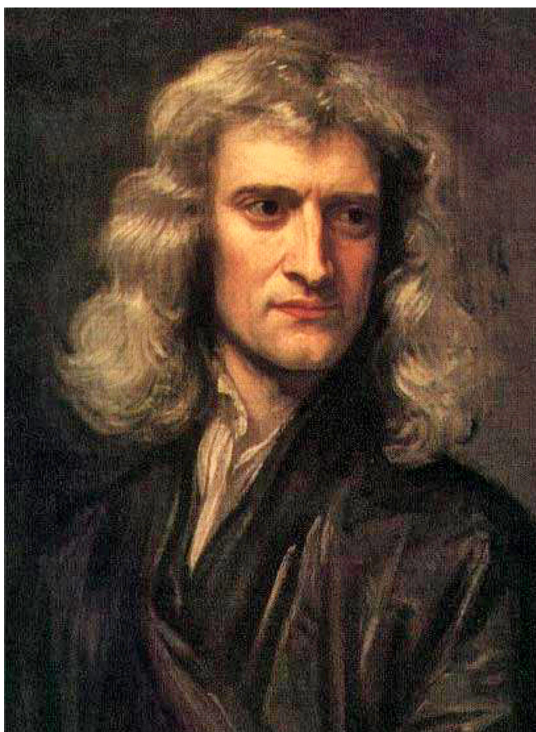
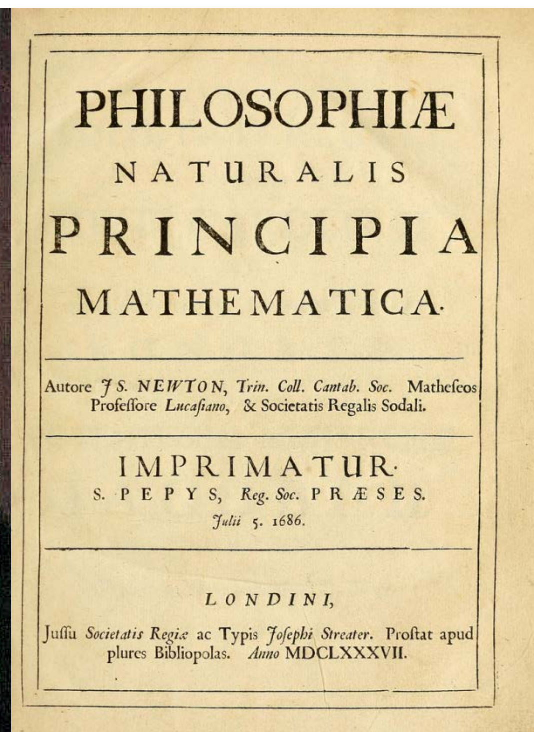
Newton and Leibniz: The development of calculus





PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore *J S.* NEWTON, *Trin. Coll. Cantab. Soc.* Matheseos Professore *Lucasiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,

Jussu *Societatis Regiæ* ac Typis *Josephi Streater.* Prostat apud plures Bibliopolas. *Anno* MDCLXXXVII.

Isaac Newton (1642-1727)

- Isaac Newton was born on Christmas day in 1642, the same year that Galileo died. This coincidence seemed to be symbolic and in many ways, Newton developed both mathematics and physics from where Galileo had left off.
- A few months before his birth, his father died and his mother had remarried and Isaac was raised by his grandmother.
- His uncle recognized Newton's mathematical abilities and suggested he enroll in Trinity College in Cambridge.

Newton at Trinity College

- At Trinity, Newton keenly studied Euclid, Descartes, Kepler, Galileo, Viete and Wallis.
- He wrote later to Robert Hooke, “If I have seen farther, it is because I have stood on the shoulders of giants.”
- Shortly after he received his Bachelor’s degree in 1665, Cambridge University was closed due to the bubonic plague and so he went to his grandmother’s house where he dived deep into his mathematics and physics without interruption.
- During this time, he made four major discoveries: (a) the binomial theorem; (b) calculus ; (c) the law of universal gravitation and (d) the nature of light.
- The binomial theorem, as we discussed, was of course known to the Chinese, the Indians, and was re-discovered by Blaise Pascal. But Newton’s innovation is to discuss it for fractional powers.

The binomial theorem

- Newton's notation in many places is a bit clumsy and he would write his version of the binomial theorem as:

The Extractions of Roots are much shortened by the Theorem

$$\overline{P+PQ}\,\Big|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \text{etc.}$$

where $P + PQ$ stands for a Quantity whose Root or Power or whose Root of a Power is to be found, P being the first term of that quantity, Q being the remaining terms divided by the first term and m/n the numerical Index of the powers of $P + PQ$ . . . . Finally, in place of the terms that occur in the course of the work in the Quotient, I shall use A, B, C, D, etc. Thus A stands for the first term $P(m/n)$; B for the second term $(m/n)AQ$; and so on.[1]

In modern notation, the left hand side is $(P+PQ)^{m/n}$ and the first term on the right hand side is $P^{m/n}$ and the other terms are:

$$B = \frac{m}{n}AQ = \frac{m}{n}P^{m/n}Q$$

$$C = \frac{m-n}{2n}BQ = \frac{(m-n)m}{(2n)n}P^{m/n}Q^2 = \frac{\left(\frac{m}{n}\right)\left(\frac{m}{n}-1\right)}{2}P^{m/n}Q^2$$

$$D = \frac{m-2n}{3n}CQ = \frac{\left(\frac{m}{n}\right)\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{3\times 2}P^{m/n}Q^3 \quad \text{and so on}$$

The binomial theorem as a Taylor series

What we see here is the Taylor series expansion of the function $(1+Q)^{m/n}$.

$$(1+Q)^{m/n} = 1 + \frac{m}{n}Q + \frac{\left(\frac{m}{n}\right)\left(\frac{m}{n}-1\right)}{2}Q^2 + \frac{\left(\frac{m}{n}\right)\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{3\times 2}Q^3 + \cdots$$

The key point is that this expression continues to make sense if m/n is replaced by any real number. This is the breakthrough! When, for instance m/n=1/2, this series gives an infinite series which converges to the square root of 1+Q.

The discovery of calculus

- The most impressive discovery of Newton is his treatise on calculus, which he called the “method of fluxions”.
- Using his new theory, Newton obtains an infinite series expression for π.

OF

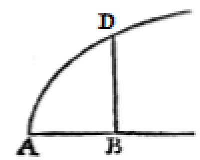
ANALYSIS

BY

Equations of an infinite Number of Terms.

1. *THE General Method, which I had deviſed ſome conſiderable Time ago, for meaſuring the Quantity of Curves, by Means of Series, infinite in the Number of Terms, is rather ſhortly explained, than accurately demonſtrated in what follows.*

2. Let the Baſe AB of any Curve AD have BD for it's perpendicular Ordinate; and call AB=x, BD=y, and let a, b, c, &c. be given Quantities, and m and n whole Numbers. Then



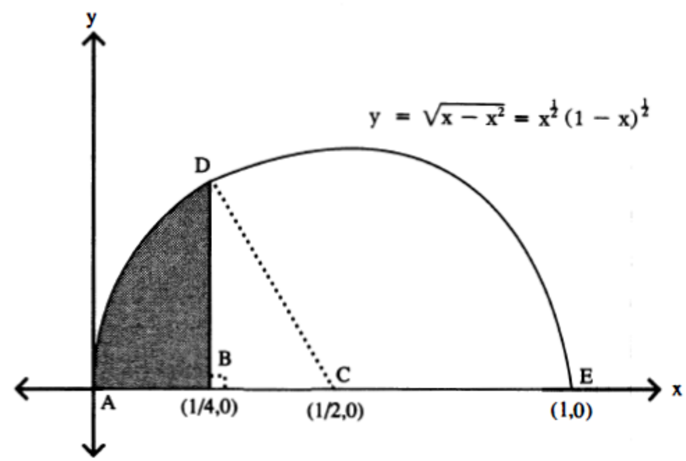
The Quadrature of Simple Curves,

RULE I.

3. If $ax^{\frac{m}{n}}=y$; it ſhall be $\frac{an}{m+n}x^{\frac{m+n}{n}}$ = Area ABD.

A series for π using the binomial theorem

- Consider the circle centered at (½, 0) of radius ½.
- We want to calculate the area of the shaded region in two different ways.



$$(x - \tfrac{1}{2})^2 + (y - 0)^2 = \tfrac{1}{2}^2 \quad \text{or} \quad x^2 - x + \tfrac{1}{4} + y^2 = \tfrac{1}{4}$$

Simplifying and solving for y gives the equation of the upper semicircle as

$$y = \sqrt{x - x^2} = \sqrt{x}\sqrt{1-x} = x^{1/2}(1-x)^{1/2}$$

Using the binomial theorem:

$$\begin{aligned} y &= x^{1/2}(1-x)^{1/2} \\ &= x^{1/2}\left(1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \frac{7}{256}x^5 - \cdots\right) \\ &= x^{1/2} - \frac{1}{2}x^{3/2} - \frac{1}{8}x^{5/2} - \frac{1}{16}x^{7/2} - \frac{5}{128}x^{9/2} - \frac{7}{256}x^{11/2} - \cdots \end{aligned}$$

Now integrate term by term:

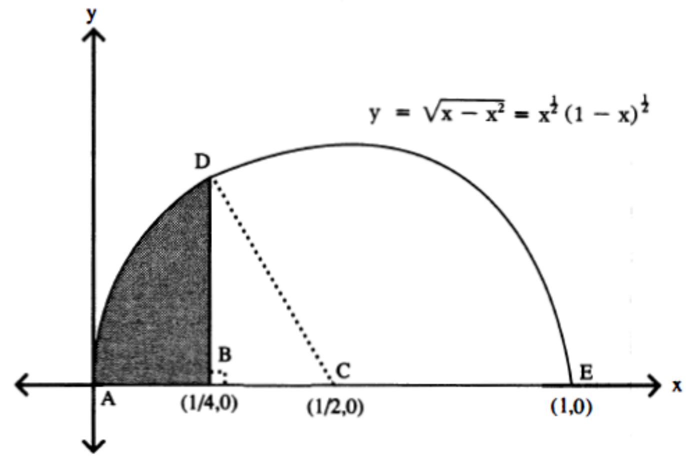
$$\begin{aligned} &\frac{2}{3}x^{3/2} - \frac{1}{2}\left(\frac{2}{5}x^{5/2}\right) - \frac{1}{8}\left(\frac{2}{7}x^{7/2}\right) - \frac{1}{16}\left(\frac{2}{9}x^{9/2}\right) - \cdots \\ &= \frac{2}{3}x^{3/2} - \frac{1}{5}x^{5/2} - \frac{1}{28}x^{7/2} - \frac{1}{72}x^{9/2} - \frac{5}{704}x^{11/2} - \cdots \end{aligned}$$

And put x=1/4 to get:

The area of the region ABD

- The first six terms of the infinite series are:

$$\frac{1}{12} - \frac{1}{160} - \frac{1}{3584} - \frac{1}{36864} - \frac{5}{1441792} \cdots - \frac{429}{163208757248}$$



On the other hand, we can use simple geometry to calculate the area of the shaded region. Note that:

$$\overline{BD} = \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2} = \sqrt{\frac{3}{16}} = \frac{\sqrt{3}}{4}$$

Hence,

$$\text{Area }(\Delta DBC) = \frac{1}{2}(\overline{BC}) \times (\overline{BD}) = \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{4}\right) = \frac{\sqrt{3}}{32}$$

$$\text{Area (sector)} = \frac{1}{3}\text{ Area (semicircle)}$$

$$= \frac{1}{3}\left(\frac{1}{2}\pi r^2\right) = \frac{1}{3}\left[\frac{1}{2}\pi\left(\frac{1}{2}\right)^2\right] = \frac{\pi}{24}$$

$$\text{Area }(ABD) = \text{Area (sector)} - \text{Area }(\Delta DBC) = \frac{\pi}{24} - \frac{\sqrt{3}}{32}$$

Calculus and the laws of motion

- Newton's innovation is in using calculus as a tool to study physics, in particular, the laws of motion.
- If a particle travels a distance of s(t) in time t, the rate of change of distance with respect to a unit of time is called the velocity.
- Using calculus terminology, the velocity at time t, denoted v(t) is the first derivative of s(t).
- What Newton realized is that the area under the curve of the graph of v(t) represents the distance travelled. He realized that differentiation and integration were inverses of each other and thus formulated the fundamental theorem of calculus.
- $\int_a^b v(t)dt = s(b) - s(a)$ since $v(t) = s'(t)$.

What is acceleration?

- Acceleration, denoted $a(t)$, is the rate of change of velocity and so is the first derivative of $v(t)$ and the second derivative of $s(t)$.
- These mathematical definitions of velocity and acceleration allowed Newton to explain the famous Galileo experiment.
- Galileo observed two different bodies, regardless of their mass, accelerate at the same rate. In other words, $v'(t) = g$ is constant. Thus, $v(t) = gt$ and $s(t) = gt^2/2$. g is called the acceleration due to gravity.

What is a force?

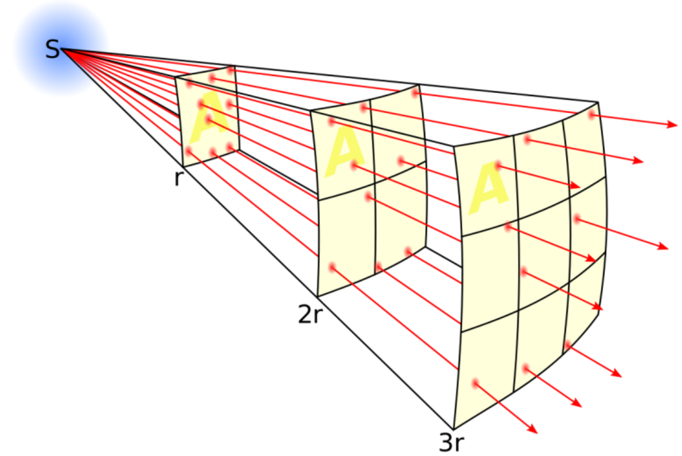
- Intuitively, a force is a push or a pull.
- Newton defined force as mass time acceleration or in symbols, F=ma.
- This allows us to make a distinction between weight and mass. If we think of an object as being composed of atomic particles, its mass should be the same wherever in the cosmos it is. But not so its' weight, which would depend on the gravitational force being exerted on it.
- Thus, the acceleration due to gravity denoted as g earlier, changes from planet to planet.

Newton's laws of motion

- First law (discovered earlier by Galileo) is the principle of inertia: a moving object undisturbed, continues to move with a constant velocity in a straight line or remains inert if it was initially inert.
- Second law: F=ma. Newton defined momentum as mass times velocity, so then F (force) is the rate of change of momentum.
- Third law (conservation of momentum): to every action, there is an opposite and equal reaction. In mathematical notation, if there are a number of interacting particles with masses $m_1, \ldots, m_r$ each moving at velocities $v_1, \ldots, v_r$ then $m_1 v_1 + \ldots + m_r v_r = \text{constant}$.

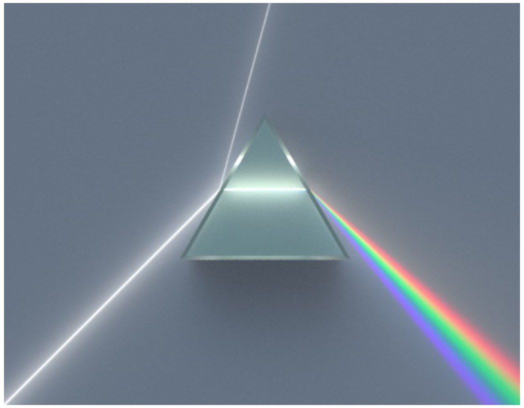
The inverse square law

- Though several scientists had surmised an inverse square law of gravitational force, it was Newton who realized it as a universal principle.
- In the diagram, we see that the point S radiates a gravitational force uniformly in all directions and this force weakens with distance r; thus the radial force is inversely proportional to the surface area which is $4\pi r^2$.
- This led Newton to his famous law of universal gravitation:
- $F=GmM/r^2$ where G is a universal constant, M and m are the two masses and r is the distance between them. The force they exert on each other is determined by this equation.



Newton called this the Law of Universal Gravitation. He was able to derive Kepler's Laws from this single law.

Newton and light



- Newton adopted a particle theory of light and analysed sunlight and showed it is composed of different frequencies of light.
- This understanding allowed him to devise a reflecting telescope instead of a refracting telescope that Galileo had invented.

- The telescope was more powerful than the ones before and it is the model on which modern telescopes are built.

Gottfried Wilhelm Leibniz (1646-1716)

- Independently of Newton, Leibniz in Germany had discovered calculus and he published his work before Newton had.
- Newton was averse to publishing his work partly because of criticism from his peers and so, he would just keep things to himself.
- Nevertheless, it looks as if Leibniz had arrived at the calculus independently from the perspective of a philosopher and mathematician and less of a physicist trying to understand the laws of motion, as Newton was aiming to do.
- Much of the modern notation that we use is due to Leibniz: the use of dy/dx, and the integral symbol are due to Leibniz, who was a master of symbolism.
- Sadly, both Newton and Leibniz were embroiled in a priority fight for most of their lives.

In retrospect, it is a combination of the contributions of both of these mathematicians that comprises the modern theory of calculus.

Leibniz and infinite series

- Leibniz was fascinated with numbers and their properties. For instance, he studied the harmonic triangle which resembled the Pascal triangle but had interesting properties of its own.
- In the harmonic triangle, each term (not in the first row or column) is the difference of the two terms directly above it and to the right.
- Thus, he could sum the series $\sum_1^\infty 2/n(n+1)$.

$$\frac{2}{n(n+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Arithmetic triangle

1	1	1	1	1	1	1 ⋯
1	2	3	4	5	6 ⋯	
1	3	6	10	15 ⋯		
1	4	10	20 ⋯			
1	5	15 ⋯				
1	6 ⋯					
1 ⋯						

Harmonic triangle

$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}\cdots$
$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{30}\cdots$	
$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{30}$	$\frac{1}{60}\cdots$		
$\frac{1}{4}$	$\frac{1}{20}$	$\frac{1}{60}\cdots$			
$\frac{1}{5}$	$\frac{1}{30}\cdots$				
$\frac{1}{6}\cdots$					

This allowed him to solve
A problem posed by Huygens.

He rediscovered the Madhava Series for π.

Other contributions of Leibniz

- Leibniz noticed a parallel between the binomial theorem and the formula for calculating the n-th derivative of a product of two functions.
- He discovered the multinomial theorem.
- He anticipated the theory of determinants needed to solve systems of simultaneous equations.
- He realized the importance of imaginary numbers in solving polynomial equations.

The theory of monads

- In 1714, Leibniz wrote a book called Monadologie, in which he described the monad as the building block of everything.
- Perhaps this is the precursor of an atomic theory.
- But Leibniz was after a grander theory and he wanted to deduce the existence of God from his theory.
- His work was perhaps inspired by the problem of interaction between mind and body posed by Descartes on the one hand, and the nature of "substance" as enunciated by Spinoza.
- Apart from his discovery of calculus, Leibniz is best known today for his insistence on symbols as an instrument for rational thought and in particular, his promotion of good notation.
- Perhaps he was searching for an "algebra of mind" or a "geometry of thought".

A page from Leibniz's manuscript of the *Monadology*