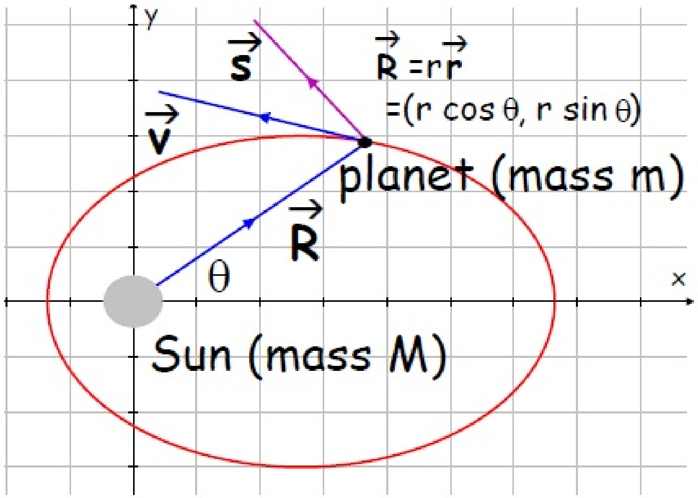
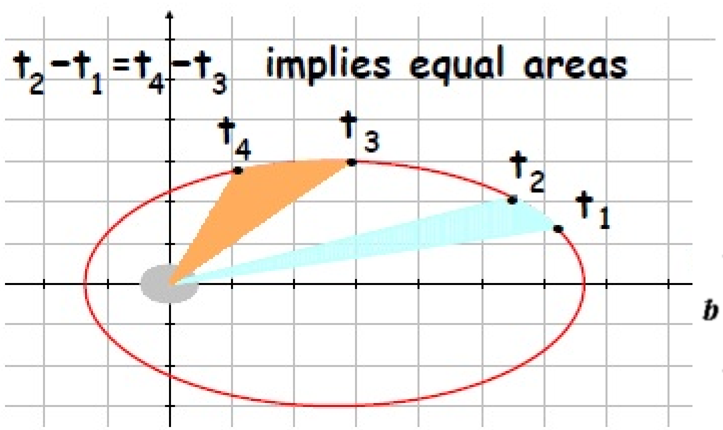
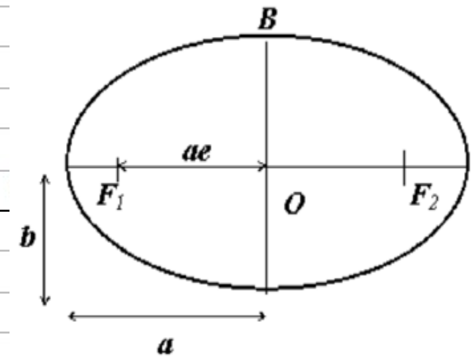
Newton's derivation of Kepler's laws







Review of Kepler's laws

- Let us first review Kepler's three laws:

Kepler's First Law:

The orbit of a planet about the Sun is an ellipse with the Sun at one focus.

Kepler's Second Law:

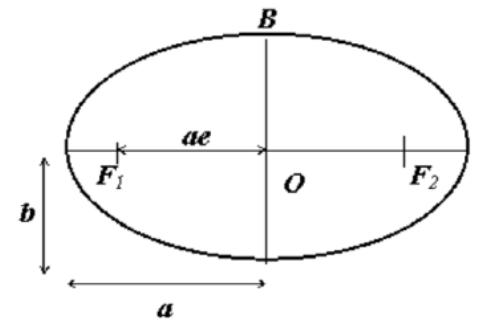
A line joining a planet and the Sun sweeps out equal areas in equal intervals of time.

Kepler's Third Law:

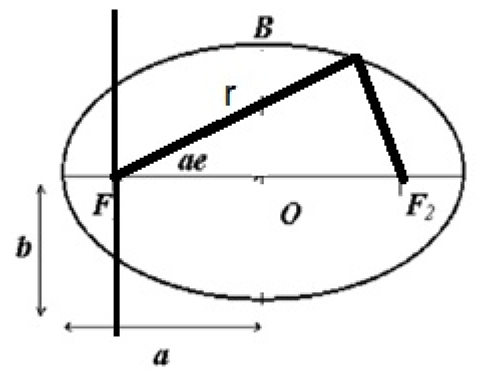
The squares of the sidereal periods of the planets are proportional to the cubes of their semimajor axes.

These laws were stated by Kepler from meticulous observations. He did not provide any theoretical explanation for them. This was done by Newton through his law of universal gravitation.

The equation for the ellipse in polar co-ordinates



- Recall that the ellipse can be described as the locus of points whose sum of the two distances to two foci F_1 and F_2 is constant. The length of the major axis is denoted a and the minor axis is b.
- We write the distance of the foci from the origin as ae and call e the eccentricity.
- Thus, the eccentricity for a circle is zero.



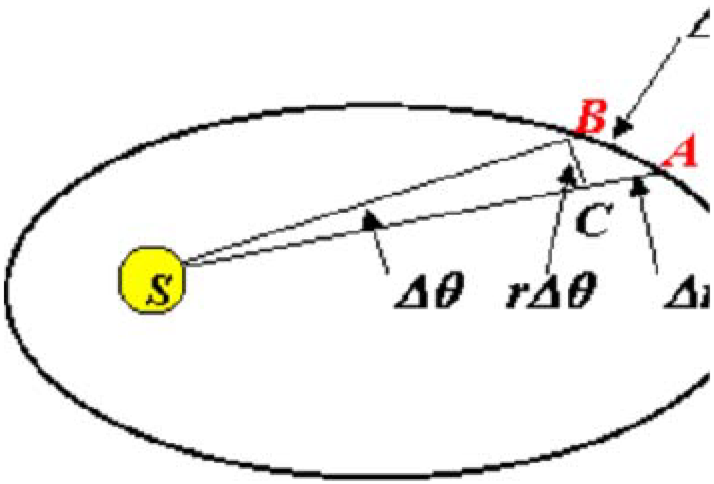
- To derive the equation for the ellipse in polar co-ordinates, it is convenient to make one of the foci as the origin as we did before. It is then clear that we have using the cosine law $r+\sqrt{(r^2 + (2ae)^2 - 4aer\cos\theta)} = a(1+e)$.
- This can be simplified as $r = A/(1+B\cos\theta)$ for certain constants A and B.
- This will be useful later in understanding Newton's derivation of Kepler's first law.

Area of the ellipse in polar co-ordinates

- It will be useful to derive the formula for the area of the segment of the ellipse swept by an angle θ at one of the focal points.

Thus, the area swept by the radial vector moving from θ_1 to θ_2 or time t_1 to time t_2 is:

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta = \frac{1}{2}\int_{t_1}^{t_2} r^2 \frac{d\theta}{dt}\, dt.$$



This will be useful in deriving Kepler's second law.

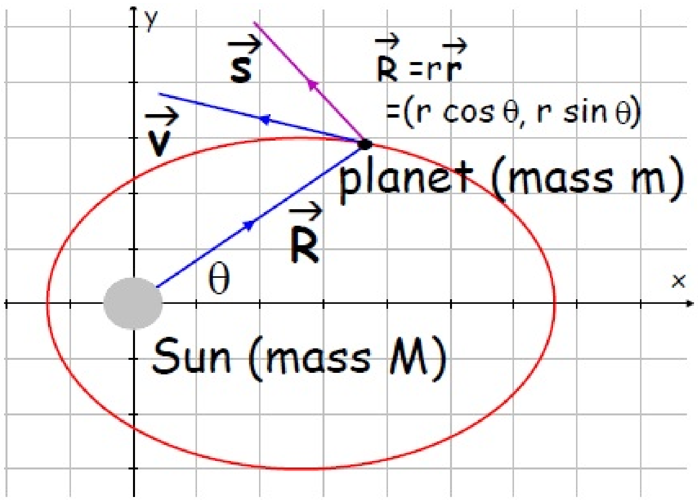
Newton's law of universal gravitation

- Recall that Newton formulated the law of universal gravitation by the equation: $F=GMm/r^2$, where G is a universal constant, M and m are the masses of the two bodies and r is the distance between them.
- Recall also Newton's second law of motion: $F=ma$, where m is the mass and a represents acceleration.
- In trying to understand planetary motion, we are led to the equality that the acceleration $a= GM/r^2$, which expresses how the acceleration changes with the radial distance.

Analysing the orbit via polar co-ordinates

- As noted earlier, in describing planetary motion, it is convenient to use polar co-ordinates instead of cartesian co-ordinates.
- In describing an ellipse in polar co-ordinates, it is convenient to use one of the foci as the origin.
- This will allow us to put the sun at one of the focal points and study the motion of the planet from this perspective.

r and θ will both be viewed as functions of time parameter t.



As depicted in the figure, the sun is at the origin (the "heliocentric" point of view), and $\overrightarrow{R}$ is the position vector of the orbiting planet. The vector $\overrightarrow{r}$ is the unit vector in the direction of $\overrightarrow{R}$. Since $\overrightarrow{r} \cdot \overrightarrow{r} = 1$, the product rule for differentiation shows that $\overrightarrow{r} \cdot \left(\frac{d}{dt}\overrightarrow{r}\right) = 0$; therefore if $\overrightarrow{s}$ is the unit vector in the dirction of $\frac{d}{dt}\overrightarrow{r}$, it follows that $\overrightarrow{r} \cdot \overrightarrow{s} = 0$, as well. All of this is depicted in Figure . In fact, from this picture, we see that $\overrightarrow{r}$ and $\overrightarrow{s}$ are given explicitly as

$$\overrightarrow{r} = (\cos\theta, \sin\theta), \quad \overrightarrow{s} = (-\sin\theta, \cos\theta),$$

from which it follows that via componentwise differentiation

$$\frac{d}{dt}\overrightarrow{\boldsymbol{r}} = \frac{d\theta}{dt}\overrightarrow{\boldsymbol{s}} \quad \text{and that} \quad \frac{d}{dt}\overrightarrow{\boldsymbol{s}} = -\frac{d\theta}{dt}\overrightarrow{\boldsymbol{r}}.$$

The vector equation dictating the motion of the orbiting planet is

$$-\left(\frac{GMm}{r^2}\right)\overrightarrow{\boldsymbol{r}} = m\frac{d^2}{dt^2}\overrightarrow{\boldsymbol{R}} \qquad (1)$$

since the force on the planet is directed back towards the sun.

In order to compute $\frac{d^2}{dt^2}\overrightarrow{\boldsymbol{R}}$ explicitly, note first that the velocity vector is given by

$$\overrightarrow{\boldsymbol{v}} = \frac{d}{dt}\overrightarrow{\boldsymbol{R}} = \frac{d}{dt}\left(r\cdot\overrightarrow{\boldsymbol{r}}\right) = \frac{dr}{dt}\overrightarrow{\boldsymbol{r}} + r\left(\frac{d}{dt}\overrightarrow{\boldsymbol{r}}\right) = \frac{dr}{dt}\overrightarrow{\boldsymbol{r}} + r\frac{d\theta}{dt}\overrightarrow{\boldsymbol{s}} \qquad (2)$$

The acceleration is the time derivative of Equation (2):

$$\begin{aligned}
\overrightarrow{\boldsymbol{a}} &= \frac{d}{dt}\overrightarrow{\boldsymbol{v}} = \frac{d}{dt}\left(\frac{dr}{dt}\overrightarrow{\boldsymbol{r}} + r\frac{d\theta}{dt}\overrightarrow{\boldsymbol{s}}\right) \\
&= \frac{d^2r}{dt^2}\overrightarrow{\boldsymbol{r}} + 2\frac{dr}{dt}\frac{d\theta}{dt}\overrightarrow{\boldsymbol{s}} + r\frac{d^2\theta}{dt^2}\overrightarrow{\boldsymbol{s}} - r\left(\frac{d\theta}{dt}\right)^2\overrightarrow{\boldsymbol{r}} \\
&= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\overrightarrow{\boldsymbol{r}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\overrightarrow{\boldsymbol{s}}
\end{aligned}$$

Deriving Kepler's second law

However, from equation (1) we see that the acceleration vector is given by

$$\overrightarrow{a} = -\frac{GM}{r^2}\overrightarrow{r}$$

from which we conclude that

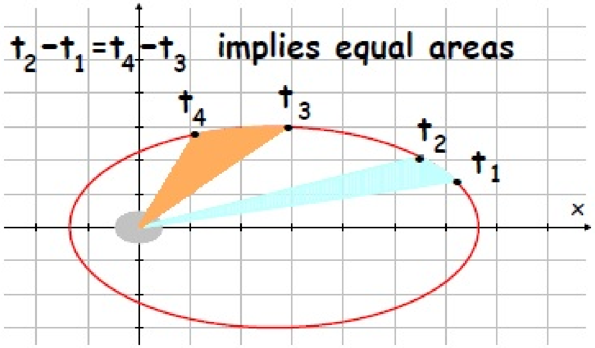
$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \qquad (3)$$

and

$$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0. \qquad (4)$$

- Using this we can derive Kepler's second law.
- Recall this states:

Area swept between times t_1 and $t_2 = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta = \frac{1}{2}\int_{t_1}^{t_2} r^2 \frac{d\theta}{dt}\, dt.$



The final step

Area swept between times t_1 and $t_2 = \dfrac{1}{2}\displaystyle\int_{\theta_1}^{\theta_2} r^2\, d\theta = \dfrac{1}{2}\int_{t_1}^{t_2} r^2 \frac{d\theta}{dt}\, dt.$

Note, however, that

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 2r\frac{dr}{dt}\frac{d\theta}{dt} + r^2\frac{d^2\theta}{dt} = r\cdot 0 = 0,$$

where we have used Equation (4), above. The upshot is that the integrand $r^2\dfrac{d\theta}{dt}$ is a constant—call it L—from which we conclude that

$$\text{Area swept between times } t_1 \text{ and } t_2 = \frac{1}{2}\int_{t_1}^{t_2} r^2\frac{d\theta}{dt}\, dt = \frac{1}{2}L(t_2 - t_1).$$

Therefore, if $t_4 - t_3 = t_2 - t_1$ the areas will be the same! This proves Kepler's Second Law.

Kepler's first law

- Recall from our discussion of Kepler's second law that:

$$L = r^2 \frac{d\theta}{dt}$$

is a constant; define the new constant

$$P = \frac{L^2}{GM},$$

and define the (dimensionless) variable

$$u = \frac{P}{r}.$$

This says that

$$\frac{d\theta}{dt} = \frac{L}{r^2} = \frac{Lu^2}{P^2}.$$

We have, using the Chain Rule, that

$$\frac{dr}{dt} = \frac{d}{d\theta}\left(\frac{P}{u}\right)\frac{d\theta}{dt} = -\frac{P}{u^2}\frac{d\theta}{dt}\frac{du}{d\theta} = -\frac{L}{P}\frac{du}{d\theta}.$$

Kepler's first law (continued)

Differentiate again and obtain

$$\frac{d^2r}{dt^2} = \frac{d}{d\theta}\left(\frac{dr}{dt}\right)\frac{d\theta}{dt} = -\frac{L}{P}\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -\frac{L^2u^2}{P^3}\frac{d^2u}{d\theta^2}$$

Next, we substitute into Equation (3) and get

$$-\frac{L^2u^2}{P^3}\frac{d^2u}{d\theta^2} - \left(\frac{P}{u}\right)\left(\frac{Lu^2}{P^2}\right)^2 = -\frac{L^2u^2}{P^3}.$$

Dividing by the common factor of $-\dfrac{L^2u^2}{P^3}$ results in the inhomogeneous second-order differential equation:

$$\frac{d^2u}{d\theta^2} + u = 1.$$

The general solution of this has the form

$$u = u(\theta) = 1 + e\cos(\theta - \theta_0),$$

where e and θ_0 are constants which can be determined from the initial conditions. In terms of the polar radius r, this becomes

$$r = \frac{P}{1 + e\cos(\theta - \theta_0)},$$

- This is the equation of an ellipse in polar co-ordinates.

Deriving Kepler's third law

- Recall that Kepler's third law says that T^2/a^3 is constant, where T is the period of the orbit and a is the length of the semi-major axis. We can deduce this from Newton's law of gravitation in five steps.

Step 1. Denoting by T the orbital period, note that the area of the ellipse swept out by the orbiting body is

$$A = \frac{1}{2}\int_{t=0}^{t=T} r^2 \frac{d\theta}{dt}\, dt.$$

However, we've seen that $r^2 \frac{d\theta}{dt}$ is a constant, denoted L. Therefore, the above area is given by $A = \frac{1}{2}LT$.

Step 2. Denoting by a the semi-major axis and by b the semi-minor axis, the area of the corresponding ellipse can be expressed by the integral

$$A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx = ab\pi.$$

Kepler's third law (continued)

Step 3. Recall from Analytical Geometry that an ellipse with semi-major axis a and eccentricity e has semi-minor axis $b = a\sqrt{1-e^2}$. (Exercise on the next assignment) Conclude from Steps 1 and 2 that

$$T = \frac{2A}{L} = \frac{2a^2\sqrt{1-e^2}\pi}{L}.$$

Step 4. Using the equation

$$r = \frac{P}{1+e\cos\theta}, \quad P = \frac{L^2}{GM},$$

together with

$$2a = r_{\min} + r_{\max} = \frac{P}{1+e} + \frac{P}{1-e}$$

leads immediately to

$$L^2 = a(1-e^2)GM.$$

Step 5. We conclude that $\dfrac{T^2}{a^3} = \dfrac{4\pi^2}{GM}$.

- This completes the proof of Kepler's third law.