Stirling and de Moivre: The development of probability theory
Abraham de Moivre

- Abraham de Moivre (1667-1734) was a French mathematician who (due to religious persecution in France) went to England and studied with Newton and Halley.
- Unable to secure a university position, he eked out an existence being a private tutor of mathematics.
- In 1718, he wrote his famous book, *Doctrine of Chances* in which he outlined the a mathematical theory of probability.
The probability integral

De Moivre was the first to recognize the importance of the probability integral:

\[ \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \]

Here is a short proof of this fact. Put:

\[ I = \int_{-\infty}^\infty e^{-x^2}dx. \]

\[ I^2 = 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dx\,dy. \]

We change \( x = ty \) in the inner integral and interchange the order, which we can do because of absolute convergence:

\[ I^2 = 4 \int_0^\infty \int_0^\infty e^{-y^2(1+t^2)}\,dy\,dt. \]

The inner integral is easily evaluated and we find:

\[ I^2 = 4 \int_0^\infty \left[ -\frac{e^{-y^2(1+t^2)}}{2(1+t^2)} \right]_{y=0}^{y=\infty} \,dt = 2 \int_0^\infty \frac{dt}{1+t^2} = 2 [\arctan t]_0^\infty = \pi. \]
The harmonic series revisited

- We saw earlier that Nicolas Oresme had shown the harmonic series diverges. But how does it diverge?
- What is the asymptotic behavior of the partial sum? This can be answered as follows.

Note that

\[ a_k := \frac{1}{k} - \int_k^{k+1} \frac{dt}{t} = \int_k^{k+1} \left( \frac{1}{k} - \frac{1}{t} \right) dt = \int_k^{k+1} \frac{t - k}{tk} dt \geq 0. \]

Also,

\[ a_k = \int_k^{k+1} \frac{t - k}{tk} dt \leq \frac{1}{k^2}, \]

since the numerator of the integrand is at most 1. Therefore

\[ C := \sum_{k=1}^{\infty} a_k < \infty. \]
Euler’s constant

By the integral test the tail is $O(1/n)$. Now
\[
\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{k} - \log(n + 1) = C + O(1/n).
\]

By the mean value theorem,
\[
\log(n + 1) = \log n + O(1/n).
\]

This proves:
\[
\sum_{k=1}^{n} \frac{1}{k} = \log n + C + O(1/n).
\]

$C$ is called Euler’s constant.

- It is unknown at present if Euler’s constant is a rational number. The conjecture is that it is irrational.
Approximating the logarithm

- We can use Taylor expansion of \( \log(1-x) \) to approximate the logarithm:

\[
\log(1 - x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}.
\]

This tells us that, for \( |x| < 1 \), we have:

\[
| - \log(1 - x) - x | \leq \sum_{k=2}^{\infty} |x|^k = \frac{|x|^2}{1 - |x|}.
\]

Changing \( x \) to \(-x\) gives us:

\[
| \log(1 + x) - x | \leq \frac{|x|^2}{1 - |x|}.
\]

By the approximation to the logarithm

\[
\log(n + 1) = \log n + \frac{1}{n} + O \left( \frac{1}{n^2} \right).
\]
De Moivre’s asymptotic formula for $n!$

In his work on probability theory, de Moivre needed an asymptotic formula for $n!$ which he obtained by simple calculus in the following way. By considering $\log n!$, we have:

$$\sum_{k=1}^{n} \log k = \sum_{k=1}^{n} \left( \log k - \int_{k}^{k+1} \log t \, dt \right) + \int_{1}^{n+1} \log t \, dt.$$  

By the approximation to the logarithm

$$\log(n + 1) = \log n + \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

The last integral is easily evaluated:

$$\int_{1}^{n+1} \log t \, dt = t \log t - t \bigg|_{1}^{n+1} = (n+1) \log(n+1) - n = n \log n - n + \log n + 1 + O(1/n).$$

The sum can be re-written as

$$- \sum_{k=1}^{n} \int_{k}^{k+1} \log \left( \frac{t}{k} \right) \, dt = - \sum_{k=1}^{n} \int_{0}^{1} \log \left( 1 + \frac{u}{k} \right) \, du.$$  

We approximate the logarithm:

$$\log \left( 1 + \frac{u}{k} \right) = \frac{u}{k} + O\left(\frac{u}{k^2}\right).$$

and the sum becomes

$$- \sum_{k=1}^{n} \frac{1}{k} \int_{0}^{1} u \, du + C_1 + O(1/n) = - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} + C_1 + O(1/n),$$

for some constant $C_1$. 
The final asymptotic

- We can now insert this in our approximation of $\log n!$. Recall that we proved:

The final answer is

$$\log n! = \sum_{k=1}^{n} \log k = n \log n - n + \frac{1}{2} \log n + C_2 + O(1/n)$$

for some constant $C_2$. This is de Moivre’s theorem. That is, as $n$ tends to infinity,

$$n! \sim \sqrt{C_3} n (n/e)^n.$$

de Moivre could not determine $C_3$. This was done by Stirling.

Let us write $n! \sim B \sqrt{n} \ (n/e)^n$
How to determine the constant?

- The simplest way to determine the constant is via Wallis’s formula, already discussed in an earlier lecture.

\[ I_n := \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta. \]

By integrating by parts, we have:

\[
I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1} \theta \sin \theta \, d\theta \\
= \sin^{n-1} \theta (-\cos \theta) \bigg|_{\theta=0}^{\theta=\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos \theta)(n-1) \sin^{n-1} \theta \cos \theta \, d\theta \\
= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta \cos^2 \theta \, d\theta \\
nI_n = (n-1)I_{n-2}
\]

\[
I_n = \frac{n-1}{n} I_{n-2}.
\]

James Stirling (1692-1770)

\[
n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n
\]

\[
\frac{I_n}{I_{n-2}} \to 1.
\]
Solving the recursion

We solve the recursion:

\[ \frac{I_n}{I_{n-2}} \to 1. \]

\[ I_{2n} = \binom{2n}{n} \frac{\pi}{2^{2n+1}} \quad \text{and} \]
\[ I_{2n+1} = \frac{2^{2n}(n!)^2}{(2n + 1)!}. \]

Now, since \( 0 \leq \sin(\theta) \leq 1 \) when \( \theta \in [0, \frac{\pi}{2}] \), it follows that
\[ I_{n-2} \geq I_{n-1} \geq I_n. \]

It follows that:
\[ \lim_{n \to \infty} \frac{I_n}{I_{n-1}} = 1. \]

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Computing along the even subsequence, we see that we are immediately led to the constant \( B \) in de Moivre’s formula:

\[ 1 = \lim_{n \to \infty} \frac{I_{2n}}{I_{2n-1}} = \lim_{n \to \infty} \frac{\pi n}{2^{4n}} \left( \frac{2n}{n} \right)^2 \]
The final steps

- We can now find B by Wallis’s formula (proved in an earlier lecture):

\[
\lim_{n \to \infty} \frac{\sqrt{2n}}{2^{2n}} \binom{2n}{n} = \frac{2}{\sqrt{2\pi}}
\]

\[
\lim_{n \to \infty} \frac{\sqrt{n} B \sqrt{2n \left(\frac{2n}{e}\right)^{2n}}}{2^{2n} \left(B \sqrt{n \left(\frac{n}{e}\right)^n}\right)^2} = \frac{1}{\sqrt{\pi}}
\]

\[
\lim_{n \to \infty} \frac{\sqrt{2}}{B} = \frac{1}{\sqrt{\pi}}
\]