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TRANSCENDENTAL NUMBERS AND ZETA FUNCTIONS

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1. Introduction

The concept of "number" has formed the basis of civilzation since time immemorial. Looking back from our vantage point of the digital age, we can agree with Pythagoras that "all is number". The study of numbers and their properties is the mathematical equivalent of the study of atoms and their structure. It is in fact more than that. The famous physicist and Nobel Laureate Eugene Wigner spoke of the "unreasonable effectiveness of mathematics in the natural sciences" to refer to the miraculous power of abstract mathematics to describe the physical universe.

Numbers can be divided into two groups: algebraic and transcendental. Algebraic numbers are those that satisfy a non-trivial polynomial equation with integer coefficients. Transcendental numbers are those that do not. Numbers such as $\sqrt{2}, \sqrt{-1}$ are algebraic, whereas, numbers like π and e are transcendental. To prove that a given number is transcendental can be quite difficult. It is fair to say that our knowledge of the universe of transcendental numbers is still in its infancy.

A dominant theme that has emerged in the recent past is the theory of special values of zeta and L-functions. In this article, we will touch only the hem of the rich tapestry that weaves transcendental numbers and values of L-functions in an exquisite way. This idea can be traced back to Euler and his work.

In 1735, Euler discovered experimentally that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}. \tag{1}$$

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He gave a "rigorous" proof much later, in 1742. Here is a sketch of Euler's proof [8]. The polynomial

$$(1 - x/r_1)(1 - x/r_2)...(1 - x/r_n)$$

has roots equal to $r_1, r_2, ..., r_n$. When we expand the polynomial, the coefficient of x is

$$-(1/r_1 + 1/r_2 + \cdots + 1/r_n).$$

Using this observation, Euler proceeded "by analogy." Supposing that $\sin \pi x$ "behaves" like a polynomial and noting that its roots are at $x = 0, \pm 1, \pm 2, ...$, Euler puts

$$f(x) = \frac{\sin \pi x}{\pi x}.$$

By l'Hôpital's rule, $f(0) = 1$. Now $f(x)$ has roots at $x = \pm 1, \pm 2, ...$ and so

$$f(x) = (1 - x)(1 + x)(1 - x/2)(1 + x/2)(1 - x/3)(1 + x/3) \cdots .$$

That is,

$$f(x) = (1 - x^2)(1 - x^2/4)(1 - x^2/9) \cdots .$$

The coefficient of x^2 on the right hand side is

$$-\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots +\right).$$

By Taylor's expansion,

$$\sin \pi x = \pi x - (\pi x)^3/3! + \cdots$$

so that comparing the coefficients, gives us formula (1).

The main question is whether all of this can be justified. Euler certainly didn't have a completely rigorous proof of his argument. To make the above discussion rigorous, one needs Hadamard's theory of factorization of entire funtions, a theory developed much later in 1892, in Jacques Hadamard's doctoral thesis.

The next question is whether Euler's result can be generalized. For example, can we evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Euler had difficulty with the first question but managed to show, using a similar argument, that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

and more generally that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \in \pi^{2k}\mathbb{Q}.$$

It is not hard to see Euler's proof can be modified to deduce the above results. Indeed, if $i = \sqrt{-1}$, then observing that

$$f(ix) = (1 + x^2)(1 + x^2/4)\cdots$$

we see that

$$f(x)f(ix) = (1 - x^4)(1 - x^4/2^4)(1 - x^4/3^4)\cdots$$

But the Taylor expansion of $f(x)f(ix)$ is

$$\left(1 - \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} - \cdots\right)\left(1 + \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} + \cdots\right).$$

Computing the coefficient of x^4 yields

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Continuing in this way, it is not difficult to see how Euler arrived at the assertion that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \in \pi^{2k}\mathbb{Q}.$$

Euler's work is the beginning of a modern theme in number theory, namely the transcendence of special values of L-series.

As explained at the outset, a complex number α is called *algebraic* if it is the root of a monic polynomial with rational coefficients. It is a well-known fact of algebra that the set of all algebraic numbers forms a field, usually denoted by $\overline{\mathbb{Q}}$. The elements of $\mathbb{C}\backslash\overline{\mathbb{Q}}$ are called *transcendental* numbers. For example, $\sqrt{2}$ is algebraic since it is the root of $x^2 - 2$. So is $5^{1/3}/3$ since it is the root of $x^3 - 5/27$. On the other hand, numbers such as π and e are transcendental and this is due to Lindemann and Hermite, respectively. Once a number is suspected to be transcendental, it is often difficult to show that it is so. For example, it is unknown if

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is transcendental, though it is conjectured to be so. In 1978, Roger Apéry [1] surprised the world by proving that it is irrational. There are other familiar numbers about whose arithmetic nature nothing is known. For instance, Euler's constant γ defined as

$$\gamma := \lim_{x\to\infty}\left(\sum_{n\le x} \frac{1}{n} - \log x\right)$$

is conjectured to be transcendental. We do not even know if it is irrational. More will be said about such questions later in this article.

Historically, it took mathematicians quite a long time to show that transcendental numbers exist. This was the question back in 1844. In that year, Liouville showed that if $\alpha \in \mathbb{C}\backslash\mathbb{Q}$ and

$$\inf\{q^n|\alpha - p/q| : p/q \in \mathbb{Q}\} = 0$$

for all natural numbers n, then α is transcendental. Using this criterion, he gave an explicit construction of certain transcendental numbers. For instance, he showed that

$$\sum_{n=1}^{\infty} 1/2^{n!}$$

is transcendental.

In retrospect, it is easy to see that a simple countability argument establishes the existence of transcendental numbers, but this argument, due to Cantor, came much later, in 1874, when he introduced notions of countability and uncountability. Incidentally, Cantor was born on March 3, 1845, one year after the publication of Liouville's paper.

In 1873, Hermite showed that e is transcendental and in 1882, Lindemann, using Hermite's technique, showed that π is transcendental. Lindemann's theorem, was important for resolving the problem of "squaring the circle." This problem, arising in ancient times, is to construct a square, using only a straightedge and compass, with area equal to that of a circle of radius 1. In other words, one must construct a line segment of length, $\sqrt{\pi}$.

One can show that if $\alpha > 0$ and we can construct a line segment of length α using only a straightedge and compass, then α must be algebraic. In fact, the field $\mathbb{Q}(\alpha)$ generated by α must have degree equal to a power of 2. More precisely, there is a tower of successive quadratic extensions

$$\mathbb{Q} = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n = \mathbb{Q}(\alpha)$$

with $[K_{i+1} : K_i] = 2$. In fact, the converse is also true and this gives a characterization of constructible numbers. From this result, we deduce that if $\sqrt{\pi}$ is constructible, then it must be algebraic. Consequently, π is algebraic, contrary to Lindemann's theorem. Thus, the problem of "squaring the cirlce" is impossible.

Let us observe that the ancient problem of doubling the cube, that is, the problem of constructing a cube whose volume equals that of a cube of length 1 is equivalent to constructing $2^{1/3}$. Since this number generates a field of degree 3 over $\mathbb{Q}$, it cannot be contained in a tower of fields as described above. Therefore, it is not constructible. A similar form of reasoning dismisses the problem of trisecting an angle using straightedge and compass. Indeed, the problem of trisecting $\pi/3$ is equivalent to constructing $\cos \pi/9$

and one can show that this number is the root of

$$4x^3 - 3x - 1/2.$$

This polynomial is irreducible over $\mathbb{Q}$ and so, $\cos \pi/9$ generates a cubic extension of the rationals.

These two problems of doubling the cube and trisecting an angle, were first treated in this algebraic setting by P. Wantzel in 1837 [21].

Shortly after Lindemann's paper appeared, Hermite obtained the following extension. For any algebraic $\alpha \neq 0$, e^α is transcendental. As a corollary, we deduce that for any algebraic number $\alpha \neq 0, 1$, $\log \alpha$ is transcendental. Since $e^{\pi i} = -1$, this shows that π is transcendental.

In his 1882 paper, Lindemann stated without proof that if $\alpha_1, ..., \alpha_n$ are distinct algebraic numbers, then

$$e^{\alpha_1}, ..., e^{\alpha_n}$$

are linearly independent over $\overline{\mathbb{Q}}$ (see page 100 of [14]). In 1885, Weierstrass published a proof of this by showing the stronger result that if $\alpha_1, ..., \alpha_n$ are linearly independent over $\mathbb{Q}$, then

$$e^{\alpha_1}, ..., e^{\alpha_n}$$

are algebraically independent over $\overline{\mathbb{Q}}$. In other words, there is no non-trivial polynomial

$$P(x_1, ..., x_n) \in \mathbb{Q}[x_1, ..., x_n]$$

such that

$$P(e^{\alpha_1}, ..., e^{\alpha_n}) = 0.$$

This is usually referred to as the Lindemann-Weierstrass theorem.

All of these results comprise the first phase in the development of transcendental number theory. The second phase begins with Hilbert's problems. In 1900, at the International Congress of Mathematicians, Hilbert posed 23 problems for the 20th century. His 7th problem asked if a number like $2^{\sqrt{2}}$ is transcendental. More generally, he asked if α is algebraic $\neq 0, 1$ and β is an algebraic irrational, then is α^β transcendental? If the answer is yes, then we can deduce that numbers like e^π are transcendental by taking $\alpha = -1 = e^{\pi i}$ and $\beta = -i$. This specific case was proved by Gelfond in 1929. Extending these techniques, Gelfond and Schneider, independently, in 1934, proved the following:

Theorem 1. (Gelfond-Schneider, 1934) *If $\alpha, \beta \in \overline{\mathbb{Q}}$, with $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$, then α^β is transcendental.*

In 1966, Baker [2] derived a generalization of this theorem. He showed that if $\alpha_1, ..., \alpha_n, \beta_0, \beta_1, ...\beta_n \in \overline{\mathbb{Q}}$ with $\alpha_1 \cdots \alpha_n \beta_0 \neq 0$, then

$$e^{\beta_0}\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$$

is transcendental. He proved this by showing that the linear form

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental. In 1970, Alan Baker was awarded the Fields medal at the ICM in France for this work.

For later reference, we state Baker's theorem precisely.

Theorem 2. *If $\alpha_1, ..., \alpha_n \in \overline{\mathbb{Q}} \backslash \{0\}$ and $\beta_1, ..., \beta_n \in \overline{\mathbb{Q}}$, then*

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental. The former case arises only if $\log \alpha_1, ...,$ $\log \alpha_n$ are linearly dependent over $\mathbb{Q}$ or if $\beta_1, ..., \beta_n$ are linearly dependent over $\mathbb{Q}$.

Proof. This is the content of Theorems 2.1 and 2.2 of [2]. Let us note that here and later, we interpret log as the principal value of the logarithm with the argument lying in the interval $(-\pi, \pi]$. □

2. **The Riemann zeta function and its' special values**

Interesting transcendental numbers arise as special values of L-series. The prototypical case is that of the Riemann zeta function. This function, originally defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\Re(s) > 1$, can be extended analytically to the complex plane and shown to satisfy the functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s),$$

where

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt,$$

for $\Re(s) > 0$ and extended as a meromorphic function to the entire complex plane via the familiar functional equation $\Gamma(s+1) = s\Gamma(s)$.

The famous theorem of Euler described in the introduction is that $\zeta(2k) \in \pi^{2k}\mathbb{Q}$ so that $\zeta(2k)$ is transcendental for all natural numbers $k \geq 1$. A natural question that arises is: what is the nature of $\zeta(3), \zeta(5), ...$? It is conjectured that the numbers $\pi, \zeta(3), \zeta(5), ...$ are algebraically independent numbers. In particular, $\zeta(3), \zeta(5), \ldots$ are all conjectured to be transcendental.

As mentioned earlier, Roger Apéry showed in 1977 that $\zeta(3)$ is irrational. His proof, involving complicated recurrences, has received considerable attention and simplification. Most recently, Rivoal [19] was able to extend his technique to show that infinitely many $\zeta(2k+1)$ for $k \geq 1$ are irrational.

3. Apéry's theorem

Here is a brief sketch of Apéry's proof. His starting point was the remarkable formula

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

Consider the recurrence

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5) u_{n-1}.$$

Let a_n be the sequence satisfying this recurrence with the initial conditions $a_0 = 0$ and $a_1 = 6$. Let b_n be the sequence satisfying this recurrence with the initial conditions $b_1 = 1, b_1 = 5$. Apéry showed that all the b_n are integers, which is rather surprising since in the recurrence, we are dividing by n^3 to get the expression for u_n. Even more remarkable is that the a_n's are rational numbers such that $2[1, 2, ..., n]^3 a_n$ is an integer for all n. (Here $[1, 2, ..., n]$ denotes the least common multiple of the numbers, $1, 2, ..., n$.) With these sequences in place, Apéry shows that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \zeta(3).$$

More precisely,

$$|\zeta(3) - \frac{a_n}{b_n}| \leq \frac{6}{b_n^2}.$$

He now invokes an elementary lemma that is easy to prove: if there are infinitely many rational numbers p_n/q_n such that

$$|\theta - p_n/q_n| \leq 1/q_n^{1+\delta},$$

for some $\delta > 0$, then θ is irrational. In our case, the denominator of a_n/b_n is easily estimated and the irrationality of $\zeta(3)$ follows from the elementary lemma.

Indeed, if $\zeta(3)$ was rational and equal to A/B (say) with A, B positive integers, then

$$|B\zeta(3)b_n - a_n| = |Ab_n - a_n| \leq 6/b_n.$$

Multiplying through by $2[1, 2, ..., n]^3$ clears all denominators. By the prime number theorem, this is asymptotic to

$$2e^{3n+o(n)}.$$

On the other hand, $b_n = \alpha^{n+o(n)}$ where $\alpha = (1+\sqrt{2})^4 = 17 + 12\sqrt{2}$, by a routine calculation. Since $e^3 < (1+\sqrt{2})^4$, we deduce that $a_n = Ab_n$ for n sufficiently large and this is quickly checked to be a contradiction.

Attempts to generalize this argument to other odd values of the zeta function have not succeeded. In this direction, Rivoal showed in 2000, that infinitely many of the numbers $\zeta(2k+1)$ are irrational. In 2001, Rivoal and Zudilin [20] showed that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational. In 2003, Ball and Rivoal [4] showed that the $\mathbb{Q}$-vector space spanned by $\zeta(3), \zeta(5), ...$ has infinite dimension.

Related to this is a curious formula of Ramanujan. In his notebooks, Ramanujan wrote

$$\zeta(3) + 2\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n}-1)} = \frac{7\pi^3}{180}.$$

Presumably, Ramanujan had a proof but the first rigorous proof was given by Grosswald [9] in 1970. Hence, at least one of the two terms on the left hand side is transcendental!

4. **Multiple zeta values**

To understand the arithmetic nature of special values of the Riemann zeta function, it has become increasingly clear that multiple zeta values (MZV's for short) must be studied. These are defined as follows:

$$\zeta(a_1, ..., a_k) = \sum_{n_1>n_2>\cdots n_k>0} \frac{1}{n_1^{a_1} n_2^{a_2} \cdots n_k^{a_k}},$$

where $a_1, a_2, ..., a_k$ are positive integers with the proviso that $a_1 \neq 1$. The last condition is imposed to ensure convergence of the series.

There are several advantages to introducing these multiple zeta functions. First, they have an algebraic structure which we describe. It is easy to see that

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).$$

Indeed, the left hand side can be decomposed as

$$\sum_{n_1,n_2} \frac{1}{n_1^{s_1} n_2^{s_2}} = \sum_{n_1>n_2} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n_2>n_1} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n_1=n_2} \frac{1}{n_1^{s_1} n_2^{s_2}}$$

from which the identity becomes evident. In a similar way, one can show that $\zeta(s_1)\zeta(s_2, ..., s_t)$ is again an integral linear combination of multiple zeta values (MZV's). More generally, the product of any two MZV's is an integral linear combination of MZV's. These identities lead to new relations, like $\zeta(2,1) = \zeta(3)$, an identity which appears in Apéry's proof of the irrationality of $\zeta(3)$.

If we let A_r be the $\mathbb{Q}$-vector space spanned by

$$\zeta(s_1, s_2, ..., s_k)$$

with $s_1 + s_2 + \cdots + s_k = r$, then the product formula for MZV's shows that

$$A_r A_s \subseteq A_{r+s}.$$

In this way, we obtain a graded algebra of MZV's. Let d_r be the dimension of A_r as a vector space over $\mathbb{Q}$. For convenience, we set $d_0 = 1$ and $d_1 = 0$. Clearly, $d_2 = 1$ since A_2 is spanned by $\pi^2/6$. Zagier [13] has made the following conjecture: $d_r = d_{r-2} + d_{r-3}$, for $r \geq 3$. In other words, d_r satisfies a Fibonacci-type recurrence relation. Consequently, d_r is expected to have exponential growth. Given this prediction, it is rather remarkable that not a single value of r is known for which $d_r \geq 2$!

In view of the identity, $\zeta(2,1) = \zeta(3)$, we see that $d_3 = 1$. What about d_4? A_4 is spanned by $\zeta(4), \zeta(3,1), \zeta(2,2), \zeta(2,1,1)$. What are these numbers? Zagier's conjecture predicts that $d_4 = d_2 + d_1 = 1 + 0 = 1$. Is this true?

Let's adapt Euler's technique to evaluate $\zeta(2,2)$. As noted in the introduction,

$$(1 - x/r_1)(1 - x/r_2) \cdots (1 - x/r_n)$$

has roots equal to $r_1, r_2, ..., r_n$. When we expand the polynomial, the coeffcient of x is

$$-(1/r_1 + 1/r_2 + \cdots + 1/r_n).$$

The coefficient of x^2 is

$$\sum_{i<j} 1/r_i r_j.$$

With this observation, we see from the product expansion

$$f(x) = \frac{\sin \pi x}{\pi x} = (1 - x^2)(1 - x^2/4)(1 - x^2/9) \cdots$$

that the coefficient of x^4 is precisely $\zeta(2,2)$. An easy computation shows that

$$\zeta(2,2) = \pi^4/5!.$$

It is now clear that this method can be used to evaluate $\zeta(2,2,...,2) = \zeta(\{2\}^m)$ (say). By comparing the coefficient of x^{2m} in our expansion of $f(x)$, we obtain that

$$\zeta(\{2\}^m) = \frac{\pi^{2m}}{(2m+1)!}.$$

We could have also evaluated $\zeta(2,2)$ using the identity

$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4),$$

but we had opted to the method above to indicate its generalization which allows us to also evaluate $\zeta(2,2,...,2)$. What about $\zeta(3,1)$? This is a bit more difficult and will not come out of our earlier work. In 1998, Borwein, Bradley, Broadhurst and Lisonek [5] showed that $\zeta(3,1) = 2\pi^4/6!$. What about $\zeta(2,1,1)$? With some work, one can show that this is equal to $\zeta(4)$. Thus, we conclude that $d_4 = 1$ as predicted by Zagier.

What about d_5? With more work, we can show that

$$\zeta(2,1,1,1) = \zeta(5); \quad \zeta(3,1,1) = \zeta(4,1) = 2\zeta(5) - \zeta(2)\zeta(3).$$

$$\zeta(2,1,1) = \zeta(2,3) = 9\zeta(5)/2 - 2\zeta(2)\zeta(3).$$

$$\zeta(2,2,1) = \zeta(3,2) = 3\zeta(2)\zeta(3) - 11\zeta(5)/2.$$

This proves that $d_5 \leq 2$. Zagier conjectures that $d_5 = 2$. In other words, $d_5 = 2$ if and only if $\zeta(2)\zeta(3)/\zeta(5)$ is irrational.

Can we prove Zagier's conjecture? To this date, not a single example is known for which $d_n \geq 2$. If we write

$$(1 - x^2 - x^3)^{-1} = \sum_{n=1}^{\infty} D_n x^n,$$

then it is easy to see that Zagier's conjecture is equivalent to the assertion that $d_n = D_n$ for all $n \geq 1$. Deligne and Goncharov [10] and (independently) Terasoma [22] showed that $d_n \leq D_n$.

5. Dirichlet L-functions

The Riemann zeta function is only a tiny fragment of a galaxy of L-series whose special values are of deep interest. The success of Euler's explicit evaluation of $\zeta(2k)$ can be extended to a class of series known as Dirichlet L-functions. These functions are defined as follows. Let q be a natural number and $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}$ be a homomorphism of the group of coprime residue classes mod q. For each natural number $n \equiv a (\operatorname{mod} q)$ with a coprime to q, $\chi(n)$ is defined to be $\chi(a)$. If n is not coprime to q, we set $\chi(n)$ to be zero. With this definition, we set

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

If χ is the trivial character, then $L(s,\chi)$ is easily seen to be

$$\zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right),$$

so that this is (upto a known factor) essentially the Riemann zeta function. If χ is not the trivial character, then it turns out that one can evaluate $L(k,\chi)$ explicitly in certain cases. To elaborate further, we say a character χ is even if $\chi(-1) = 1$ and odd if $\chi(-1) = -1$. For $k = 1$, $L(1,\chi)$ has been studied extensively and summarized in the celebrated formulas of Dirichlet.

For $k \geq 2$, it turns out that $L(k, \chi) \in \pi^k \mathbb{Q}$ if k and χ are both even or both odd. When k and χ are of opposite parity, the arithmetic nature is a complete mystery, as enigmatic as the values of the Riemann zeta function at odd arguments. The simplest case of this mystery is to determine if

$$G := 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

is irrational. If χ is the non-trivial character (mod 4), then χ is odd and $G = L(2, \chi)$. Attempts to generalize Apéry's argument to show the irrationality of G have failed. However, Rivoal and Zudilin [20] have shown that at least one of $L(2k, \chi)$ with $1 \leq k \leq 10$ is irrational.

6. Chowla's conjecture

Inspired by the nature of Dirichlet's L-functions and the mystery surrounding their special values, Sarvadaman Chowla [6] considered Dirichlet series of the form

$$L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

where f is an algebraic-valued periodic function, with period q. It is not hard to see that the sum converges at $s = 1$ if and only if

$$\sum_{a=1}^{q} f(a) = 0.$$

In [6], Chowla asked if there exists a rational-valued function f, not identically zero, with prime period, such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0. \tag{2}$$

In [3], Baker, Birch and Wirsing answered this question using Baker's theory of linear forms in logarithms. In the general case when q is not necessarily prime, and f is algebraic-valued, they showed that the sum (6) in question can be written as a $\overline{\mathbb{Q}}$-linear form in logarithms of algebraic numbers. They considered functions f satisfying $f(a) = 0$ whenever $1 < (a, q) < q$. In the case that the q-th cyclotomic polynomial is irreducible over the field generated by the values of f, they showed that the sum is nonzero. By Baker's theorem, Theorem 2, we deduce the sum is transcendental. Several interesting corollaries can be deduced from this work, as noted in [16]. As indicated there, Chowla's question can be connected with the theory of the Hurwitz zeta function in the following way. The Hurwitz zeta function, defined for $0 < a \le 1$, as

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

has an analytic continuation to the entire complex plane with a simple pole at $s = 1$. Its' Laurent expansion at $s = 1$ is given by

$$\zeta(s,a) = \frac{1}{s-1} - \psi(a) + O(s-1),$$

where $\psi(a)$ is the digamma function, $\Gamma'(a)/\Gamma(a)$. As proved in [16], we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = -\frac{1}{q} \sum_{a=1}^{q} f(a)\psi(a/q).$$

By choosing appropriate test functions f, one can show that there is at most one a/q such that $\psi(a/q)$ is algebraic.

7. Generalized Euler constants

Following Lehmer [15], we can define generalized Euler's constants $\gamma(a,q)$ by

$$\gamma(a,q) = \lim_{x\to\infty} \left(\sum_{\substack{n \le x \\ n \equiv a (\bmod q)}} \frac{1}{n} - \frac{\log x}{q} \right).$$

Then, it is not hard to show that

$$q\gamma(a,q) = \gamma - \sum_{b=1}^{q-1} e^{-2\pi i b a/q} \log(1 - e^{2\pi i b/q}).$$

Apart from the γ term on the right hand side, this expresses the generalized Euler constants as a $\mathbb{Q}$-linear form of logarithms of algebraic numbers. Thus, Baker's theorem can be applied to study them. In addition, we can relate these constants to Chowla's question. Indeed, as noted in [16], we have:

Lemma 3. *If f is as above and*

$$\sum_{a=1}^{q} f(a) = 0,$$

then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{a=1}^{q} f(a)\gamma(a,q).$$

This result allows us to state that all of the generalized Euler constants are transcendental with at most one possible exception.

8. The Chowla-Milnor conjecture

In [7], Paromita and Sarvadaman Chowla, considered the question of non-vanishing of $L(s, f)$ at $s = 2$ for rational valued functions f. In the case q is prime, they conjectured that $L(2, f) \neq 0$ unless

$$f(1) = f(2) = \cdots f(q-1) = \frac{f(q)}{1-q^2}.$$

This can be formulated equivalently as the following conjecture on special values of the Hurwitz zeta function, as noted by Milnor [12]: the numbers

$$\zeta(2, 1/q), \zeta(2, 2/q), ..., \zeta(2, (q-1)/q)$$

are linearly independent over $\mathbb{Q}$. This suggested to Milnor the more general conjecture: for any natural number q, and $k \geq 2$, the numbers

$$\zeta(k, a/q), \quad 1 \leq a < q, \quad (a, q) = 1$$

are linearly independent over $\mathbb{Q}$. We refer to this as the Chowla-Milnor conjecture.

The difficulty of this conjecture is partly seen by the following theorem:

Theorem 4. (S. Gun, M. Ram Murty and P. Rath, 2008) *The Chowla-Milnor conjecture for the single modulus $q = 4$ is equivalent to the irrationality of $\zeta(2k+1)/\pi^{2k+1}$ for all natural numbers $k \geq 1$.*

Proof. See [11]. □

There are other implications of the Chowla-Milnor conjecture to many classical questions related to special values of Dirichlet L-functions and these are expanded upon in [11].

9. Concluding remarks

We hope that the reader is convinced that this is only a shadow of a richer theory that is yet to unfold. We refer the reader to [11] for yet more connections between transcendental numbers and zeta functions that emerge from Chowla's conjectures and their generalizations.

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