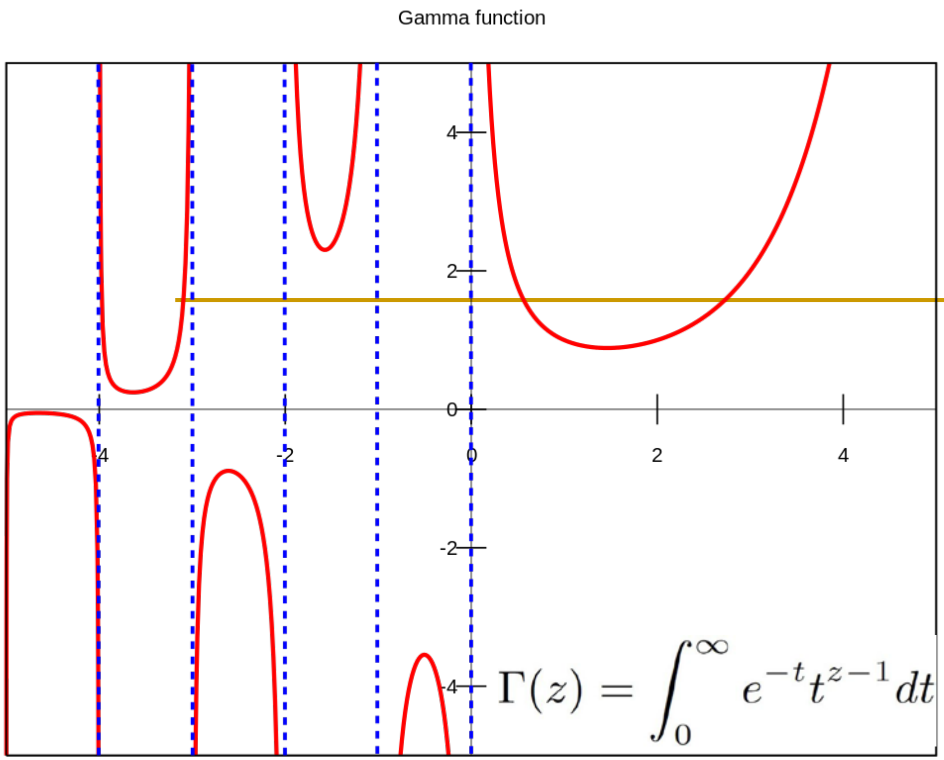
The Age of Euler



Euler and the Bernoullis

- Leonhard Euler (1707-1783) was born in Basel, Switzerland and educated by the Bernoullis.
- In 1730, he obtained a position as a professor of natural philosophy at the St. Petersburg Academy in Russia.
- There, he married and settled down to raise a family of thirteen children.
- He started a journal in which he contributed articles regularly and to this date, Euler is considered the most prolific of mathematicians with over a thousand research papers.

At the age of 28, he lost the sight of his right eye, but this did not diminish his research output!

The move to Berlin and aftermath

- In 1741, King Frederick invited him to join the Berlin Academy and Euler accepted. But relations between Frederick and Euler worsened especially when Frederick complained that Euler had no scintillating personality and teased him by calling him a "mathematical cyclops" with obvious reference to Euler's dysfunctional eye.
- So in 1766, he returned to St. Petersburg where he worked until his death in 1783, at the age of 76.
- For the last 17 years of his life, he was literally blind because he lost the sight of his other eye due to cataract and he would dictate mathematics to his children who would write up his results.
- The end came sadly when he was sipping tea in the company of one of his grandchildren.

$e^{\pi i} + 1 = 0$

- Euler was a master of notation and many of the symbols we use today are due to him.
- For example, he introduced the letter e perhaps suggested by the word "exponential" to designate the natural base for the logarithm.
- He also used π for the ratio of the circumference to the diameter of a circle, though William Jones had suggested this earlier.
- He used the symbol i for √-1. Using this, he related the exponential function and the trigonometric functions via the formulas:

$$\sin x = \frac{e^{\sqrt{-1}x} - e^{-\sqrt{-1}x}}{2\sqrt{-1}}$$

$$\cos x = \frac{e^{\sqrt{-1}x} + e^{-\sqrt{-1}x}}{2}$$

$$e^{\sqrt{-1}x} = \cos x + \sqrt{-1}\sin x$$

From this last equation, he deduced by setting $x = \pi$, his famous formula relating the constants π, e, i, 0 and 1: $e^{\pi i} + 1 = 0$.

Euler was so enamoured by this equation, he had it engraved on his tombstone!

From polynomials to functions

- Euler realized the importance of the exponential function, and its relations to the sine and cosine functions.
- He also realized the importance of complex numbers and the need to study a function through the study of the complex zeros of that function.
- For instance, he was aware (but did not prove) what is often called the fundamental theorem of algebra, that a polynomial of degree n has n complex roots and these roots determine the polynomial.
- A natural question is if this extends to other functions like the sine function:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Polynomials and roots

Suppose $P(x)$ is a polynomial of degree n having as its n roots $x = a$, $x = b$, $x = c, \ldots,$ and $x = d$; in other words, $P(a) = P(b) = P(c) = \cdots = P(d) = 0$. Suppose further that $P(0) = 1$. Then Euler knew that $P(x)$ factors into the product of n linear terms as follows:

$$P(x) = \left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{b}\right)\left((1 - \frac{x}{c}\right) \cdots \left(1 - \frac{x}{d}\right)$$

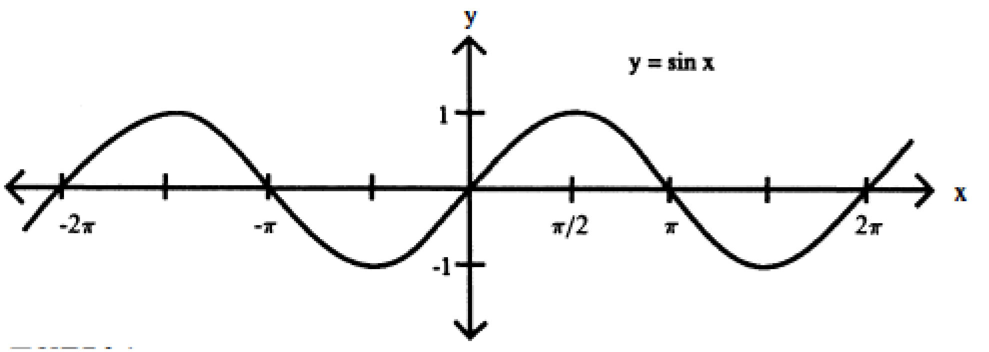
- Now consider f(x)=(sin x)/x and let us treat it as a "polynomial".

$$f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$$

To Euler, $f(x)$ was just an infinite polynomial with $f(0) = 1$ (as is immediately apparent). Thus, it can be factored, in the manner developed above, provided we determine the roots of the equation $f(x) = 0$.

The zeros of the sine function

- What are the zeros of $f(x)=(\sin x)/x$? Let us look at the graph of $\sin x$:



We see that all the real zeros are at integer multiples of π. But are there any other zeros?

To show that there are no other zeros, we need to use Euler's formula relating the trig functions and the exponential function.

First we need to prove that the exponential function has no zeros: write $z=x+iy$
So that $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)=0$ implies $\cos y=\sin y = 0$, which is impossible.

Recall that $\sin z= (e^{iz} – e^{-iz})/2i$ so if $\sin z =$, then $e^{2iz} = 1$ and the only way this can happen is if z is an integral multiple of π by the de Moivre – Euler formula.

A "factorization" of sin x

- Now that we have determined all the zeros of f(x)= (sin x)/x, we can pretend that it is a "polynomial" and write:

$$f(x) = \left(1 - \frac{x}{\pi}\right)\left(1 - \frac{x}{-\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{-2\pi}\right) \left(1 - \frac{x}{3\pi}\right)\left(1 - \frac{x}{-3\pi}\right)\cdots$$

$$= \left[\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\right]\left[\left(1 - \frac{x}{2\pi}\right) \left(1 + \frac{x}{2\pi}\right)\right]\left[\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\right]\cdots$$

which amounts to

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots = \left[1 - \frac{x^2}{\pi^2}\right]\left[1 - \frac{x^2}{4\pi^2}\right]\left[1 - \frac{x^2}{9\pi^2}\right]\left[1 - \frac{x^2}{16\pi^2}\right]\cdots$$

Comparing coefficients and the solution to the Basel problem

- We can compare the coefficients of both sides:

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$$
$$= \left[1 - \frac{x^2}{\pi^2}\right]\left[1 - \frac{x^2}{4\pi^2}\right]\left[1 - \frac{x^2}{9\pi^2}\right]\left[1 - \frac{x^2}{16\pi^2}\right]\cdots$$
$$= 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right)x^2 + (\ldots)\,x^4 - \cdots$$

$$-\frac{1}{3!} = -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right)$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

This solves the problem posed by Pietro Mengoli a century earlier, as well as the Bernoullis. However, it will be noted that not all steps are rigorous.

Euler and the zeta function

- A century before Riemann introduced the zeta function, Euler had studied it and derived even a "functional equation". But he did not study it is a function of a complex variable, which is what Riemann had done. Beginning with:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

Euler noted that this series converges for s>1 and because of unique factorization, it can be expressed as an infinite product over prime numbers:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

This allowed him to give a new proof of the infinitude Of primes by letting s→1+. The left hand side diverges and so the right must also diverge, so there are infinitely man primes.

This idea was later useful to Dirichlet when he showed there are infinitely many primes in an arithmetic progression.

$$\varsigma(2k) = B_k(2\pi i)^k/k!$$

Euler was able to extend his evaluation of $\varsigma(2)$:

where B_k is the k-th Bernoulli number.

The Gamma Function

- An important discovery of Euler is the Gamma function which he found by first noting that the following integral interpolates the factorials:

$\int_0^\infty e^{-x} x^n \mathrm{dx} = n!$

which is easily proved by induction.

The amazing next step of Euler is to realize that the left hand side makes sense for n replaced by any s>1.

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt,$$

He noticed that integrating by parts, we have $\Gamma(s+1)=s\Gamma(s)$ and this gives us an "analytic continuation" of the function to the entire complex plane, an idea developed later by Riemann.

Euler and number theory

- Euler made fundamental contributions to number theory as was mentioned in earlier lectures.
- The most notable is the case of n=3 of Fermat's conjecture.
- He also introduced the φ-function and generalized Fermat's little theorem.
- After discovering the exponential function, he introduced the sinh and cosh, the hyperbolic functions.
- He also showed that e is an irrational number.

The irrationality of e

- To show that e is irrational we proceed as follows.

PROOF. It is equivalent to show that $\alpha = e - 2 = \sum_{n=2}^{\infty} \frac{1}{n!}$ is irrational. For every $n \geq 2$, $n!\alpha = k_n + s_n$ with $k_n = n!(\sum_{i=2}^{n} \frac{1}{i!})$ is

an integer, and

$$0 < s_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots$$
$$< \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots = \frac{1}{n}.$$

If α is rational, then choosing n large will kill the denominator on the left hand side. This contradiction shows e is irrational. This was Euler's proof.
The fact that e is transcendental was proved a century later by Charles Hermite.