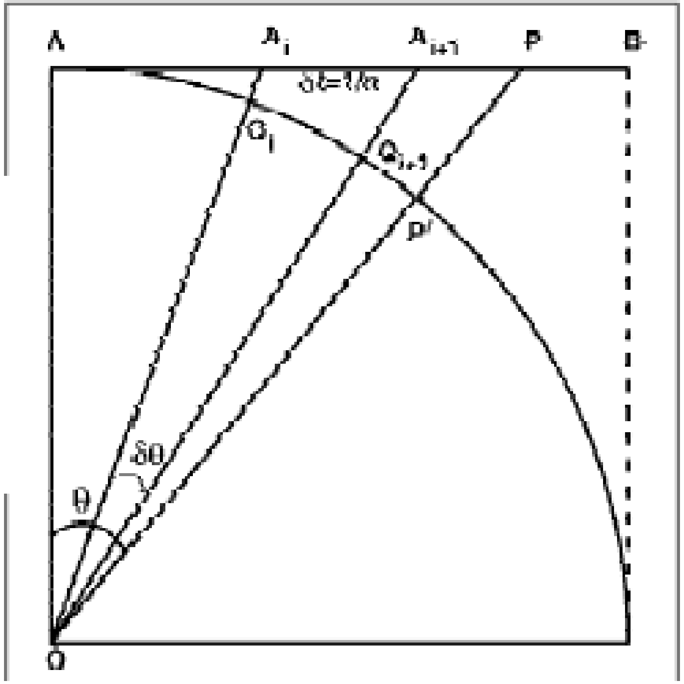
From Madhava to Wallis: Prelude to calculus

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \ \ldots$$



Madhava and the Kerala school

- There is now considerable documentary evidence that many of the ideas needed for the development of calculus were already written down by what is now called the Kerala school of mathematics in south western India in the 14th century.
- Madhava (1340-1425 CE) was the foremost member of this school who developed the theory of the trigonometric functions and derived the familiar Taylor series expansions for them.
- His work was described in detail with proofs by Jyesthadeva who wrote the text called Yuktibhasha in the middle of the 16th century.

Madhava series for the trigonometric functions

- Studying the sine and cosine function, Madhava derived the following well-known formulas:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots.$$

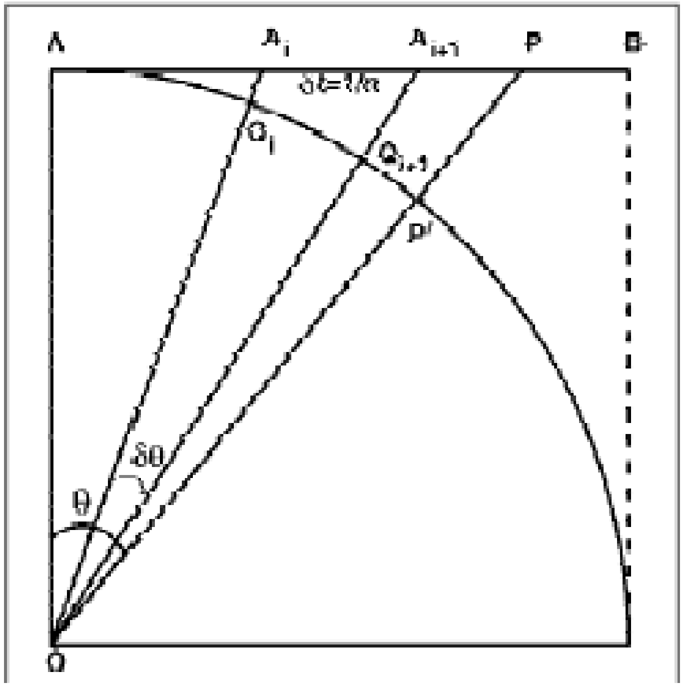
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots.$$

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \ldots.$$

The last is the famous arctan series re-discovered by Gregory several centuries later. When $\theta=\pi/4$, we get the famous Madhava-Gregory-Leibniz series for π.

The series for arctan x

- Madhava began by considering a quadrant of unit circle of radius 1. Thus OA=1. Let θ be the angle AOP so that AP=tan θ=t (say).
- Divide AP into n parts. Joining the center to each of the n points on AP subdivides the sector AP' into n parts. We want to determine the length of the arc AP'=θ.
- Now O_iO_{i+1} is trapped between the lengths h and k of the perpendicular from A_i to OA_{i+1} and O_i to OA_{i+1}.
- Using similar triangles, he sees that h/(1/n) =1/OA_{i+1}. But k/h =1/OA_i. Thus, k=(1/n) 1/OA_iOA_{i+1}. Since k=δθ, we find adding up the parts that $\sum_{i=0}^{n} \frac{1}{n\left(1+\left(\frac{i}{n}\right)^2\right)}$.
- This is the Riemann sum for the integral for arctan x.



Evaluation of the sum

- Each term in the summand is then expanded as a geometric series:
- $(1+\left(\frac{i}{n}\right)^2)^{-1} = 1 - (\frac{i}{n})^2 + \ldots$
- This now requires a formula for the sum of k-th powers of the natural numbers, but only in the limit.
- The limit needed is $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(\frac{i}{n})^k$ which is easily evaluated to be $\frac{1}{k+1}$.

The migration of ideas and Father Mersenne

- According to George Joseph, author of the Crest of the Peacock, it is quite possible that many of the ideas of the Kerala school migrated through Jesuit missionaries to Europe.
- Most notable among the Jesuits was Father Marin Mersenne (1588-1648) who was a close friend of both Descartes and Fermat (in fact, many conjectures attributed to Fermat appear in his letters to Mersenne).
- Thus many of the ideas of calculus were "in the air".

The work of Cavalieri, Fermat, Mengoli and Gregory

- Bonaventura Cavalieri (1598-1647) was an assistant of Galileo who discovered (in modern notation) that $\int_0^a x\,dx = a^2/2$ and more generally, $\int_0^a x^n\,dx = a^{n+1}/(n+1)$.
- Fermat (1601-1665) discovered the concept of a derivative of a function and used it to determine maxima and minima of various functions.
- Pietro Mengoli (1625-1686) investigated infinite series and showed $1-1/2 + 1/3 - 1/4 + \ldots = \log 2$ and he also posed the famous Basel problem of evaluating the sums of the reciprocals of the squares: $\sum_{n\geq 1} n^{-2}$ which was later solved by Euler in 1736.
- James Gregory (1638-1675) discovered the Taylor series for the arctan function that Madhava had found several centuries earlier: $\int_0^a (1+x^2)^{-1}\,dx = \arctan x = x - x^3/3 + x^5/5 - \ldots$ from which we deduce the Madhava-Gregory-Leibniz series for π:
- $\pi = 1 - 1/3 + 1/5 - \ldots$

Mengoli and log 2

- Mengoli discovered that 1-1/2 + 1/3 -1/4 + … converges to log 2. How would he have proved this? Recall: $(1-r)(1+r+\cdots+r^n) = 1-r^{n+1}$,

$$\begin{aligned}\sum_{n=1}^{m}\frac{(-1)^{n+1}}{n} &= \sum_{n=1}^{m}(-1)^{n+1}\int_0^1 x^{n-1}\,\mathrm{d}x\\ &= \int_0^1\sum_{n=1}^{m}(-1)^{n+1}x^{n-1}\,\mathrm{d}x\\ &= \int_0^1\sum_{n=1}^{m}(-x)^{n-1}\,\mathrm{d}x\\ &= \int_0^1\frac{1-(-x)^m}{1+x}\,\mathrm{d}x\\ &= \int_0^1\frac{\mathrm{d}x}{1+x}-\int_0^1\frac{(-x)^m}{1+x}\,\mathrm{d}x\\ &= \ln 2-(-1)^m\int_0^1\frac{x^m}{1+x}\,\mathrm{d}x.\end{aligned}$$

Now, for $0\le x\le 1$, we have

$$\int_0^1\frac{x^m}{1+x}\,\mathrm{d}x \le \int_0^1 x^m\,\mathrm{d}x = \frac{1}{m+1}$$

since $1+x\ge 1$ and, therefore,

$$\lim_{m\to\infty}\int_0^1\frac{x^m}{1+x}\,\mathrm{d}x = 0.$$

There was a sudden interest in infinite series. Mengoli posed the famous Basel problem of evaluating the sum of the reciprocals of the squares, later solved by Euler.

Gregory and the arctan function

- In 1668, Gregory published which represent rediscovery of the works of Madhava's series for arctan.

$$\int_0^x \frac{dx}{1+x^2} = \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

The method we used to derive the formula for log 2 can be used here also.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

$$\int_0^x \frac{1}{1+y^2}\,\mathrm{d}y = \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

John Wallis and calculus

- John Wallis (1616-1703) was an English clergyman and mathematician. He is given partial credit for the discovery of infinitesimal calculus.
- He introduced the symbol ∞ $\ and \frac{1}{\infty}$ for an infinitesimal.
- He was a contemporary of Newton.

Wallis's formula for π

- In1650, Wallis discovered:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$$

This is equivalent to: showing the following limit equals 1. (Exercise)

$$\lim_{n\to\infty} \frac{\pi n}{2^{4n}} \binom{2n}{n}^2$$

The proof

- The limit can be evaluated using trigonometric integrals:

$$I_n := \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta.$$

By integrating by parts, we have:

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta \sin\theta \, d\theta \\ &= \sin^{n-1}\theta(-\cos\theta)\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos\theta)(n-1)\sin^{n-2}\theta\cos\theta \, d\theta \\ &= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\cos^2\theta \, d\theta \\ nI_n &= (n-1)I_{n-2} \end{aligned}$$

$$I_n = \frac{n-1}{n} I_{n-2}.$$

$$\frac{I_n}{I_{n-2}} \to 1.$$

$I_0 = \frac{\pi}{2}$ and $I_1 = 1$.

$$I_{2n} = \binom{2n}{n} \frac{\pi}{2^{2n+1}} \text{ and}$$

$$I_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

The final limit

Now, since $0 \leqslant \sin(\theta) \leqslant 1$ when $\theta \in [0, \frac{\pi}{2}]$, it follows that

$$I_{n-2} \geqslant I_{n-1} \geqslant I_n.$$

It follows that:

$$\lim_{n\to\infty} \frac{I_n}{I_{n-1}} = 1.$$

$$1 = \lim_{n\to\infty} \frac{I_{2n}}{I_{2n-1}} = \lim_{n\to\infty} \frac{\pi n}{2^{4n}} \binom{2n}{n}^2$$

- because

$$I_{2n} = \binom{2n}{n} \frac{\pi}{2^{2n+1}} \text{ and}$$

$$I_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$