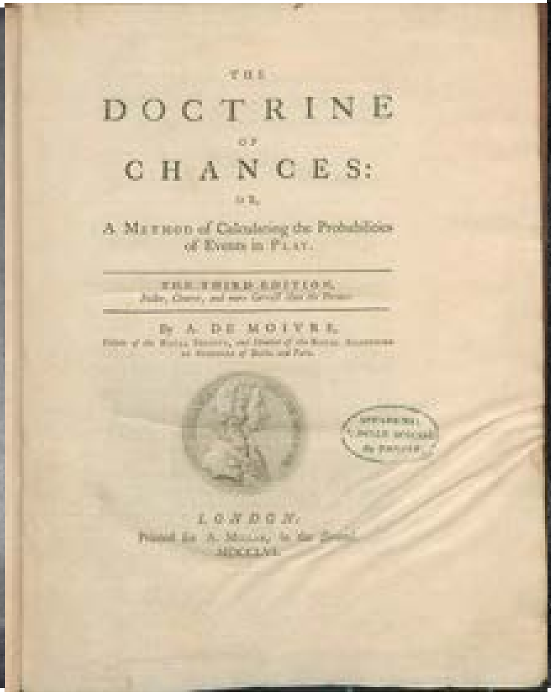
Stirling and de Moivre: The development of probability theory



THE

DOCTRINE

OF

CHANCES:

OR,

A METHOD of Calculating the Probabilities of Events in PLAY.

THE THIRD EDITION.

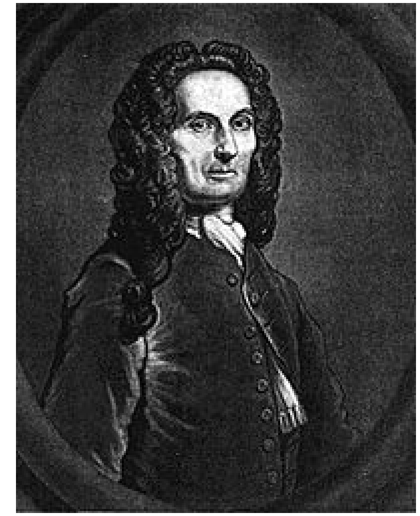
By A. DE MOIVRE.

LONDON:

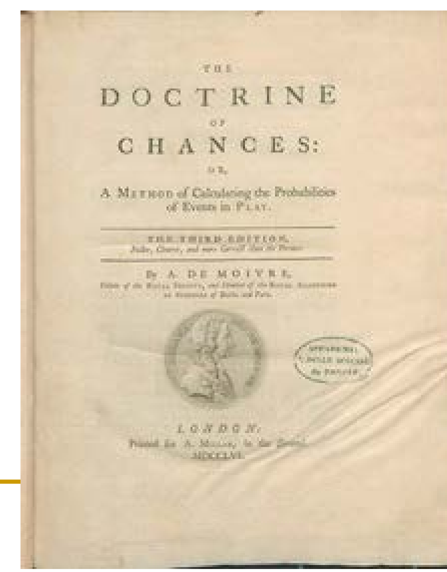
MDCCLVI.

Abraham de Moivre

- Abraham de Moivre (1667-1734) was a French mathematician who (due to religious persecution in France) went to England and studied with Newton and Halley.
- Unable to secure a university position, he eked out an existence being a private tutor of mathematics.
- In 1718, he wrote his famous book, *Doctrine of Chances* in which he outlined the a mathematical theory of probability.



Abraham de Moivre
(1667-1754)



The probability integral

- De Moivre was the first to recognize the importance of the probability integral:

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Here is a short proof of this fact. Put:

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx.$$

$$I^2 = 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy.$$

We change $x = ty$ in the inner integral and interchange the order, which we can do because of absolute convergence:

$$I^2 = 4 \int_0^{\infty} \int_0^{\infty} e^{-y^2(1+t^2)} y \, dy \, dt.$$

The inner integral is easily evaluated and we find:

$$I^2 = 4 \int_0^{\infty} \left[\frac{-e^{-y^2(1+t^2)}}{2(1+t^2)} \Big|_{y=0}^{y=\infty} \right] dt = 2 \int_0^{\infty} \frac{dt}{1+t^2} = 2 \left[\arctan t \right]_0^{\infty} = \pi.$$

The harmonic series revisited

- We saw earlier that Nicolas Oresme had shown the harmonic series diverges. But how does it diverge?
- What is the asymptotic behavior of the partial sum? This can be answered as follows.

Note that

$$a_k := \frac{1}{k} - \int_k^{k+1} \frac{dt}{t} = \int_k^{k+1} \left(\frac{1}{k} - \frac{1}{t} \right) dt = \int_k^{k+1} \frac{t-k}{tk} dt \geq 0.$$

Also,

$$a_k = \int_k^{k+1} \frac{t-k}{tk} dt \leq \frac{1}{k^2},$$

since the numerator of the integrand is at most 1. Therefore

$$C := \sum_{k=1}^{\infty} a_k < \infty.$$

Euler's constant

By the integral test the tail is $O(1/n)$. Now

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) = C + O(1/n).$$

By the mean value theorem,

$$\log(n+1) = \log n + O(1/n).$$

This proves:

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + C + O(1/n).$$

C is called Euler's constant.

- It is unknown at present if Euler's constant is a rational number. The conjecture is that it is irrational.

Approximating the logarithm

- We can use Taylor expansion of log (1-x) to approximate the logarithm:

$$\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

This tells us that, for $|x| < 1$, we have:

$$|-\log(1-x) - x| \leqslant \sum_{k=2}^{\infty} |x|^k = \frac{|x|^2}{1-|x|}.$$

Changing x to $-x$ gives us:

$$|\log(1+x) - x| \leqslant \frac{|x|^2}{1-|x|}.$$

By the approximation to the logarithm

$$\log(n+1) = \log n + \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

De Moivre's asymptotic formula for n!

- In his work on probability theory, de Moivre needed an asymptotic formula for n! which he obtained by simple calculus in the following way. By considering log n!, we have:

$$\sum_{k=1}^{n} \log k = \sum_{k=1}^{n} \left(\log k - \int_{k}^{k+1} \log t \, dt \right) + \int_{1}^{n+1} \log t \, dt.$$

By the approximation to the logarithm

$$\log(n+1) = \log n + \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

The last integral is easily evaluated:

$$\int_{1}^{n+1} \log t \, dt = t \log t - t \Big|_{1}^{n+1} = (n+1)\log(n+1) - n = n \log n - n + \log n + 1 + O(1/n).$$

The sum can be re-written as

$$-\sum_{k=1}^{n} \int_{k}^{k+1} \log \frac{t}{k} dt = -\sum_{k=1}^{n} \int_{0}^{1} \log\left(1 + \frac{u}{k}\right) du.$$

We approximate the logarithm:

$$\log\left(1 + \frac{u}{k}\right) = \frac{u}{k} + O\left(\frac{u}{k^2}\right)$$

and the sum becomes

$$-\sum_{k=1}^{n} \frac{1}{k} \int_{0}^{1} u \, du + C_1 + O(1/n) = -\frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} + C_1 + O(1/n),$$

for some constant C_1.

The final asymptotic

- We can now insert this in our approximation of log n! Recall that we proved:

The final answer is

$$\log n! = \sum_{k=1}^{n} \log k = n \log n - n + \frac{1}{2} \log n + C_2 + O(1/n)$$

for some constant C_2. This is de Moivre's theorem. That is, as n tends to infinity,

$$n! \sim \sqrt{C_3 n}(n/e)^n.$$

de Moivre could not determine C_3. This was done by Stirling.

Let us write $n! \sim B\sqrt{n}\,(n/e)^n$

How to determine the constant?

- The simplest way to determine the constant is via Wallis's formula, already discussed in an earlier lecture.

$$I_n := \int_0^{\frac{\pi}{2}} \sin^n \theta \, \mathrm{d}\theta.$$

By integrating by parts, we have:

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta \sin\theta \, \mathrm{d}\theta \\ &= \sin^{n-1}\theta(-\cos\theta)\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos\theta)(n-1)\sin^{n-2}\theta\cos\theta \, \mathrm{d}\theta \\ &= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\cos^2\theta \, \mathrm{d}\theta \\ nI_n &= (n-1)I_{n-2} \end{aligned}$$

$$I_n = \frac{n-1}{n} I_{n-2}.$$

James Stirling (1692-1770)

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\frac{I_n}{I_{n-2}} \to 1.$$

Solving the recursion

- We solve the recursion:

$$I_{2n} = \binom{2n}{n}\frac{\pi}{2^{2n+1}} \text{ and}$$

$$I_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

$$\frac{I_n}{I_{n-2}} \to 1.$$

Now, since $0 \leqslant \sin(\theta) \leqslant 1$ when $\theta \in [0, \frac{\pi}{2}]$, it follows that

$$I_{n-2} \geqslant I_{n-1} \geqslant I_n.$$

It follows that:

$$\lim_{n\to\infty} \frac{I_n}{I_{n-1}} = 1.$$

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Computing along the even subsequence, we see that we are immediately led to the constant B in de Moivre's formula:

$$1 = \lim_{n\to\infty} \frac{I_{2n}}{I_{2n-1}} = \lim_{n\to\infty} \frac{\pi n}{2^{4n}} \binom{2n}{n}^2$$

The final steps

- We can now find B by Wallis's formula (proved in an earlier lecture):

$$\lim_{n\to\infty} \frac{\sqrt{2n}}{2^{2n}} \binom{2n}{n} = \frac{2}{\sqrt{2\pi}}$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{2^{2n}} \frac{B\sqrt{2n}(\frac{2n}{e})^{2n}}{(B\sqrt{n}(\frac{n}{e})^{n})^2} = \frac{1}{\sqrt{\pi}}$$

$$\lim_{n\to\infty} \frac{\sqrt{2}}{B} = \frac{1}{\sqrt{\pi}}$$