Ramanujan's Proof of Bertrand's Postulate
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Theorem 3. *Let* $f = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ *be such that* $a_0 \geq a_1 \geq \cdots \geq a_n > 0$ *and* $\gamma(f) = 1$. *Let* ν *be the number of prime divisors counted with their multiplicities of* a_0. *Then for any* $s \geq 1$ *we have that* $f(x^s)$ *is the product of at most* ν *nonunit polynomials in* $\mathbb{Z}[x]$.

Proof. By the first lemma, any root θ satisfies $\theta = 1$ or $|\theta| > 1$. Since $f(1) = a_0 + a_1 + \cdots + a_n > 0$, every root θ of f satisfies $|\theta| > 1$. Let $s \geq 1$ and let α be a root of $f(x^s)$. Thus α^s is a root of f, so $|\alpha^s| > 1$ and hence $|\alpha| > 1$. Now our result follows from the second lemma. ■

We conclude with the following corollary.

Corollary 4. *Let* $f = p + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$, *where* p *is a prime and* $p \geq a_1 \geq \cdots \geq a_n \geq 1$. *The following statements are equivalent:*

1. $f = p + a_1 x + \cdots + a_n x^n$ *is irreducible in* $\mathbb{Z}[x]$;
2. *for any* $s \geq 1$, $f(x^s) = p + a_1 x^s + \cdots + a_n x^{ns}$ *is irreducible in* $\mathbb{Z}[x]$;
3. *the list* $(p, a_1, \ldots, a_n)$ *does not consist of* $(n+1)/d$ *consecutive constant lists of length* $d > 1$; *and*
4. $\gamma(f) = 1$.

Proof. (3) and (4) are plainly equivalent. (4) ⇒ (2) is a consequence of our theorem. (2) ⇒ (1) is a fortiori. We noted earlier that if $d = \gamma(f) > 1$, then f would factor as

$$f = (x^{d-1} + \ldots + 1)(p + \cdots + b_t x^{td}).$$

Since p is a prime, this would give a nontrivial factorization of f. Hence (1) ⇒ (4). ■

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Jaban Meher and M. Ram Murty

Abstract. We present Ramanujan's proof of Bertrand's postulate and in the process, eliminate his use of Stirling's formula. The revised proof is elegant and elementary so as to be accessible to a wider audience.

1. INTRODUCTION. In 1845, Joseph Bertrand conjectured that between x and $2x$, there is always a prime number for every $x > 1$. Chebyshev proved this in 1850, and his proof is often presented in introductory courses after deriving some standard tools of analytic number theory. An excellent historical account can be found in [**2**]. A proof

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by Erdős, which is calculus-free, is given in the celebrated "Proofs from The Book" [**1**]. In 1919, Ramanujan [**3**] gave a short and elegant proof of Bertrand's postulate, which uses Stirling's formula. We are unable to find a calculus-free derivation of Stirling's formula. The purpose of this note is to eliminate the use of Stirling's formula from his proof. The revised proof now is so elegant that it qualifies to be included in "Proofs from The Book". We hope that our presentation and arrangement makes Ramanujan's proof more widely known and accessible to a larger community.

We replace Ramanujan's use of Stirling's formula with the following lemmas.

Lemma 1. *For* $x > 1$, *let* $R(x) = [x]!/[x/2]!^2$. *Then*

$$\frac{2^{x-1}}{x+1} \le R(x) \le 2^{x-1}(x+1).$$

Proof. If $[x] = 2k$ is even, then

$$R(x) = \binom{2k}{k}$$

is the largest binomial coefficient in the expansion of $(1+1)^{2k}$. So

$$\frac{2^{2k}}{2k+1} \le R(x) \le 2^{2k}, \tag{1}$$

from which the stated inequality is immediate. If $[x] = 2k+1$ is odd, then

$$R(x) = \binom{2k+1}{k}(k+1).$$

Now,

$$2\binom{2k+1}{k} = \binom{2k+1}{k} + \binom{2k+1}{k+1} \le 2^{2k+1},$$

so that

$$2^{2k} \le R(x) \le 2^{2k}(k+1),$$

and the result is now immediate. ■

As pointed out by the referee, a result similar to Lemma 1 can be found in [**1**].

Lemma 2. $R(x) \le 6^{x/2}$ *for all* $x \ge 1$.

Proof. If $[x]$ is even, the result is clear from (1). If $[x] = 2k+1$ is odd, we need only to check that

$$2^x(1+k)/2 \le 2^x\left(1+\frac{1}{2}\right)^k < 6^{x/2}.$$ ■

The von Mangoldt function Λ is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some } k \geq 1,\ p \text{ prime} \\ 0 & \text{otherwise.} \end{cases}$$

The unique factorization property of the natural numbers implies

$$\log n = \sum_{d|n} \Lambda(d).$$

Let θ be the function, defined by

$$\theta(x) = \sum_{p \leq x} \log p, \text{ where the sum is over primes } p \leq x.$$

To prove Bertrand's postulate, it suffices to show that $\theta(x) - \theta(x/2) > 0$, for any $x \geq 2$.

Theorem 3. *For $x > 1$, there is at least one prime between x and $2x$.*

Proof. As in Ramanujan [**3**], we have

$$\log\,[x]! = \sum_{n \leq x} \log n = \sum_{de \leq x} \Lambda(d) = \sum_{e \leq x} \psi(x/e), \text{ where } \psi(x) = \sum_{n \leq x} \Lambda(n). \tag{2}$$

The above equation implies that

$$\log\,[x]! - 2\log\,[x/2]! = \psi(x) - \psi(x/2) + \psi(x/3) - \psi(x/4) + \cdots .$$

Since the right-hand side is an alternating series of a decreasing function, we deduce, using the notation of the lemma,

$$\psi(x) - \psi(x/2) \leq \log R(x) \leq \psi(x) - \psi(x/2) + \psi(x/3), \tag{3}$$

which implies, by Lemma 2,

$$\psi(x) - \psi(x/2) \leq \frac{x}{2} \log 6. \tag{4}$$

Changing x to $x/2, x/4, x/8, \ldots$ in the above equation and adding up all the inequalities, we get

$$\psi(x) < x \log 6. \tag{5}$$

Then using (3) and Lemma 1, we obtain

$$(x-1)\log 2 - \log\,(x+1) \leq \psi(x) - \psi(x/2) + \psi(x/3).$$

Using (5) in the above inequality, we get

$$\psi(x) - \psi(x/2) \geq (x/3)\log\,(4/3) - \log 2(x+1). \tag{6}$$

It is easy to see that the relation between the functions ψ and θ is given by

$$\psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \cdots, \tag{7}$$

so that

$$\psi(x) - 2\psi(x^{1/2}) = \theta(x) - \theta(x^{1/2}) + \theta(x^{1/3}) - \theta(x^{1/4}) + \cdots.$$

Since the right-hand side is an alternating series of a decreasing function, we deduce as before,

$$\psi(x) < \theta(x) + 2\psi(x^{1/2}).$$

Now using (5) and the fact that $\theta(x) \le \psi(x)$, we get

$$\psi(x) - \psi(x/2) \le \theta(x) + 2\psi(x^{1/2}) - \theta(x/2) < \theta(x) - \theta(x/2) + 2\sqrt{x}\log 6.$$

Using (6) we get, with $A = \frac{1}{3}\log(4/3)$, $B = -2\log 6$, and $C = -\log 2$, that

$$\theta(x) - \theta(x/2) > (Ax + B\sqrt{x} + C) - \log(x+1).$$

We can write $(Ax + B\sqrt{x} + C) = A(\sqrt{x} + a)(\sqrt{x} - b)$ with a and b positive and $b \doteq 37.562$. Thus, for $\sqrt{x} > b + 1/A \doteq 47.98$, we need only to check that $e^{\sqrt{x}} > 1 + x$. But this is evidently the case for $x > 36$, since $e^{\sqrt{x}} > 1 + \sqrt{x} + x/2 + x^{3/2}/6$. This establishes the result for $x > 1151$. For smaller values of x, we need only to observe that

2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1259

is a sequence of primes in which each member is less than twice its predecessor. From this, Bertrand's postulate is easily verified for $x \le 1151$. This completes the proof. ■

Ramanujan uses Stirling's formula to show that $R(x) < e^{3x/4}$ for all $x \ge 1$, and for $x > 300$, $R(x) > e^{2x/3}$. Using basic calculus, we can show that $R(x) < e^{.93x}$ for all $x \ge 1$ and for $x \ge 450$, $R(x) > e^{.69x}$, and this leads to a more streamlined proof more in line with Ramanujan's proof. Our approach above was motivated by the desire to show that Ramanujan's method leads to a calculus-free "bare hands" derivation of the result. We also remark that his proof gives Chebyshev-type upper and lower bounds of the right order for the functions $\psi(x)$, $\theta(x)$, as well as the prime counting function $\pi(x)$.

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