Cauchy, Fourier and Chebycheff

Cauchy, complex analysis & group theory

- Even though Gauss introduced the field of complex numbers, it was Cauchy who realized that one extend the ideas of calculus to this realm. He singlehandedly created complex analysis.
- He also initiated the study of group theory with emphasis on permutation groups which he encountered in his study of determinants.
- Augustin Louis Cauchy (1789-1857) was born in Paris in well to do family during the Napoleonic era. Lagrange was a friend of the Cauchy family and we may assume that he had a formative influence in the mathematical development of the young Cauchy.
- Cauchy was educated to be an engineer, but field work did not suit him. In 1812, at the age of 23, he fell ill and had to retire from engineering. He became more and more enamored by the beauty of abstract mathematics.

Cauchy as Professor at Ecole Polytechnique

Cauchy prior to 1857

- Cauchy soon accepted a professorship at Ecole Polytechnique in Paris and began his research into mathematics there.
- It is said Cauchy was a poor lecturer, cramming too much material into his lectures and often assuming that his students were familiar with the concepts as he was.
- Still, his work on complex analysis started in earnest, and his famous residue theorem in complex analysis was soon to find important applications.

Cauchy's theory of complex analysis

- For a function of a complex variable, Cauchy defined the notion of differentiability using the familiar notions from calculus, but he soon realized that this notion led to more structure and a remarkable cosmos unfolded before his eyes.

Let Ω be an open set of the complex plane $\mathbb{C}$. Suppose that $f : \Omega \to \mathbb{C}$ is differentiable. That is, the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for every $z_0 \in \Omega$. Writing $z = x + iy$, we can decompose $f(z)$ as

$$u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real valued functions of x and y.

Restricting the limiting process to the two orthogonal directions, namely the x and y directions, we get

$$\lim_{x \to x_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0} = \lim_{y \to y_0} \frac{f(x_0 + iy) - f(x_0 + iy_0)}{i(y - y_0)}.$$

The first limit is

$$\lim_{x \to x_0} \frac{u(x, y_0) + iv(x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{x - x_0} = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]_{x=x_0, y=y_0}.$$

The second limit is

$$-i \lim_{y \to y_0} \frac{u(x_0, y) + iv(x_0, y) - u(x_0, y_0) - iv(x_0, y_0)}{y - y_0} = -i\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]_{x=x_0, y=y_0}.$$

The Cauchy-Riemann equations

- This leads to the important set of equations called the Cauchy-Riemann equations.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

The reader should understand that although the definition of differentiability of f is similar to the usual derivative of a function of a real variable, the complex variable case is much richer. This is underscored by the Cauchy-Riemann equations. For instance, we see that for an analytic function $f(z) = u(x, y) + iv(x, y)$, the functions $u(x, y)$ and $v(x, y)$ of real variables satisfy **Laplace's equation**:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Differentiable functions of a complex variable are called analytic to distinguish them from functions of a real variable. If a function is analytic in the whole complex plane, it is called entire. Cauchy found that any such function can be expanded as a power series about any point z_0:

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

Joseph Fourier

- Around the same time, Joseph Fourier (1768-1830) was developing his mathematical theory of heat, and to study this, he developed what is now called the theory of Fourier series.
- His question was:

Suppose $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is a continuous function. We would like to investigate if f can be written as a Fourier series

$$f(x) \stackrel{?}{=} \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}.$$

$$\int_0^1 e^{2\pi i(m-n)\theta}$$

$=1$ if $m=n$ and 0 otherwise.

If such a series exists, we see that proceeding formally

$$\int_0^1 \left(\sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x} \right) e^{-2\pi i m x}\, dx = a_m.$$

In other words,

$$a_m = \int_0^1 f(x) e^{-2\pi i m x}\, dx.$$

These numbers are called the Fourier Coefficients of f.

Complex analysis and Fourier series

- Writing our complex number z in polar co-ordinates, we see that $z=re^{2\pi i\theta}$ so that now, our function can be thought of as a function of the real variable r and the angle θ.
- The power series for any analytic function then can be viewed as a Fourier series if we write $z=a+re^{2\pi i\theta}$ so that

$$f(a+re^{i\theta}) = \sum_{n=0}^{\infty} c_n r^n e^{in\theta} \qquad \sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{i\theta})|^2 \, d\theta.$$

This allows us to deduce an important theorem in complex analysis called Liouville's theorem: a bounded entire function is constant. Here is the proof. Suppose $|f(z)|<M$. Then,

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{i\theta})|^2 \, d\theta < M^2.$$

If we let $r \to \infty$, we get a contradiction unless $c_1 = c_2 = \cdots = 0$. Thus, f is constant. □

The Fundamental Theorem of Algebra

- The fundamental theorem of algebra, first proved by Gauss, is the theorem that any polynomial of degree n with complex coefficients has n roots over the complex number field.
- This should be seen as the culmination of the efforts of ancient mathematicians trying to understand roots of quadratic, cubic, quartic equations that we have already studied from earlier civilizations.
- We can use Liouville's theorem to prove this. If $P(z)$ is a polynomial of degree n, we see that $|P(z)|$ tends to infinity as $|z|$ tends to infinity.
- If $P(z)$ had no roots, then $1/P(z)$ is analytic everywhere and so entire. But as $|P(z)|$ tends to infinity as $|z|$ to infinity, we see $1/P(z)$ is a bounded entire function, and hence constant by Liouville's theorem, which is a contradiction.
- Therefore, any polynomial of degree n has a root.
- Now, by the division algorithm, we can factor out that root to obtain a polynomial of degree n-1 so we see an induction argument proves the theorem.
- Many mathematicians have objected to the use of analysis to prove a theorem in algebra. But they forget that analysis and algebra are human constructs that don't exist in "nature".

Chebycheff and Prime Numbers

- The name of Chebycheff is spelled in many ways in the literature and we adopt here the one that starts with a C!
- Pafnuti Chebycheff (1821-1894) was a Russian mathematician who worked in number theory and probability theory.
- His most notable contribution in number theory is the proof of Bertrand's postulate: there is always a prime number between n and 2n.
- Chebycheff proved this using very elementary arguments involving binomial coefficients.
- Though he didn't prove the prime number theorem conjectured by Gauss, he came very close and showed that there are constants A and B such that $Ax/\log x < \pi(x) < Bx/\log x$.
- He also showed that if $\pi(x)(\log x)/x$ has a limit as x tends to infinity, then this limit must be 1.

Chebycheff and Bertrand's postulate

- Let us give a brief description of Chebycheff's proof of Bertrand's postulate.
- His proof is based on three elementary observations:

1. $\frac{2^{2n}}{2n} \le \binom{2n}{n} \le 2^{2n}$: this comes from $(1+1)^{2n} = \sum_{m=0}^{2n} \binom{2n}{m}$.
2. $\binom{2n}{n}$ is not divisible by any primes $p > 2n$.
3. $\binom{2n}{n}$ is divisible by all primes $n < p \le 2n$.

Properties (2) and (1) of the middle binomial coefficient imply that

$$n^{\pi(2n)-\pi(n)} \le \prod_{n<p\le 2n} p \le \binom{2n}{n} \le 2^{2n}.$$

Taking the log gives $\pi(2n) - \pi(n) \le 2\log 2 \frac{n}{\log n}$. Using induction we now easily see that

$$\pi(2^k) \le 3 \cdot \frac{2^k}{k}.$$

The inductive argument and the upper bound

In fact, this is checked directly for $k \le 5$; for $k > 5$ we find

$$\pi(2^{k+1}) \le \pi(2^k) + \frac{2^{k+1}}{k} \le \frac{3 \cdot 2^k}{k} + \frac{2 \cdot 2^k}{k} = \frac{5 \cdot 2^k}{k} \le \frac{3 \cdot 2^{k+1}}{k+1}.$$

Now we exploit the fact that $f(x) = \frac{x}{\log x}$ is monotonely increasing for $x \ge e$. Thus if $4 \le 2^k < x \le 2^{k+1}$, then

$$\pi(x) \le \pi(2^{k+1}) \le 6\frac{2^k}{k+1} \le 6\log 2\frac{2^k}{\log 2^k} \le 6\log 2\frac{x}{\log x}.$$

Since $\pi(x) \le 6\log 2\frac{x}{\log x}$ for $x \le 4$, the proof is now complete.

- This proves the upper bound for $\pi(x)$.

The lower bound

- We write the unique factorization of n! as a product of primes p, with $v_p(n!)$ being the power of p that divides it. Here is the formal lemma due to Legendre:

Lemma *Let $v_p(n)$ denote the exponent of the maximal power of p dividing n. Then*

$$v_p(n!) = \sum_{m \geq 1} \left\lfloor \frac{n}{p^m} \right\rfloor.$$

Proof. Among the numbers $1, 2, \ldots, n$, exactly $\lfloor \frac{n}{p} \rfloor$ are multiples of p and thus contribute 1 to the exponent; moreover, exactly $\lfloor \frac{n}{p^2} \rfloor$ are multiples of p^2 and contribute another 1 to the exponent ... $\square$

Now put $N = \binom{2n}{n}$. By the lemma above we have

$$v_p(N) = \sum_{m \geq 1} \left(\left\lfloor \frac{2n}{p^m} \right\rfloor - 2 \left\lfloor \frac{n}{p^m} \right\rfloor \right).$$

For all $x \in \mathbb{R}$ we have $\lfloor 2x \rfloor - 2\lfloor x \rfloor \in \{0, 1\}$.

Proof. Write $x = \lfloor x \rfloor + \{x\}$. If the fractional part $\{x\} < \frac{1}{2}$, then $2x = 2\lfloor x \rfloor + \{2x\}$, hence $\lfloor 2x \rfloor - 2\lfloor x \rfloor = 0$. If $\{x\} \geq \frac{1}{2}$, then we get $\lfloor 2x \rfloor - 2\lfloor x \rfloor = 1$. $\square$

It is also clear that if $m > \frac{\log 2n}{\log p}$, then $\lfloor \frac{2n}{p^m} \rfloor - 2\lfloor \frac{n}{p^m} \rfloor = 0$. Thus we find $v_p(N) \leq \lfloor \frac{\log 2n}{\log p} \rfloor$, and now

$$\begin{aligned}
2n \log 2 - \log 2n &\leq \log \binom{2n}{n} && \text{because } \frac{2^{2n}}{2n} \leq \binom{2n}{n} \\
&\leq \sum_{p \leq 2n} \left\lfloor \frac{\log 2n}{\log p} \right\rfloor \log p && \text{because } N = \prod p^{v_p(N)} \\
&\leq \sum_{p \leq 2n} \log 2n && \text{because } \lfloor x \rfloor \leq x \\
&= \pi(2n) \log 2n
\end{aligned}$$

This yields the lower bound

$$\pi(2n) \geq \log 2 \frac{2n}{\log 2n} - 1.$$

Some fine tuning

- We can fine tune this to get a lower bound for π(x) valid for all x. We just showed:

$$\pi(2n) \geq \log 2 \frac{2n}{\log 2n} - 1.$$

$$\pi(x) \leq 6 \log 2 \frac{x}{\log x}$$

We claim that this implies

$$\pi(x) \geq \frac{\log 2}{2} \frac{x}{\log x}$$

for all $x \geq 2$. This inequality can be checked directly for $x \leq 16$, hence it is sufficient to prove it for $x > 16$. Pick an integer n with $16 \leq 2n < x \leq 2n + 2$. Then

$$\frac{2n}{\log 2n} - \frac{n+1}{\log 2n} = \frac{n-1}{\log 2n} \geq \frac{7}{4 \log 2} \geq \frac{1}{\log 2},$$

hence

$$\pi(x) \geq \pi(2n) \geq \log 2 \frac{2n}{\log 2n} - 1 \geq \frac{(n+1)\log 2}{\log(2n+2)} \geq \frac{\log 2}{2} \frac{x}{\log x}.$$

Putting both estimates together, we have shown that there are constants A and B such that Ax/log x < π(x) < Bx/log x, which is Chebycheff's theorem.
Chebycheff did some further fine tuning to show two things: (1) Bertrand's postulate that there is always a prime between n and 2n and (2) that if π(x)(log x)/x tends to a limit, then the limit must be 1. Both of these are "easy" given our current knowledge and so we give a quick indication of how these two results were derived by Chebycheff.

Prelude to Bertrand's postulate

- Chebycheff found that it is a bit easier if instead of looking at π(x), we look at a related function, $\theta(x) = \prod_{p \le x} p$. Then log θ(x) weights each prime by log p instead of by the weight 1. This seemingly minor change simplifies our work.

$\Theta(x) \le 4^x$.

Proof. It is clearly sufficient to prove this for integers $x \ge 4$. Observe that $\binom{2m+1}{m} = \binom{2m+1}{m+1} < 2^{2m} = 4^m$. This gives

$$\prod_{m+2 \le p \le 2m+1} p \mid \binom{2m+1}{m} < 4^m.$$

Now we prove the claim by induction; assume it is true for all $n < k$. If k is even, then $\Theta(k) = \Theta(k-1) < 4^{k-1} < 4^k$ by induction assumption and the fact that k is not prime. Assume therefore that $k = 2m+1$. Then

$$\Theta(k) = \Theta(m+1) \prod_{m+2 \le p \le 2m+1} p < 4^{m+1} \cdot 4^m = 4^k.$$

We have already seen that $N = \binom{2n}{n}$ is divisible by all primes p with $n < p \le 2n$ (if there are any). Now we claim that the primes p with $\frac{2}{3}n < p \le n$ do not divide N. In fact we have $2n < 3p \le p^2$, hence $2 \le \frac{2n}{p} < 3$, hence $\lceil \frac{2n}{p} \rceil = 2$ and $\lceil \frac{n}{p} \rceil = 1$, and this implies that $v_p(N) = 2 - 2 = 0$.

Proof of Bertrand's postulate: there is always a prime p between n and 2n

Now we prove Bertrand's postulate by contradiction. Assume there is an integer n such that the interval $(n, 2n]$ does not contain any prime. By the discussion above this implies that $N = \binom{2n}{n}$ is not divisible by any prime $p > \frac{2}{3}n$.

Now consider primes $p \mid N$ with $v_p(N) > 1$. They satisfy $p^2 \le p^{v_p} \le 2n$, hence we must have $p \le \sqrt{2n}$ for such primes. The number of such primes is clearly bounded by $\sqrt{2n}$. Now we find

$v_p(N) \le \lfloor \frac{\log 2n}{\log p} \rfloor$;

$$\frac{2^{2n}}{2n} \le \binom{2n}{n} \le \prod_{v_p > 1} p^{v_p} \cdot \prod_{v_p = 1} p \le (2n)^{\sqrt{2n}} \cdot \Theta\left(\frac{2n}{3}\right) \le (2n)^{\sqrt{2n}} 2^{4n/3}.$$

- Here we are using our earlier bound for θ(x).

$$\Theta(x) \le 4^x.$$

Taking the log we get

$$2n \log 2 \le 3(1 + \sqrt{2n}\,) \log 2n.$$

Since $\frac{x}{\log x}$ is monotonically increasing for $x > 3$, this inequality must be false for all sufficiently large values of n. In fact, it is false for $n \ge 512$. For $n < 512$, Bertrand's postulate is proved by looking at the sequence of primes 7, 13, 23, 43, 83, 163, 317, 631.

Stirling's formula and Chebycheff

- Chebycheff noticed that Stirling's formula implies that if $\pi(x) \sim Cx/\log x$, then $C=1$.
- Define the function $\Lambda(n)$ as $\log p$ if n is a power of a prime p and zero otherwise.
- From the unique factorization of natural numbers as product of distinct prime powers, it is easy to see that $\log k = \Sigma_{d|k} \Lambda(d)$. (Exercise)
- We sum both sides of this equation for $k \leq n$. The left hand side is $\log n!$ which by Stirling's formula is $n\log n - n + O(\log n)$.
- The right hand side becomes $\Sigma_{d \leq n} \Lambda(d)[n/d] = n\Sigma_{d \leq n} \Lambda(d)/d + O(n)$, using our bound for the prime counting function.
- Dividing through by n gives us that $\Sigma_{d \leq n} \Lambda(d)/d = \log n + O(1)$.
- Now let $\psi(n) = \Sigma_{d \leq n} \Lambda(d)$. A simple partial summation argument shows that $\pi(n) \sim n/\log n$ is equivalent to $\psi(n) \sim n$. If $\psi(n) \sim Cn$ for some constant C then we would get $\Sigma_{d \leq n} \Lambda(d)/d = C\log n + O(1)$. Hence $C=1$.
- So the difficult part in the proof of the prime number theorem is to show the limit exists. A way to address this was discovered by Riemann which we discuss in a later lecture.