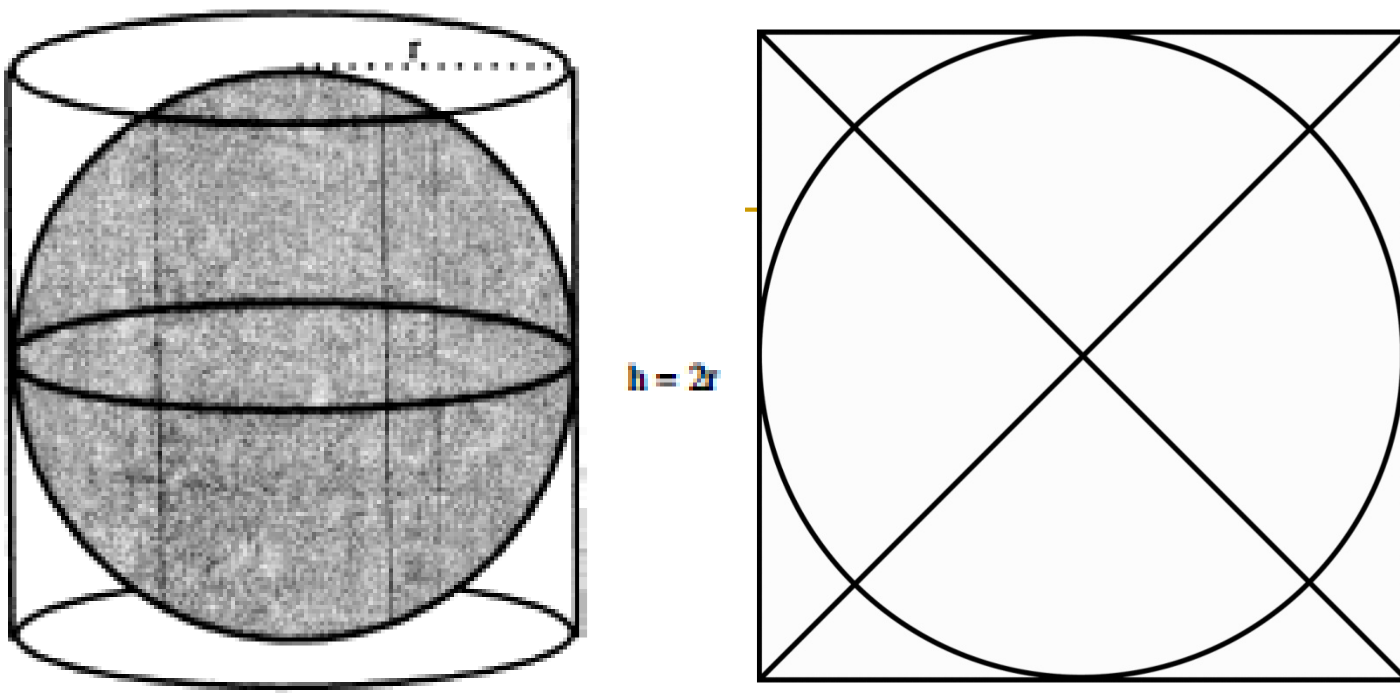
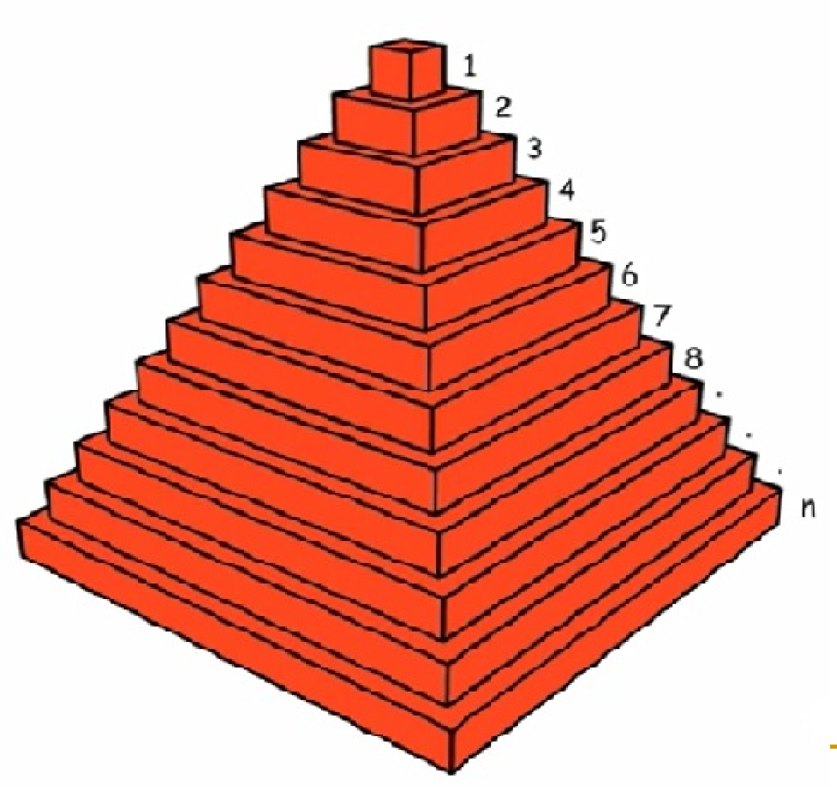
Archimedes and the Sphere



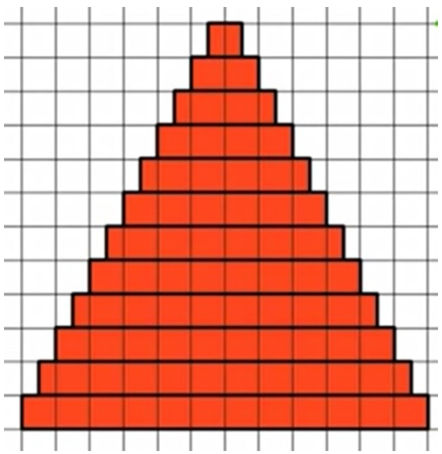
Volume of a pyramid

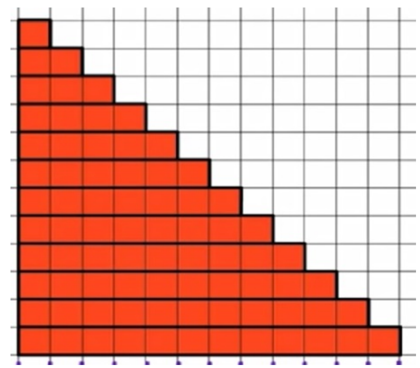
- Volume of a pyramid is easily calculated by decomposing it into rectangular slabs.



The cross-sectional method

- The calculation is easy if we look at the cross section.

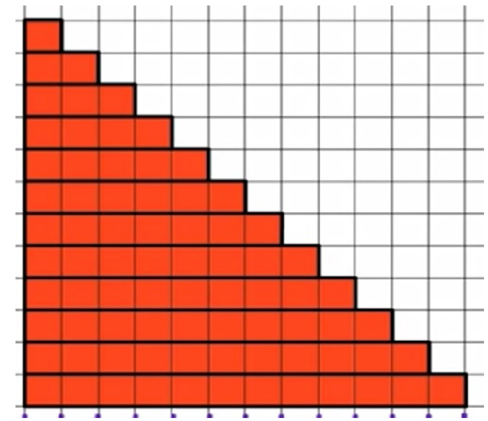




Suppose that each slab has height 1 and the base is a square.
The total volume is the sum of the first n squares.

$$\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right) = \left(\frac{n(n+1)(2n+1)}{6}\right)$$

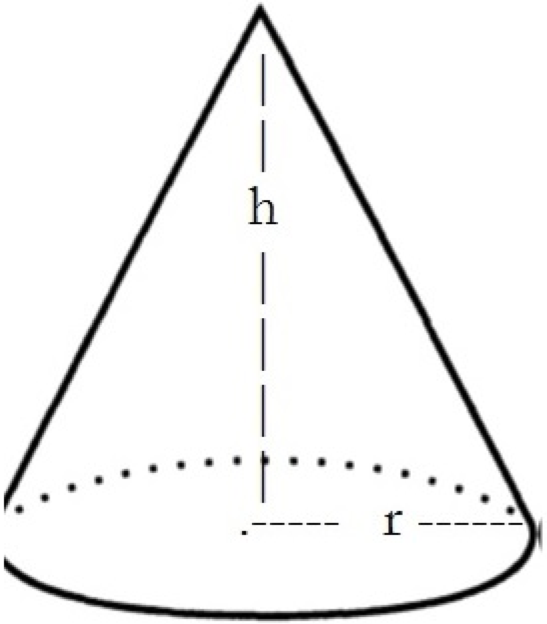
V=(Area of base)h/3



- We subdivide the base b into n parts and the height h into n parts.
- At the k-th level, the volume is $(h/n)(kb/n)^2$.
- We sum this with k=1 to n and use the sum of the squares formula and take the limit as n tends to infinity.
- Thus, $V=hb^2/3$.

Volume of a cone

- The same method can be used to calculate the volume of a cone with height h and base circle of radius r.
- $V = (1/3)h\pi r^2$.

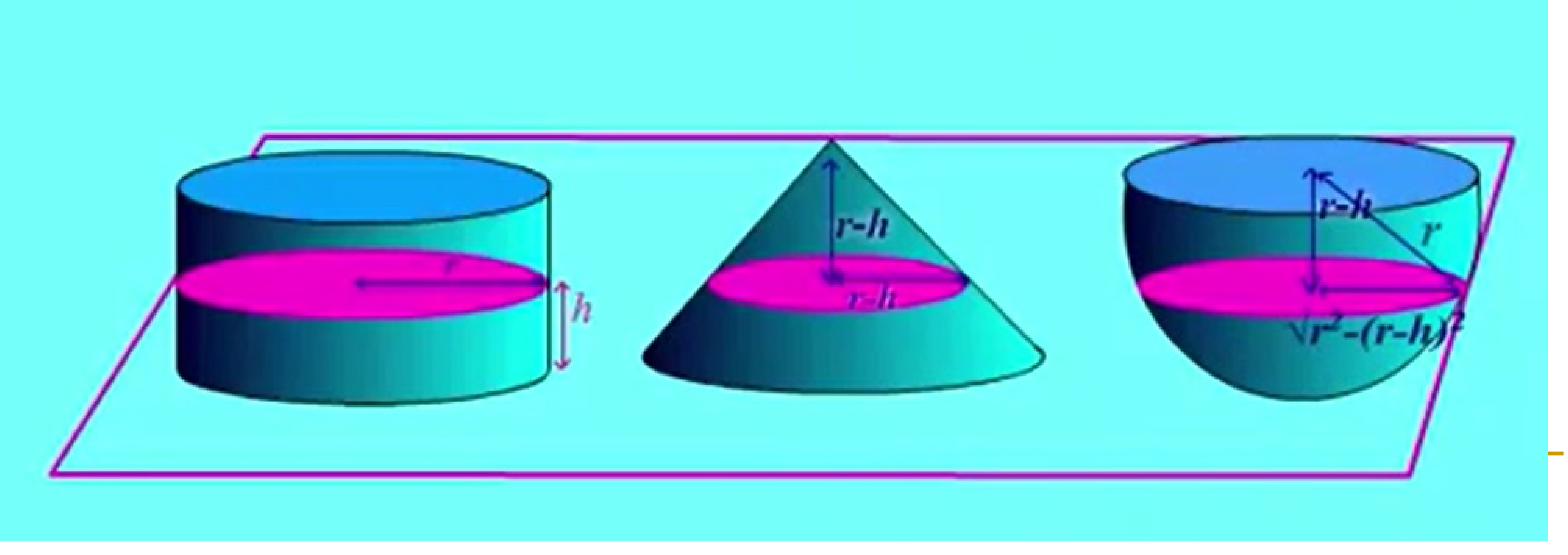


The volume of the sphere

- Archimedes extended his method relating the area and circumference of a circle to relate the volume and surface area of a sphere.
- The idea is to visualize the surface of the sphere as being divided into squares by the longitudinal and latitudinal lines.
- These squares, when joined to the center form a pyramid whose volume is (area of square)r/3.
- Volume of sphere = (r/3)(Surface area of sphere)

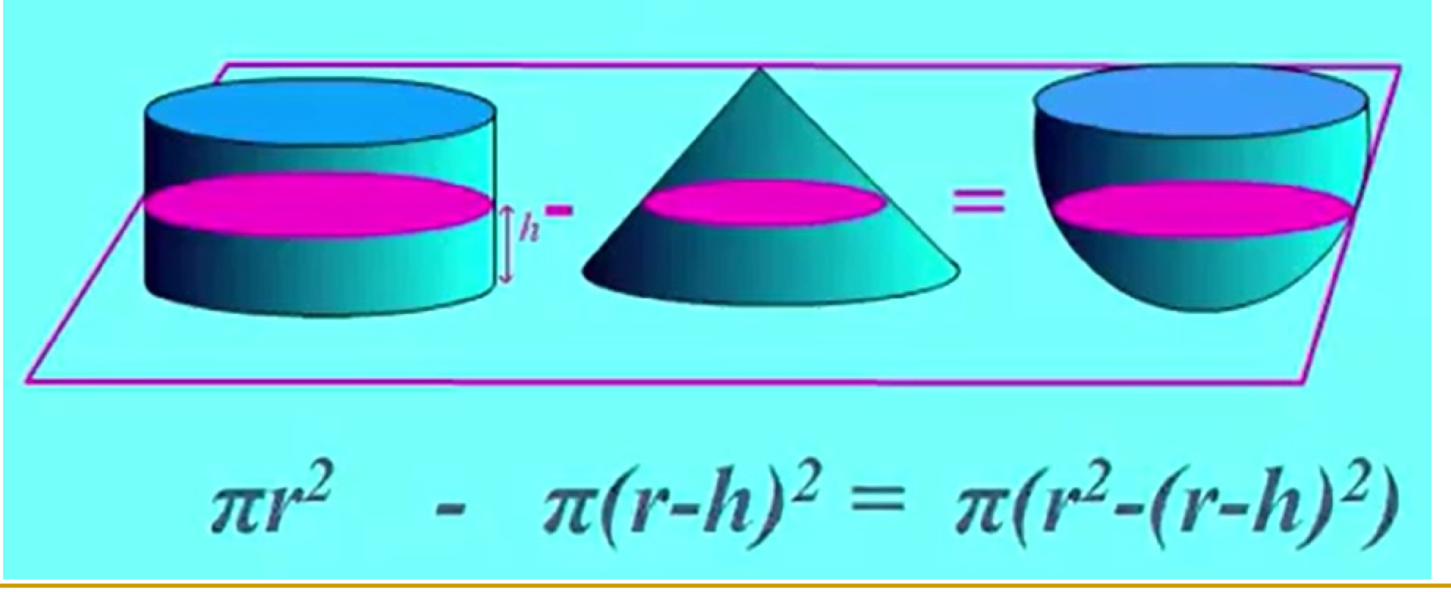
The cross-sectional method

- Archimedes used his cross-sectional method to calculate the volume of a sphere as follows.
- He lined up a cylinder, a cone and a hemisphere as indicated below.



Cross-sectional areas

- By examining the cross sectional areas, he deduced that the volume of the cylinder = volume of the cone + volume of the hemi-sphere.

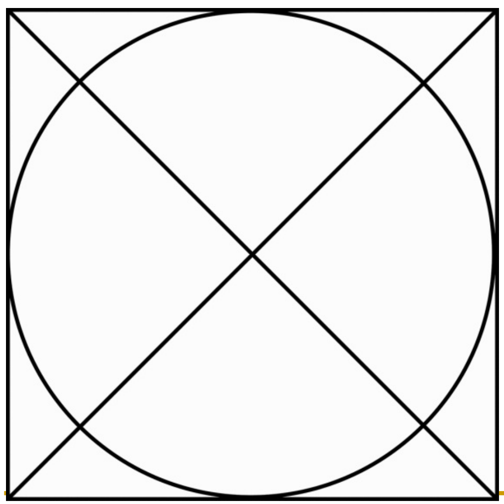


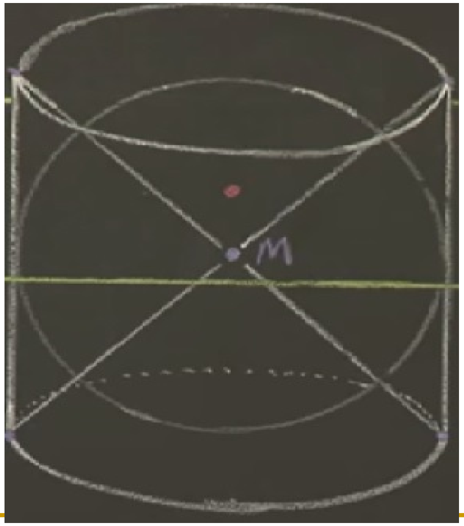
Volume of a sphere

- The volume of a cylinder of height r and radius of the base circle r is πr^3.
- The volume of a cone of height r and radius of the base circle r is $(1/3)\pi r^3$.
- Thus the volume of the hemisphere is $(2/3)\pi r^3$.
- Therefore the volume of the sphere is $(4/3)\pi r^3$.
- Corollary: the surface area of a sphere of radius r is $4\pi r^2$.

Volumes of revolution

- How would Archimedes get the idea of these relationships between the sphere, cone and cylinder?
- If you rotate the planar figure below about the vertical axis through M, you get the three solids of revolution.





Archimedes was so proud of this discovery that he wanted this figure to be carved on his tombstone.

The sine and the cosine

- Though he did not formally define the sine and the cosine functions, Archimedes derived geometrically many statements that in modern language we recognize as “the integral of the sin x is –cos x.”
- Here is how Archimedes wrote this:

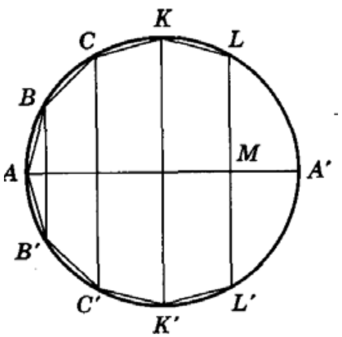
If a polygon be inscribed in a segment of a circle LAL' so that all its sides excluding the base are equal and their number even, as $LK \ldots A \ldots K'L'$, A being the middle point of the segment; and if the lines BB', CC', ... parallel to the base LL' and joining pairs of angular points be drawn, then $(BB' + CC' + \cdots + LM):AM = A'B:BA$, where M is the middle point of LL' and AA' is the diameter through M (Fig. 8.6).

The trigonometric equation

This is the geometrical equivalent of the trigonometric equation

$$\sin\frac{\theta}{n} + \sin\frac{2\theta}{n} + \cdots + \sin\frac{n-1}{n}\theta + \tfrac{1}{2}\sin\frac{n\theta}{n} = \frac{1-\cos\theta}{2}\cot\frac{\theta}{2n}$$

From this theorem it is easy to derive the modern expression $\int_0^\phi \sin x\,dx = 1 - \cos\phi$ by multiplying both sides of the equation above by θ/n and taking limits as n increases indefinitely. The left-hand side becomes



$$\lim_{n\to\infty}\sum_{i=1}^{n}\sin x_i \Delta x_i$$

where $x_i = i\theta/n$ for $i = 1, 2, \ldots n$, $\Delta x_i = \theta/n$ for $i = 1, 2, \ldots n-1$, and $\Delta x_n = \theta/2n$. The right-hand side becomes

$$(1-\cos\theta)\lim_{n\to\infty}\frac{\theta}{2n}\cot\frac{\theta}{2n} = 1 - \cos\theta.$$

Approximations to π

- In his work, "Measurement of the Circle", Archimedes devoted some effort to approximate the value of π.
- By inscribing a regular 360-gon in a circle of diameter 1, he computed π=3.1416 which is correct to three decimal places.
- π is a transcendental number and so, its decimal expansion does not terminate.
- Approximation to π has been a recurring theme of mathematics since ancient times.