Kepler as his assistant.

Tycho Brahe



Sadly, he died at the age of 54 due to his bladder bursting. According to Kepler, he was dining at a banquet hosted by the King and he thought it would be a breach of etiquette if he left to relieve himself. The bladder story has been verified in 1901 to determine the true cause of his death.



Valentin Naboth's drawing of Martianus Capella's geo-heliocentric astronomical model (1573)

Kepler's Laws

- Using the extensive data of Tycho Brahe, Kepler was able to discern three laws which seemed to explain the extensive data.

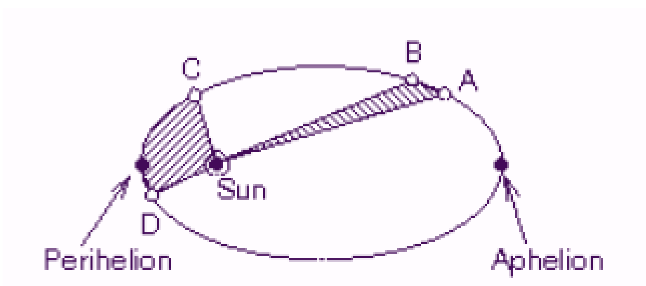
Johannes Kepler (1571-1630)

Kepler's First Law:

The orbit of a planet about the Sun is an ellipse with the Sun at one focus.

Kepler's Second Law:

A line joining a planet and the Sun sweeps out equal areas in equal intervals of time.



Kepler didn't prove these laws. He only stated them based on the data of Tycho Brahe. The proof was given later by Isaac Newton.

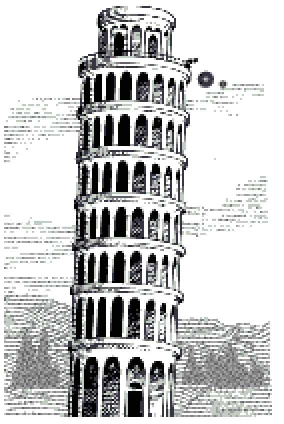
Kepler's Third Law:

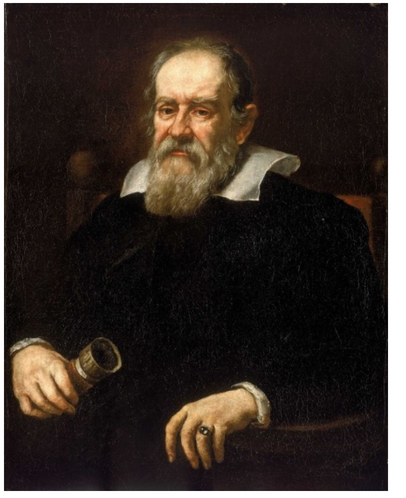
The squares of the sidereal periods of the planets are proportional to the cubes of their semimajor axes.

Kepler trying to explain his discovery

Galileo and the birth of the scientific revolution

- Galileo Galilee (1564-1642) stands out as the nexus between the old way of thinking and the new way based on reason and experiment.
- He is credited with many innovations. The foremost is his refutation of the time-honoured teaching of Aristotle about gravity who said that if two objects are dropped from a given height, the heavier object falls first and the lighter object second.
- Before Galileo, no one had even thought of checking this via experiment.

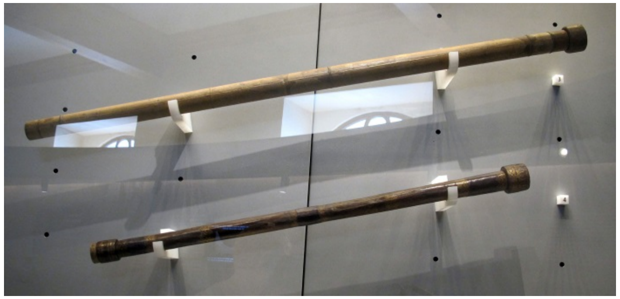




Galileo’s paradox and infinite sets

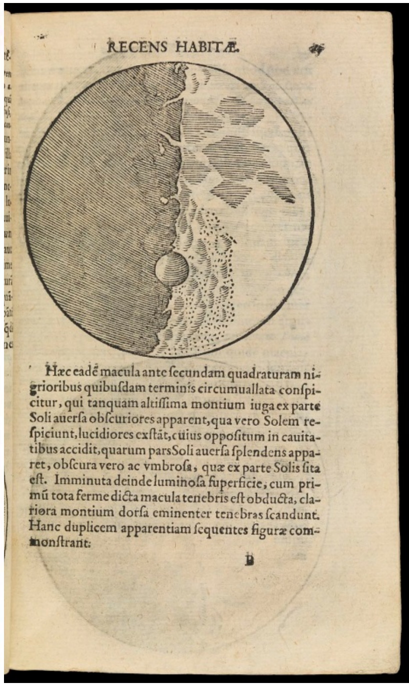
- In a fundamental work, Galileo discusses the notion of an infinite set and concludes that infinite sets cannot be “compared”.
- His example was the set of natural numbers and the set of perfect squares. Both of these sets are in one-to-one correspondence and yet one is a proper subset of the other. He concludes that we cannot talk about “greater” or “lesser” or even “equal” when it comes to infinite sets. This is often called Galileo’s paradox and resembles Zeno’s paradox where the notion of limit was missing. Here a proper notion of cardinality of infinite sets is missing, something that Cantor discusses two centuries later.

Galileo and the telescope



- Perhaps the most significant of Galileo’s discoveries is the telescope.
- Originally he devised it as a tool for the army but later realized he can turn it towards the heavens and understand the motion of stars and planets.
- This really is the beginning of the scientific revolution.
- He observed the moon and its craters, the phases of Venus, and the moons of Jupiter. These observations supported a heliocentric theory of the solar system.

It was on this page that Galileo first noted an observation of the moons of Jupiter. This observation upset the notion that all celestial bodies must revolve around the Earth. Galileo published a full description in *Sidereus Nuncius* in March 1610



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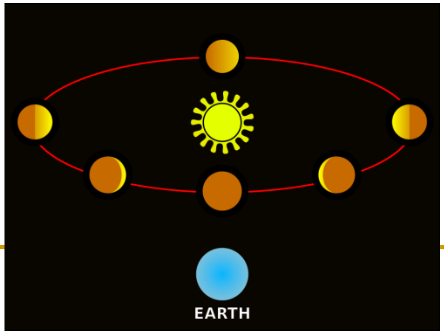

Galileo and the Church

- Galileo always viewed his science as only supporting the existence of God since science only revealed the wonder of God's creation.
- This was a prevalent attitude for many of the scientists of the Renaissance period.
- Yet the Church viewed his works as heretical since it went against Biblical teaching.

Galileo showing his telescope to the Doge of Venice (fresco by G. Bertini)

Since he would not retract the heliocentric theory, Galileo was placed under house arrest after an inquisition By the Church in 1615. He died there in 1642 at the age of 77.

Newton was born in 1642 and it seems the center of scientific development now moves away from Italy.