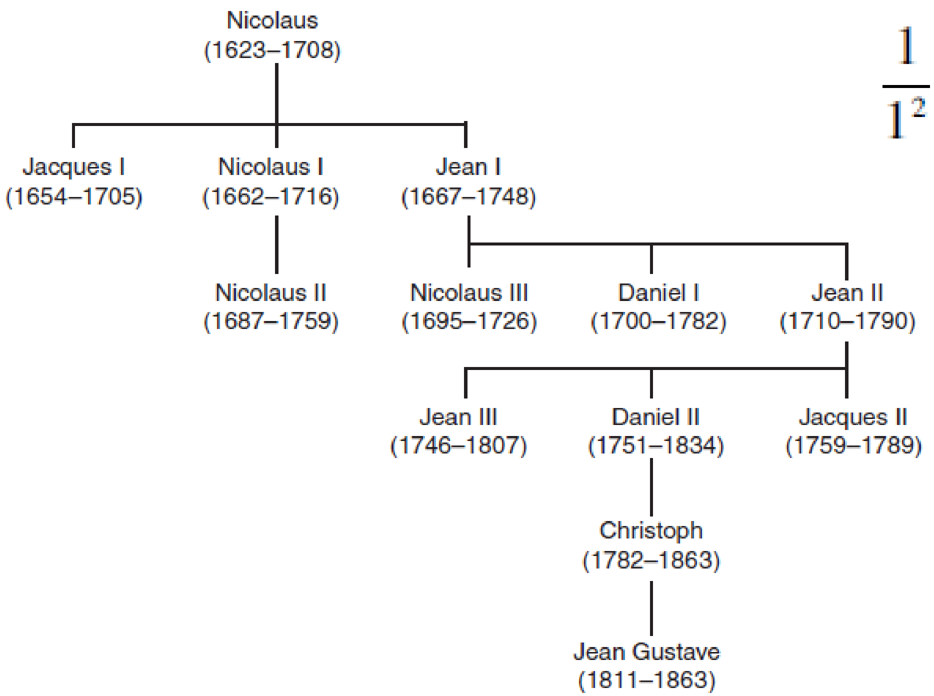
The Bernoulli family and the development of analysis



The mathematical Bernoullis: a genealogical chart

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

for $x \geq 0$, as $n \to \infty$,

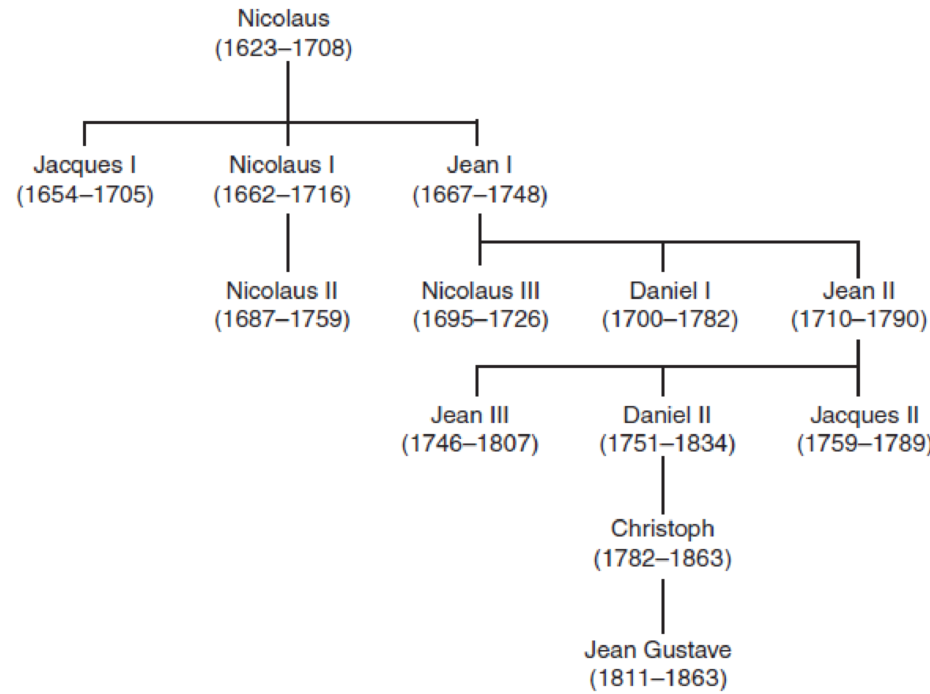
$$\sum_{\substack{k: \\ |k-\frac{1}{2}n| \leq \frac{1}{2}x\sqrt{n}}} \binom{n}{k} \sim 2^n \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du,$$

Leibniz and *Acta Eruditorum*

- Though Newton and Leibniz discovered calculus independently, and there were priority fights, the development of calculus in England was slow while the development in continental Europe was faster since Leibniz communicated his ideas freely through his journal *Acta Eruditorum*.
- Two Swiss brothers, Jacques (or James) Bernoulli (1654-1705) and Jean (or John) Bernoulli (1667-1748) became devoted disciples of Leibniz by studying and learning from his papers published in *Acta Eruditorum*.

The Bernoulli genealogy

- In a single century, the family produced eight mathematicians.
- Jacques (or James) Bernoulli is perhaps the most famous, with his solution of the brachistochrone problem and the isoperimetric problem, as well as his discovery of the "law of large numbers" in probability theory.
- James taught his brother Jean mathematics, but there seemed to be some sibling rivalry between them.
- The Bernoulli family also taught the great Euler and many of them held professorships at the University of Basel in Switzerland.



The mathematical Bernoullis: a genealogical chart

Several more recent prominent scholars are also descended from the family, including:

- Johann Jakob Bernoulli (1831–1913), art historian and archaeologist; noted for his *Römische Ikonographie* (1882 onwards) on Roman Imperial portraits
- Ludwig Bernoully (1873 – 1928), German architect in Frankfurt
- Hans Bernoulli (1876–1959), architect and designer of the Bernoullihäuser in Zurich and Grenchen SO
- Elisabeth Bernoulli (1873-1935), suffragette and campaigner against alcoholism

The Basel problem

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

- Jacques Bernoulli was fascinated by the problem of determining the sum of the reciprocals of the squares, which became known as the Basel problem.
- He knew the series was convergent but that it equals $\pi^2/6$ is a famous theorem of Euler of 1736, some thirty years after the death of Jacques.

The Bernoulli inequality

- The inequality $(1+x)^n > 1+nx$ when $x > -1$ and n is a natural number > 1 is often attributed to Jacques Bernoulli but has since been found in the 1670 papers of Isaac Barrow, the teacher of Isaac Newton.
- Nowadays, with the calculus, the proof is simple enough: consider the function $f(x) = (1+x)^n - 1 - nx$. We want to show that $f(x) > 0$ if $x > -1$. For $x > 0$, the result is clear by the binomial theorem. So we focus on $-1 < x < 0$.
- We differentiate with respect to x, and get $f'(x) = n(1+x)^{n-1} - n$ which is negative if $-1 < x < 0$.
- Thus, the function $f(x)$ is a decreasing function on $(-1,0)$.
- Hence $f(x) > f(0) = 0$ for $n > 1$.

Compound interest and *e*

- Long before Euler discovered *e*, Jacques Bernoulli encountered it in dealing the problem of compound interest.

Jacob Bernoulli

One example is an account that starts with $1.00 and pays 100 percent interest per year. If the interest is credited once, at the end of the year, the value is $2.00; but if the interest is computed and added twice in the year, the $1 is multiplied by 1.5 twice, yielding $\$1.00 \times 1.5^2 = \$2.25$. Compounding quarterly yields $\$1.00 \times 1.25^4 = \$2.4414\ldots$, and compounding monthly yields $\$1.00 \times (1.0833\ldots)^{12} = \$2.613035\ldots$.

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

The Bernoulli numbers

- These numbers first arose in Jacob Bernoulli's book *Ars Conjectandi* which laid the foundations of probability theory.
- Bernoulli was seeking a formula for the k-th powers of the first (n-1) numbers:

$$s_k(n) = 1^k + 2^k + 3^k + \cdots + (n-1)^k.$$

Bernoulli discovered the general formula is a polynomial of degree k+1 in n:

$$\frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c(c-1)(c-2)}{2\cdot 3\cdot 4}Bn^{c-3}$$
$$+ \frac{c(c-1)(c-2)(c-3)(c-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}Cn^{c-5}\cdots,$$

The generating function of the Bernoulli numbers

- We can package the Bernoulli numbers into a power series so that their definition becomes evident:

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} \frac{B_j t^j}{j!}$$

An expansion (via long division) shows that the Bernoulli numbers are rational numbers.

From this definition, we also see that they satisfy a simple recursion formula.

A simple and (modern) derivation

- Here is a simple way of deriving this formula using the method of generating functions, which we have already seen in understanding recurrence sequences.

We consider the power series

$$\sum_{k=0}^{\infty} \frac{s_k(n)t^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\Big(\sum_{j=0}^{n-1} j^k\Big)$$
$$= \sum_{j=0}^{n-1} e^{tj} = \frac{e^{nt}-1}{e^t-1}.$$

Writing

$$\frac{e^{nt}-1}{e^t-1} = \frac{e^{nt}-1}{t} \cdot \frac{t}{e^t-1}$$
$$= \Big(\sum_{k=1}^{\infty} \frac{n^k t^{k-1}}{k!}\Big)\Big(\sum_{j=0}^{\infty} \frac{B_j t^j}{j!}\Big)$$

and comparing coefficients of both sides gives the result

$$(k+1)s_k(n) = \sum_{i=0}^{k} \binom{k+1}{i} B_i n^{k+i-i}.$$

L'Hospital's rule

- Jean Bernoulli tutored a young French marquis, G.F.A. de l'Hospital (1661-1704) in the new calculus.
- It seems that under the agreements of this tutelage, L'Hospital would (or could) publish any results of Bernoulli under his name, which he did.
- The most famous of these is the l'Hospital rule (1694) used by calculus students today the world over.

Bernoulli's chief contributions, dating from 1694, has ever since been known as L'Hospital's rule on indeterminate forms. Jean Bernoulli had found that if $f(x)$ and $g(x)$ are functions differentiable at $x = a$ such that $f(a) = 0$ and $g(a) = 0$ and

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof: the limit is

$$\lim_{x \to a} \frac{(f(x) - f(a))/x}{(g(x) - g(a))/x}$$

The l'Hospital – Bernoulli controversy

- L\Hospital's rule appears in the book *Analyse des infinment petits* written by him in 1696.
- The work was a great success and went into numerous editions in the next century.
- In 1704, after l'Hospital's death, Bernoulli publicly accused him of plagiarism and contemporaries regarded these accusations as unfounded. So the name stuck!
- But after the Bernoulli- l'Hospital correspondence was discovered, it is now realised that much of the work is due to Jean Bernoulli.

Bernoulli trials and coin tossing

If we flip a fair coin a 100 times say, we expect that "roughly" half the time we will get heads and half the time we will get tails.

Any experiment with only two possible outcomes is called a Bernoulli trial.

Can we make this mathematically precise?

Consider the set:

$$\mathcal{S}_N := \{\sigma = (a_1, a_2, ..., a_N) : \quad a_i = \pm 1\}.$$

Evidently, $|\mathcal{S}_N| = 2^N$. For each $\sigma \in \mathcal{S}_N$, we define

$$s(\sigma) = a_1 + a_2 + \cdots + a_N.$$

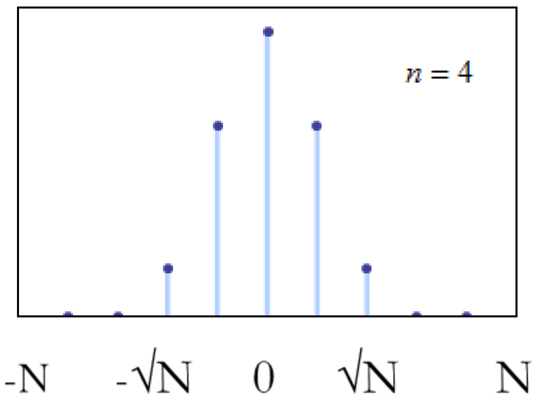
Since each of ± 1 is taken with probability $1/2$, we would expect $s(\sigma)$ to be zero "on average". In fact,

$$\sum_{\sigma \in \mathcal{S}_N} s(\sigma) = \frac{1}{2} \sum_{\sigma \in \mathcal{S}_N} (s(\sigma) + s(-\sigma)) = 0.$$

This models the coin-tossing experiment with +1 for heads and -1 for tails so that s(σ) should be on average zero.

The standard deviation

- The possible values of s(σ) range from –N to N.
- If we plot all the 2^N possibilities, we will get a graph of the following shape:



Writing $\sigma = (a_1(\sigma), \ldots, a_N(\sigma))$, we also see that

$$\begin{aligned}\sum_{\sigma\in\mathcal{S}_N} s(\sigma)^2 &= \sum_{\sigma\in\mathcal{S}_N}\sum_{1\le i,j\le N} a_i(\sigma)a_j(\sigma)\\ &= \sum_{\sigma\in\mathcal{S}_N}\left(N+\sum_{i\neq j} a_i(\sigma)a_j(\sigma)\right)\\ &= N2^N + \sum_{\sigma\in\mathcal{S}_N}\sum_{i\neq j} a_i(\sigma)a_j(\sigma).\end{aligned}$$

For each $\sigma \in \mathcal{S}_N$ and $i \neq j$, define $\check{\sigma}$ to be the same as σ except $a_i(\check{\sigma}) = -a_i(\sigma), a_j(\check{\sigma}) = a_j(\sigma)$. It is then transparent that interchanging sums in the last summation and pairing σ with $\check{\sigma}$, the sum vanishes and we obtain

$$\sum_{\sigma\in\mathcal{S}_N} s(\sigma)^2 = N2^N.$$

This says that | s(σ) | deviates from 0 by "roughly" √N.

Towards the central limit theorem and the law of large numbers

- Jacques Bernoulli certainly understood what was needed to be studied in the problem of Bernoulli trials.
- Though he set up the problem, he could not solve it.
- It was solved later by de Moivre using Stirling's formula for the asymptotic behaviour of n!
- The final theorem is:

for $x \geq 0$, as $n \to \infty$,

$$\sum_{\substack{k: \\ |k-\frac{1}{2}n| \leq \frac{1}{2}x\sqrt{n}}} \binom{n}{k} \sim 2^n \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du,$$