Euclid of Alexandria

PROPOSIT. XLVII.

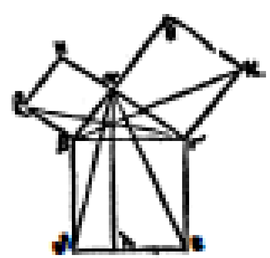
Theorema.

ΕΝ τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ, τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν τετραγώνοις.

In triangulis rectangulis: quadratum lateris angulum rectum subtendentis, est æquale quadratis laterum, rectum angulum continentium.

ἡ ἔκθεσις.

Sit triangulus rectangulus αβγ, habens an-



gulum βαγ rectum. ὁ διορισμός. Dico quod quadratum lateris βγ, est æquale quadratis laterum βα, αγ. ἡ κατασκευή. Describatur à linea βγ, quadratum βδεγ. Item à linea βα quadratum βη. Præterea à linea αγ quadratum γθ. Ducatur per punctum α, alterutri linearum βδ, γε, æquedistans recta linea αλ. Ducantur duæ lineæ rectæ αδ, ζγ.

Proposition I 47 from 1566 edition of the *Elements* (photograph courtesy of The Ohio State University Libraries)

The Elements

- After the death of Alexander “the Great” , the Egyptian portion of the Greek empire was ruled by Ptolemy I who established a research institute called “Museum” at the port city of Alexandria.
- There he assembled a group of scholars, the foremost among them was Euclid, who wrote the book called, “The Elements” a compendium of the essential theorems of geometry and number theory known at that time.

Who was Euclid?

- Not much is known about Euclid and legends describe him as a kind and gentle old man who when asked by Ptolemy if one can learn geometry quickly, replied "there is no royal road to geometry."
- The Elements consists of 13 books or chapters.
- The first six are on geometry, the next three on number theory, the tenth on irrationals, and the last three on solid geometry.
- What emerges from the writing is an axiomatic treatment of mathematics, especially with reference to geometry.

The First Book

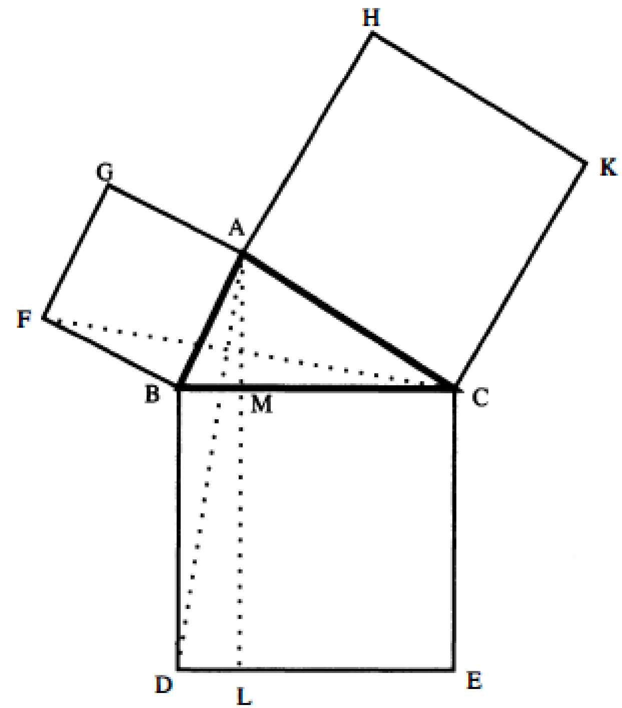
- The First Book of Euclid begins abruptly with axioms. Here is a sample:

- **Definition 1** A *point* is that which has no part.
- **Definition 2** A *line* is breadthless length.
- **Definition 4** A *straight line* is a line which lies evenly with the points on itself.

Though we may find these definitions somewhat circular and inadequate, the greatness of the work emanates from the recognition that geometry and mathematics should be formulated axiomatically.

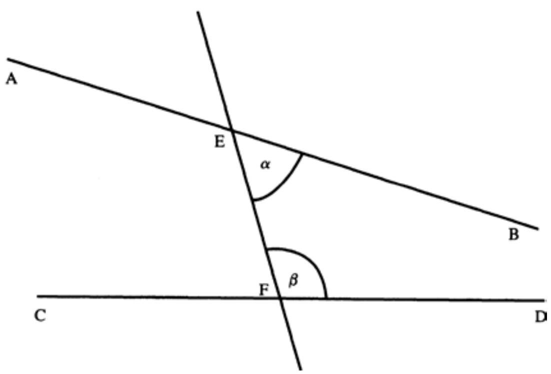
Euclid’s proof of the Pythagorean theorem

- BAC is a right angle.
- CAG is a straight line.
- Triangles ABD and FBC have the same areas.
- This is the same area as triangle BDL.
- Therefore the area of rectangle BMLD is the area of the square ABFG.
- By symmetry the area of the rectangle MCEL is the area of the square ACKH.



The postulates of geometry

- 1. Any two points can be joined by a straight line.
- 2. A straight line segment can be extended as a straight line.
- 3. Given a point C and length r, we can draw a circle with center C and radius r.
- 4. All right angles are equal.
- 5. If two straight lines are not parallel, then they meet at a point.



The parallel postulate

- The fifth postulate, often called the parallel postulate, was presumed to be a theorem and not an axiom.
- The fact that it cannot be deduced from the other four postulates is a theorem of the 19th century.
- Gauss, Bolyai and Lobachevsky, independently constructed valid geometries in which the first four axioms were valid but not the fifth.
- Such geometries are called non-Euclidean geometries.

Euclid and number theory

- Boox VII discusses the famous division algorithm: given two natural numbers a and b, we can find a quotient q and a remainder r such that $a=bq+r$ with $r=0$ or $r<b$.
- This algorithm leads to the notion of the greatest common divisor of a and b, denoted gcd(a,b).
- Boox IX introduces prime numbers and shows there are infinitely many.
- The proof is by contradiction.
- Euclid however did not state the unique factorization theorem.

Geometric progressions and perfect numbers

- Proposition 35 of Boox IX evaluates the sum $1 + r + r^2 + \ldots + r^{n-1} = (1-r^n)/(1-r)$.
- A number n is called perfect if the sum of all its divisors is 2n.
- For example, 6 is a perfect number since $1+2+3+6=12$.
- Euclid noticed that any number of the form $2^{p-1}(2^p-1)$ with 2^p-1 prime, is a perfect number.

Mersenne primes and perfect numbers

- One can prove that any even perfect number must have the form described by Euclid.
- This leads to the question of whether there are infinitely many perfect numbers. This is unknown.
- It is believed there are infinitely many primes of the form 2^p-1 and these are called Mersenne primes, after Father Mersenne, who was a Jesuit minister of the 17th century.
- It is conjectured that there are no odd perfect numbers.

Even Perfect Numbers

Let n be an even perfect number.

Write $n = 2^{\alpha-1} m$ with m odd

Write $\sigma(n) = \sum_{d|n} d$. Then

$\sigma(n) = \sigma(2^{\alpha-1} m) = (2^{\alpha}-1)\sigma(m)$

If n is perfect, $\sigma(n) = 2n$.

Thus $2^{\alpha} m = (2^{\alpha}-1)\sigma(m)$

$\Rightarrow (2^{\alpha}-1) \mid m$. Write $m = (2^{\alpha}-1) m_0$.

Then $2^{\alpha}(2^{\alpha}-1) m_0 = (2^{\alpha}-1)\sigma(m)$

$\Rightarrow \sigma(m) = 2^{\alpha} m_0$.

But m_0 and $(2^{\alpha}-1) m_0$ both divide m.

Therefore $\sigma(m) \geq m_0 + (2^{\alpha}-1) m_0 = 2^{\alpha} m_0$.

But 1 is also a divisor of m so we deduce

$m_0 = 1$. $\Rightarrow m = 2^{\alpha}-1$.

Even perfect numbers

Let n be an even perfect number.

Write $n = 2^{\alpha-1} m$ with m odd

Write $\sigma(n) = \sum_{d|n} d$. Then

$\sigma(n) = \sigma(2^{\alpha-1} m) = (2^\alpha - 1)\sigma(m)$

If n is perfect, $\sigma(n) = 2n$.

Thus $2^\alpha m = (2^\alpha - 1)\sigma(m)$

$\Rightarrow (2^\alpha - 1) \mid m$. Write $m = (2^\alpha - 1) m_0$.

Then $2^\alpha (2^\alpha - 1) m_0 = (2^\alpha - 1)\sigma(m)$

$\Rightarrow \sigma(m) = 2^\alpha m_0$.

But m_0 and $(2^\alpha - 1) m_0$ both divide m.

Therefore $\sigma(m) \geq m_0 + (2^\alpha - 1) m_0 = 2^\alpha m_0$.

But 1 is also a divisor of m so we deduce

$m_0 = 1$. $\Rightarrow m = 2^\alpha - 1$.

Theorem. Any even perfect number has the form $2^{\alpha-1}(2^\alpha - 1)$.

The influence of the Elements

- Written in 300 BCE, the "Elements" has been considered the most influential scientific treatise of all time.
- It was handed down by being copied from generation to generation. Inevitably errors crept in.
- The first printed version appeared in Venice in 1482.
- Most of the books were preserved by Arab scholars and translated into Arabic and that is how we have them today.