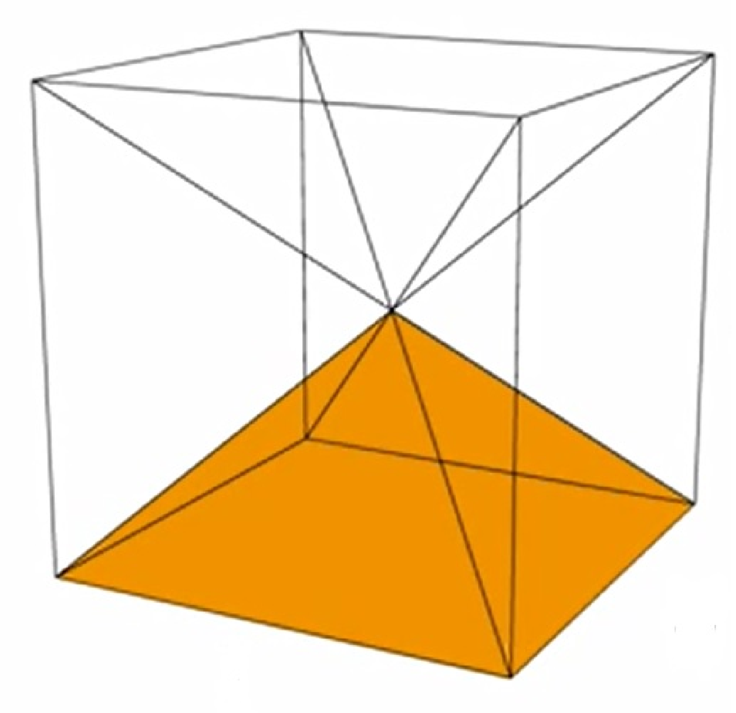
Mesopotamia

Volume of a pyramid

- Very likely, the Egyptians calculated first the volume of a pyramid with a square base b and height b/2 using visual geometry as indicated at right.
- Each pyramid then has volume $b^3/6$ as the cube has six faces.
- This can be re-written as $(1/3)b^2\ (b/2)$ where b/2 is the height.
- From this, they deduced that a pyramid with square base of length b and height h has volume $b^2h/3$.
- This formula was also known to the Babylonians.



Cuneiform records of Babylon

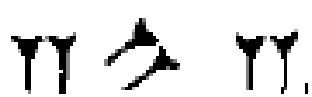
- The civilization that developed along the Euphrates river is often referred to as Babylonian.
- Archeologists have discovered countless clay tablets dating back to around 1800 BCE.
- This period of human history marks the beginnings of systems of writing and the discovery of the wheel, as well as the use of various metals.
- Surprisingly, the Babylonians seemed to have used base 60 for their numerical records.

Why base 60?

- This is not clear. Some experts say it was due to astronomical considerations.
- Others say it was because 60 his “highly divisible” and thus can easily be divided.
- Perhaps it was a combination of both.
- The sexagesimal (base 60) system seems to have had a long life. We see it even today, with 60 seconds to the minute and 60 minutes to the hour.

The lack of a zero symbol

- Until about 300 BCE, the Babylonians did not have a symbol for zero and this created quite a bit of confusion in their number system.
- After 300 BCE, they seemed to have used two wedges placed obliquely to denote zero.
- Thus the number 202 in sexagesimal notation would be:



Fractions and square root of 2

- It seems the Babylonians extended the positional notation to represent fractions also.
- One tablet calculates the square root of 2 in base 60.
- Translated into modern notation, it gives $\sqrt{2}$ as 1.414222 which differs by about .000008 from the real value!
- Their general method was later rediscovered by Newton and appears as Newton's algorithm.

Algorithm for $\sqrt{a}$

- Let a_1 be any approximation of $\sqrt{a}$. Let $b_1=a/a_1$.
- Let $a_2= (a_1+b_1)/2$ and put $b_2=a/a_2$.
- This gives a better approximation to $\sqrt{a}$. Why?
- We have a sequence of numbers a_n constructed recursively as $a_{n+1}= (a_n +a/a_n)$.
- The limit exists by the Bolzano-Weierstrass theorem and is easily seen to be $\sqrt{a}$.

Quadratic equations

- In 1930, Otto Neugebauer found evidence that the Babylonians knew how to solve quadratic equations.
- This is not surprising since they had an effective method to calculate square roots and only one more step is needed to solve a quadratic equation.
- The Yale cuneiform tablet asks the following:
- $X+Y = A$ and $XY = B$. Find X and Y.

The Babylonian method for the roots of the quadratic

- $(X+Y)^2 - 4XY = (X-Y)^2$.
- Thus, $X-Y = \sqrt{(A^2-4B)}$.
- Therefore $2X = A + \sqrt{(A^2-4B)}$.
- $2Y = A - \sqrt{(A^2-4B)}$.
- This is the familiar quadratic formula for the roots of $T^2 - AT + B = 0$.

Plimpton 322 & Pythagorean triples

- The Plimpton collection of tablets at Columbia University dates back to about 1900 – 1600 BCE.
- #322 has received considerable attention since it lists the first 15 Pythagorean triples, a 1000 years before Pythagoras!

Plimpton 322.

What is a Pythagorean triple?

- Any triple of natural numbers (a,b,c) such that $a^2 + b^2 = c^2$ is called a Pythagorean triple.
- A triple is called primitive if the gcd (a,b,c)=1.
- Plimpton 322 seems to suggest that the Babylonians knew that all such primitive triples can be generated by the following recipe: for p, q of opposite parity, all triples are given by:

$$a = p^2 - q^2 \text{ and } b = 2pq \text{ and } c = p^2 + q^2.$$

How to prove this?

- The first rigorous proof was given by Euclid in his Elements around 300 BCE.

$$a^2 + b^2 = c^2.$$

Observation 1: a and b have opposite parity.
Without loss of generality, say b is even. Then a, c are both odd.

Observation 2:

$$b^2 = c^2 - a^2 = (c+a)(c-a).$$

Dividing by4:

$$\frac{b^2}{4} = \left(\frac{c+a}{2}\right)\left(\frac{c-a}{2}\right).$$

The final step

$$\frac{b^2}{4} = \left(\frac{c+a}{2}\right)\left(\frac{c-a}{2}\right).$$

Observation 3: The two terms on the right are relatively prime.
By unique factorization, each factor must be a square.
So we can write

$$\frac{c+a}{2} = p^2, \quad \frac{c-a}{2} = q^2.$$

Thus,

$$c - p^2 + q^2, \quad a = p^2 - q^2, \quad b = 2pq.$$

The Babylonian legacy

- Many of these results are remarkable for an age where no formal mathematics existed.
- One deficiency seems to be a lack of distinction between approximate solutions and exact solutions.
- One could say that the Babylonians were the inventors of mathematical tables.
- The extensive collection of clay tablets archeologists have discovered is a testament to this fact.