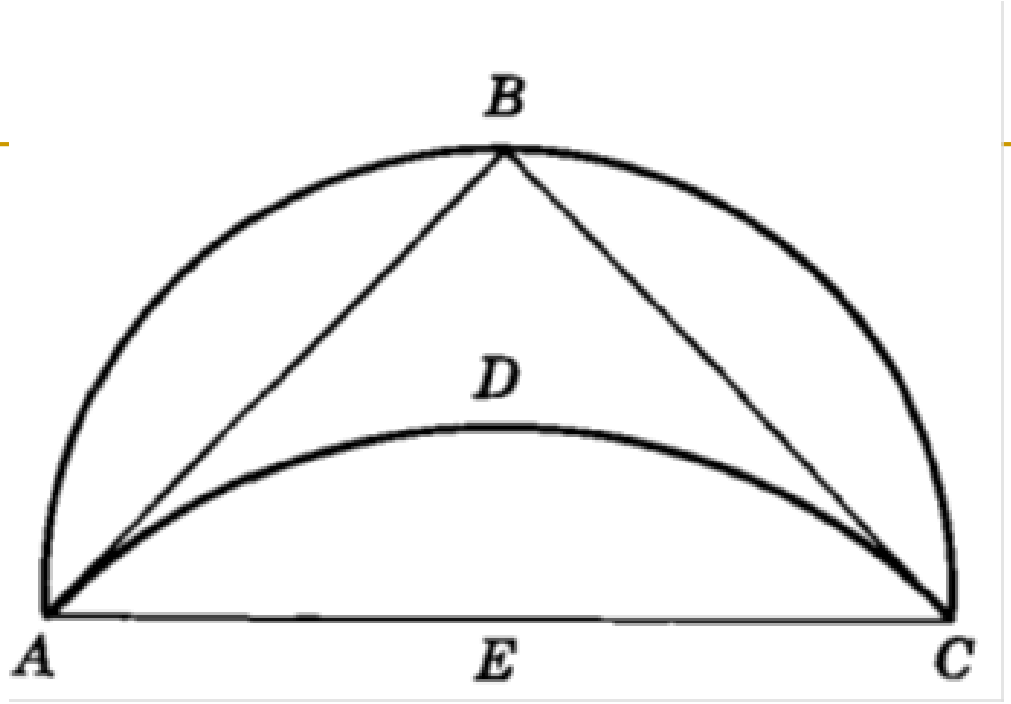
Three problems of antiquity and quadrature of the lune



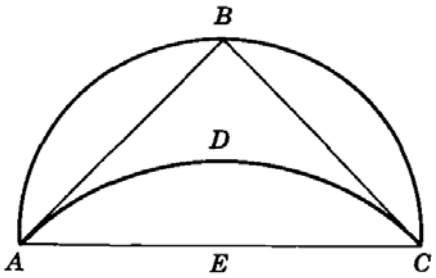
Anaxagoras

- Around 400 BCE, one can discern the presence of a group of scholars who began to have what we may call a scientific world view.
- Anaxagoras was imprisoned in Athens for asserting that the sun was not a deity, but a huge red-hot stone and the moon borrowed its light from the sun.
- His book, On Nature, was the first scientific best-seller.
- In prison, he worked on the problem of squaring the circle.

Three famous problems of antiquity

- 1. Squaring the circle: construct a square whose area is equal to the area of a circle of radius one unit length.
- 2. Duplication of the cube: construct a cube whose volume is equal to 2.
- 3. The trisection of the angle: given any angle, using straight edge and compass, trisect the angle.
- All three problems were solved only 2200 years later.

Quadrature of the Lunes

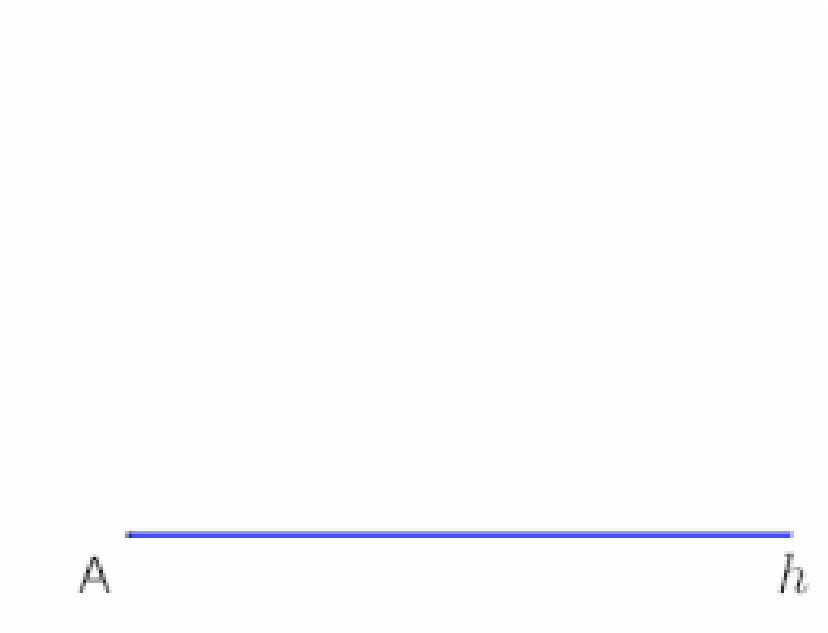


- Hippocrates seems to have written his "Elements of Geometry" a century before Euclid.
- In this work, Hippocrates considers the problem of quadrature of the lune, which is a planar region bounded by two circular arcs, which looks like a crescent, and hence the name.
- The quadrature of a plane figure means to construct using only a compass and straightedge, a square whose area equals that of the plane figure.

Three ingredients of proof

- The Pythagorean theorem
- An angle inscribed in a semi-circle is a right angle.
- The ratio of areas of two circles of radius r and R respectively is r^2/R^2.
- The second assertion was known to Thales and its proof is simple, as we have already seen.
- The first assertion seems to have been common knowledge from the Pythagorean school.
- The last assertion was proved later in Euclid's Elements and it is doubtful that Hippocrates had a rigorous proof.

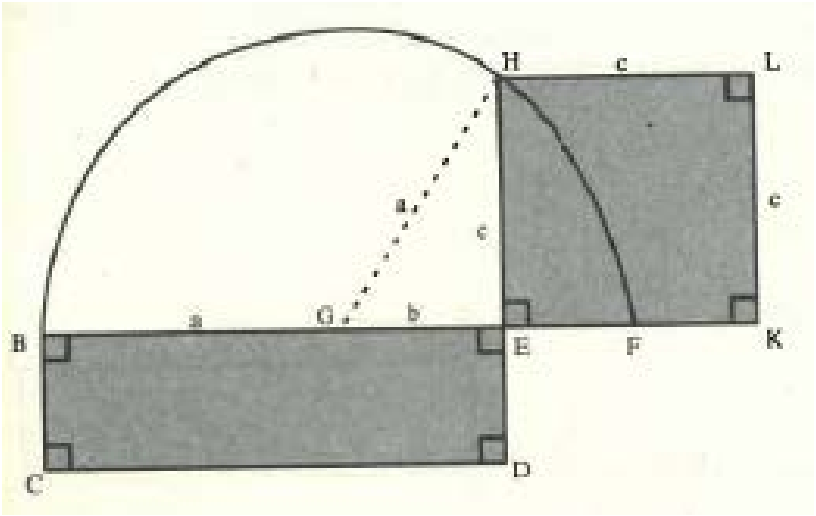
Drawing a perpendicular



Perpendicular bisector

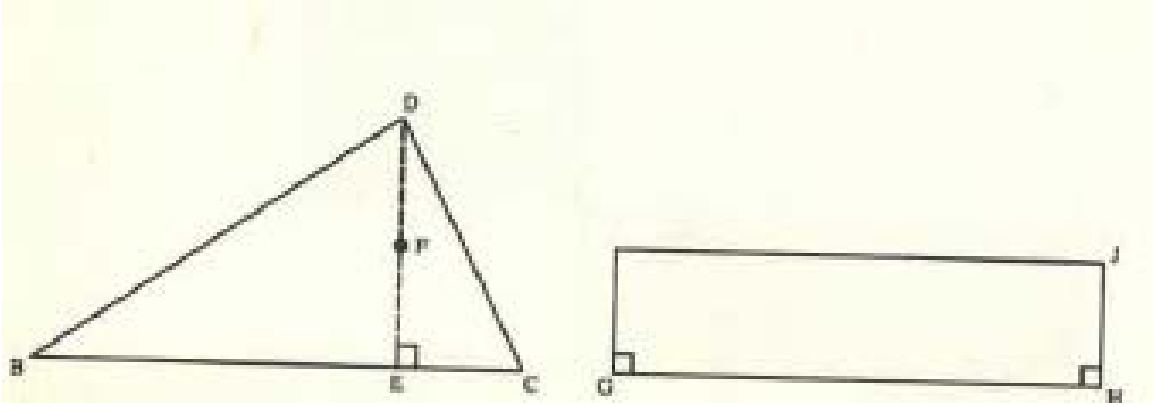
Squaring the rectangle

- 1. Extend BE to the right and mark off EF=ED.
- 2. Bisect BF at G and draw a semi-circle centered at G of radius BG.
- 3. Extend ED to meet the circle at H.
- 4. The square of length EH has area equal to BCDE which is (a-b)(a+b).



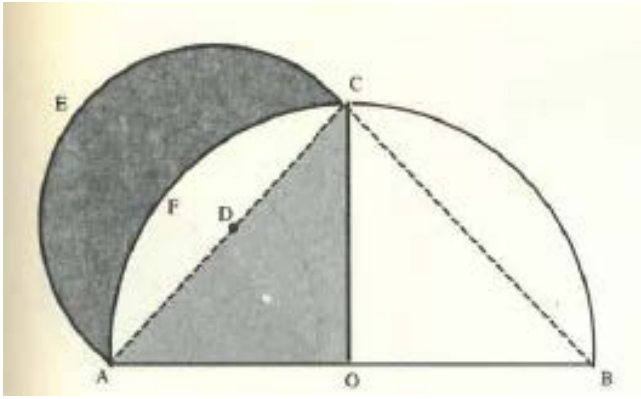
Squaring the triangle

- Drop the perpendicular from D to BC intersecting at E.
- Bisect DE at F and construct a square of length BC and height EF.



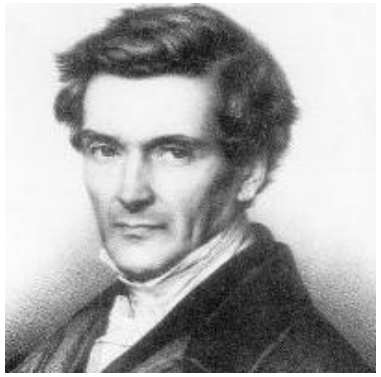
Hippocrates quadrature of the lune

- 1. The angle ACB is a right angle.
- 2. Triangles AOC and BOC are congruent so AC=BC.
- 3. By Pythagoras, $(AB)^2=2(AC)^2$.
- 4. Area of semi-circle AEC: Area of semi-circle ACB = ½.
- 5. Area of semi-circle AEC=Area of quadrant AFCO.
- 6. Area of the lune = area of quadrant AFCO – Area (AFCD)= area of triangle AOC.



The relative difficulty of the three problems of antiquity

- 1. Doubling a cube using straightedge and compass is impossible. (Wantzel, 1837)
- 2. It is impossible to trisect any given angle using straightedge and compass. (Wantzel, 1837)
- 3. It is impossible to square the circle. (Lindemann, 1882).
- ALL theorems require a knowledge of algebraic numbers and the last is essentially the statement that π is transcendental.



Pierre Wantzel 1814-1848)

F. Lindemann (1852-1939)

Zeno and infinity

- The nature of infinity was poorly understood in ancient times.
- This is root of Zeno's paradox.
- The paradox argues there is no motion as follows.
- Before an object can travel a given distance, it must first travel half this distance; but before it can cover this, it must travel the first quarter, and so on, through an infinite number of subdivisions.
- How can there be an infinite number of events in a finite amount of time? Hence there is no motion!

The dawn of deductive reasoning

- This early Greek period of mathematics shows an emergence of the deductive method of reasoning.
- The negative feature of Greek thought was its insistence of the use of whole numbers or rational numbers along with its geometric approach.
- As we will see later, the algebraic approach emerges out of India and Arabia at a later date.
- However, one person, Democritus, stands out from this period for his atomic theory. But this theory seems to have been prevalent in other cultures also.