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AN ANALOGUE OF THE ERDÖS-KAC THEOREM FOR FOURIER COEFFICIENTS OF MODULAR FORMS

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Let f be a cusp form of weight k on $\Gamma_0(N)$ which is a normalized eigenform for the Hecke operators, and suppose that f does not have complex multiplication. Write $f = \Sigma a_n e^{2\pi i n z}$ for the Fourier expansion at ∞ and suppose $a_n \in \mathbb{Z}$ for all n. Denote by $\nu(n)$ the number of distinct prime divisors of n. Assuming the Riemann Hypothesis for all Artin L-functions, we show that

$$\text{card}\,\{p \leqslant X : a_p \neq 0 \text{ and } \nu(a_p) - \log\log p \leqslant \alpha (\log\log p)^{1/2}\}$$

$$\sim \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-t^2/2}\, dt\right) \pi(X).$$

An analogous result is obtained for $\Omega(a_p)$, the total number of prime factors of a_p. We also obtain the distribution function for $\nu(a_n)$ and $\Omega(a_n)$.

1. INTRODUCTION

Let f be a normalized eigenform of the Hecke operators for $\Gamma_0(N)$. Suppose that f does not have complex multiplication (in the sense of Ribet) and that

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

with $a_n \in Z$. It was shown in Murty and Murty (1984) that, assuming a certain generalized Riemann hypothesis (GRH),

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$$\sum_{\substack{p \leqslant x \\ a_p \neq 0}} (\nu(a_p) - \log\log p)^2 \ll \frac{x \log\log x}{\log x} \qquad \ldots(1)$$

and

$$\sum_{\substack{n \leqslant x \\ a_n \neq 0}} \left(\nu(a_n) - \frac{1}{2}(\log\log n)^2\right)^2 \ll x (\log\log x)^{3+\epsilon} \qquad \ldots(2)$$

where $\nu(n)$ denotes the number of prime factors of n and in (1), p denotes a prime number.

These results led to the interesting conclusion that

$$|a_p| \geqslant \exp((\log p)^{1-\epsilon}) \qquad \ldots(3)$$

for almost all primes p (that is, apart from $o(x/\log x)$ primes $p \leqslant x$, the inequality (3) holds.) It is a classical conjecture of Lehmer, that in the case $f = \Delta$, the Ramanujan cusp form,

$$\tau(p) \neq 0.$$

Here τ denotes the Ramanujan function. Serre and Atkin have further conjectured that

$$|\tau(p)| \geqslant p^{9/2-\epsilon}$$

for all p sufficiently large. If we write $\tau(p) = 2p^{11/2} \cos \theta_p$ and assume the Sato-Tate conjecture, namely that θ_p's are equidistributed in $[-\pi, \pi]$ with respect to the measure $(2/\pi) \sin^2 \theta \, d\theta$, then it is easy to see that

$$|\tau(p)| \geqslant p^{11/2-\epsilon}$$

for almost all prime numbers p.

(1) and (2) can be viewed as the "modular" analogue of the classical theorem of Hardy and Ramanujan concerning the function $\nu(n)$. It was also noted in Murty and Murty (1984) that the assumption of GRH in (1) and (2) can be replaced by a (milder) inequality of Bombieri-Vinogradov type. The exact formulation will be given below.

The purpose of this paper is to prove that $\nu(\tau(n))$ and $\nu(\tau(p))$ obey a normal distribution law very much analogous to $\nu(n)$ (as described by the celebrated Erdös-Kac theorem).

More generally, let g be a non-zero multiplicative function taking rational integer values. Suppose that $g(n) \neq 0$ for any natural number n, and
(H_0) For some $\beta > 0$, $|g(n)| \leqslant n^{\beta}$ for all n.

Define,

$$\pi(x; g, d) = \text{card}\,(p \leqslant x : g(p) \equiv 0 \pmod d)$$

where (here and elsewhere in this paper) p denotes a prime number.

We suppose that there is a function $\delta(d)$ such that :
(H_1) For some $\theta > 0$,

$$\sum_{d \leqslant x^\theta} |\, \pi(x; g, d) - \delta(d)\,\pi(x)\,| \ll \frac{x}{\log x}$$

where $\pi(x)$ denotes the number of primes $p \leqslant x$.

We make the following hypothesis purely for the sake of clarity of exposition. More general results can be proved under less restrictive conditions. Nevertheless, we assume :

(H_2) For prime powers p^α, q^β, $(p \neq q)$,

$$\delta(p^\alpha) = p^{-\alpha} + O(p^{-\alpha-1})$$

and

$$\delta(p^\alpha q^\beta) = (p^{-\alpha} + O(p^{-\alpha-1}))\,(q^{-\beta} + O(q^{-\beta-1}))$$

where the implied constants are absolute.

Define :

$$N(x, \alpha) = \text{card}\left(n \leqslant x : \frac{\nu(g(n)) - \frac{1}{2}(\log\log n)^2}{(\log\log n)^{3/2}} \leqslant \frac{\alpha}{\sqrt{3}}\right).$$

We can now state the main theorem :

Theorem 1—Let g be a non-vanishing multiplicative function satisfying (H_0), (H_1) and (H_2) above. Then

$$\lim_{x\to\infty} \frac{N(x,\alpha)}{x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-t^2/2}\, dt.$$

Now let f be a Hecke eigenform on $\Gamma_0(N)$, with rational integer coefficients a_n. We modify the coefficients by defining :

$$\tilde{a}_n = \prod_{\substack{p^\alpha \| n \\ a_{p^\alpha} \neq 0}} a_{p^\alpha}$$

The empty product is taken to be equal to 1.

Define

$$N_f(x, \alpha) = \text{card}\left(n \leqslant x : \frac{\nu(\tilde{a}_n) - \frac{1}{2}(\log\log n)^2}{(\log\log n)^{3/2}} \leqslant \frac{\alpha}{\sqrt{3}}\right).$$

Theorem 2—Let f be a normalized Hecke eigenform on $\Gamma_0(N)$ with rational integer coefficients and without complex multiplication. Then

$$\lim_{x\to\infty} \frac{N_f(x, \alpha)}{x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-t^2/2}\, dt.$$

2. The Theorem of Kubilius-Shapiro

An additive arithmetical function h is called strongly additive if $h(p^\alpha) = h(p)$. There is a general theorem of Kubilius and Shapiro which allows one to deduce a normal distribution law for such functions satisfying certain conditions. We would like to apply this theorem to $\nu(g(n))$, but a little thought reveals that $\nu(g(n))$ is not additive. Our basic idea is to approximate this function by a strongly additive function and apply the general theory. Finally, we show that the error thus introduced is small enough to deduce our main results.

Given any strongly additive function h, define

$$A(x) = \sum_{p\leqslant x} \frac{h(p)}{p}$$

and

$$B(x) = \left(\sum_{p\leqslant x} \frac{h^2(p)}{p}\right)^{1/2}.$$

Suppose that for each fixed $\epsilon > 0$,

$$\sum_{\substack{p<x \\ |h(p)| > \epsilon B(x)}} \frac{h^2(p)}{p} = o(B^2(x)).$$

Then, setting

$$H(x; \alpha) = \text{card}\left(n \leqslant x : \frac{h(n) - A(x)}{B(x)} \leqslant \alpha\right)$$

the theorem of Kubilius-Shapiro (see Elliott 1980b) implies

$$\lim_{x\to\infty} \frac{H(x; \alpha)}{x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-t^2/2}\, dt.$$

3. The First Approximation

Let $\Omega(n)$ denote the total number of prime factors of n, counted with multiplicity. (Note that Ω is 'not' strongly additive.) Define

$$h(n) = \sum_{p|n} \Omega(g(p)).$$

Clearly h is strongly additive. We have

$$\sum_{p\leqslant x} \frac{h(p)}{p} = \sum_{p\leqslant x} \frac{\Omega(g(p))}{p}$$

$$= \sum_{q^\alpha \leqslant x^\theta} \sum_{\substack{p\leqslant x \\ q^\alpha \mid g(p)}} \frac{1}{p} + O(\log\log x)$$

since the number of prime power divisors q^α of $g(p)$ with $q^\alpha > x^\theta$ is absolutely bounded by virtue of (H_0). Partial summation gives that the first sum is

$$\sum_{q^\alpha \leqslant x^\theta} \left\{ \frac{1}{x} \pi(x; g, q^\alpha) + \int_2^x \pi(t; g, q^\alpha) \frac{dt}{t^2} \right\} = \Sigma_1 + \Sigma_2 \text{ (say)}.$$

Now, by (H_1) and (H_2), we have

$$\sum_1 \ll \frac{\log\log x}{\log x}.$$

To handle the other sum, we need the following lemmas :

Lemma 1—If $u \leqslant x^\theta$, then

$$\sum_{d\leqslant u} \pi(x; g, d) = \pi(x) \log\log u + O(\pi(x)).$$

If $u > x^\theta$, then

$$\sum_{d\leqslant u} \pi(x; g, d) = \pi(x) \log\log x + O(\pi(x)).$$

Proof · This is easily established utilising (H_0) and (H_1).

By lemma 1, we find

$$\sum_2 = \int_2^x \{\pi(t) \log\log t + O(t/\log t)\} \frac{dt}{t^2}.$$

Therefore,

$$\sum_{p\leqslant x} \frac{h(p)}{p} = \frac{1}{2} (\log \log x)^2 + O(\log \log x).$$

Now we treat $B(x)$ for the function h. We find,

$$B^2(x) = \sum_{p\leqslant x} \frac{1}{p} \sum_{\substack{q_1^\alpha / g(p) \\ q_2^\gamma / g(p) \\ q_1 \neq q_2}} 1 + O(x(\log \log x)^2).$$

Again by partial summation,

$$B^2(x) = \sum_{\substack{q_1^\alpha \leqslant x^{\theta/2} \\ q_2^\gamma \leqslant x^{\theta/2}}} \left\{ \frac{\pi(x; g, q_1^\alpha q_2^\gamma)}{x} + \int_2^x \pi(t; g, q_1^\alpha q_2^\gamma) \frac{dt}{t^2} \right\}$$

$$= \int_2^x \sum_{\substack{q_1^\alpha \leqslant t^{\theta/2} \\ q_2^\gamma \leqslant t^{\theta/2}}} \delta(q_1^\alpha q_2^\gamma) \pi(t) \frac{dt}{t^2} + O((\log \log x)^2)$$

by Lemma 1 and (H_1). We find easily that

$$B^2(x) = \int_2^x \frac{(\log \log t)^2}{t \log t} dt + O((\log \log x)^2)$$

utilising (H_2). Therefore,

$$B(x) = \frac{1}{\sqrt{3}} (\log \log x)^{3/2} + O(\log \log x).$$

We need one more calculation. We have to check that:

$$\sum_{\substack{p\leqslant x \\ \Omega(g(p)) > \epsilon B(x)}} \frac{\Omega^2(g(p))}{p} = o(B^2(x)). \qquad \ldots(4)$$

Define

$$\alpha(p) = \begin{cases} 1 & \text{if } \Omega(g(p)) > \epsilon B(x) \\ 0 & \text{if not.} \end{cases}$$

It is not difficult to see (by Turan's method and utilising (H_0), (H_1) and (H_2)) that

$$\sum_{p\leqslant x} (\Omega(g(p))-\log\log p)^2 = O(x\log\log x).$$

This was in fact established in Murty and Murty (1984).

Therefore,

$$\sum_{p<x} \alpha(p) = O(x/(\log\log x)^2). \qquad \ldots(5)$$

Hence

$$\left|\sum_{p\leqslant x} \alpha(p).\ \frac{\Omega^2(g(p))}{p}\right| \leqslant \left(\sum_{p\leqslant x}\frac{\alpha(p)}{p}\right)^{1/2}\left(\sum_{p\leqslant x}\frac{\Omega^4(g(p))}{p}\right)^{1/2}$$

$$\ll \left(\sum_{p<x}\frac{\Omega^4(g(p))}{p}\right)^{1/2}$$

utilising (5) and partial summation.

We readily find, under (H_0), (H_1) and (H_2), the estimate

$$\sum_{p\leqslant x}\frac{\Omega^4(g(p))}{p} \ll (\log\log x)^5.$$

In fact, one can establish an asymptotic formula by the preceding method, but it is not necessary for our purpose. We therefore deduce that (4) does indeed hold.

This prove that

$$\text{card}\left(n\leqslant x : \frac{h(n)-\frac{1}{2}(\log\log n)^2}{(\log\log n)^{3/2}} \leqslant \frac{\alpha}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} e^{-t^2/2}\,dt + o(1)\right)x.$$

4. The Turan-Kubilius Inequality

We want information concerning $\nu(g(n))$, and $\Omega(g(n))$. The first part of Theorem 1, for $N(x, \alpha)$ will be established if we can show that

$$\nu(g(n)) - h(n) = o((\log\log x)^{3/2})$$

for all but $o(x)$ numbers $n \leqslant x$.

For this purpose, we need :

Lemma 2 (Turan-Kubilius inequality)—Let $k(n)$ be a complex-valued additive function defined on the natural numbers. If

$$E(x) = \sum_{p^\alpha \leqslant x} \frac{k(p^\alpha)}{p^\alpha}\left(1 - \frac{1}{p}\right)$$

and

$$D(x) = \left(\sum_{p^\alpha \leqslant x} \frac{|k(p^\alpha)|^2}{p^\alpha}\right)^{1/2}$$

then

$$\sum_{n \leqslant x} |k(n) - E(x)|^2 \leqslant 32x\, D^2(x).$$

PROOF : See Elliott (1980a, p. 147).

First, we establish that for almost all n,

$$\Omega(g(n)) - \Omega_y(g(n)) = \nu(g(n)) - \nu_y(g(n))$$

where $\nu_y(n)$ and $\Omega_y(n)$ are the number of prime factors of n less than y counted without multiplicity and with multiplicity respectively.

The function $\Omega(g(n))$ is additive. Setting

$$E(x) = \sum_{p^\alpha \leqslant x} \frac{\Omega(g(p^\alpha))}{p^\alpha}\left(1 - \frac{1}{p}\right)$$

$$= \sum_{p \leqslant x} \frac{\Omega(g(p))}{p} + O(1)$$

and $D(x) = B(x)$, Lemma 2 gives

$$\sum_{n \leqslant x}\left(\Omega(g(n)) - \frac{1}{2}(\log\log n)^2\right)^2 \ll x(\log\log x)^3.$$

If we consider $\Omega_y(g(n))$ with $y = \log\log x$, it is easy to see that the corresponding $E(x)$ and $D(x)$, which we shall denote by $E_y(x)$ and $D_y(x)$, satisfy

$$E_y(x) = (\log_2 x)(\log_4 x) + O(\log_2 x)$$

and

$$D_y(x)^2 = (\log_2 x)(\log_4 x)^2 + O((\log_2 x)(\log_4 x))$$

where we have used the notation of the iterated logarithm : $\log_1 x = \log x$, $\log_k x = \log(\log_{k-1} x)$, for $k \geq 2$. Again, by Lemma 2,

$$\sum_{n \leqslant x} \Big(\Omega_y (g(n)) - (\log_2 x)(\log_4 x) \Big)^2 \ll x (\log_2 x)(\log_4 x)^2.$$

Therefore,

$$\Omega_y (g(n)) < 2 (\log_2 x)(\log_4 x) \qquad \ldots(6)$$

for almost all $n \leqslant x$. Hence

$$0 \leqslant \Omega_y (g(n)) - \nu_y (g(n)) \leqslant 2 (\log_2 x)(\log_4 x)$$

for almost all $n \leqslant x$.

Now suppose $p > y$ and $p \mid g(n)$. Then $p \mid g(q^\alpha)$ for some prime power $q^\alpha \| n$. The quantity

$$\Omega (g(n)) - \Omega_y (g(n))$$

counts prime divisors p of $g(n)$, $p > y$ with multiplicity, and

$$\nu (g(n)) - \nu_y (g(n))$$

counts the same without multiplicity. Suppose $p^2 \mid g(n)$ and $p > y$. Then there are two possibilities :

(a) there is a $q^\alpha \| n$ such that $g(q^\alpha) \equiv O \pmod{p^2}$

or

(b) there are q_1^α, $q_2^\gamma \| n$ such that $g(q_1^\alpha) \equiv O \pmod{p}$ and $g(q_2^\gamma) \equiv O \pmod{p}$.

In case (a), the number of $n \leqslant x$ is

$$\ll \sum_{y < p < x^\beta} \sum_{p^2 \mid g(q^\alpha)} \frac{x}{q^\alpha} = o(x)$$

by partial summation, (H_0) and (H_1). In case (b), we get that the number of such natural numbers is

$$\ll \sum_{y < p < x^\beta} \sum_{\substack{p \mid g(q_1^\alpha) \\ p \mid g(q_2^\gamma)}} \frac{x}{q_1^\alpha \, q_2^\gamma} = o(x)$$

by a similar method. Therefore, for almost all n,

$$\Omega(g(n)) - \Omega_y(g(n)) = \nu(g(n)) - \nu_y(g(n))$$

and the dispersion method of Lemma 2 yields (6) and therefore, if the theorem is true for $\Omega(g(n))$, it is certainly true for $\nu(g(n))$.

To this end, we establish that

$$\Omega(g(n)) - \Omega_y(g(n)) = h(n) - h_y(n)$$

for almost all $n \leqslant x$. Here,

$$h_y(n) = \sum_{\substack{p \mid n \\ p<y}} \Omega(g(p)).$$

We begin by noting that

$$\Omega(g(n)) - h(n) = \sum_{\substack{p^\alpha \| n \\ \alpha \geqslant 2}} \Big(\Omega(g(p^\alpha)) - \Omega(g(p)) \Big).$$

As $h(n)$ and $\Omega(g(n))$ are additive functions, the dispersion Lemma 2 applies. If we set

$$j(n) = \Omega(g(n)) - h(n)$$

we find that if

$$c = \sum_{\substack{p^\alpha \\ \alpha \geqslant 2}} \frac{j(p^\alpha)}{p^\alpha}\left(1 - \frac{1}{p}\right)$$

then

$$\sum_{n \leqslant x} (j(n) - c)^2 \ll x.$$

Therefore, for almost all n,

$$\Omega(g(n)) = h(n) + o((\log\log n)^{3/2}).$$

5. Proofs of Theorems 1 and 2

By the above, we have established that

$$Pr\left(n : \frac{\Omega(g(n)) - \frac{1}{2}(\log\log n)^2}{(\log\log n)^{3/2}} \leqslant \frac{\alpha}{\sqrt{3}}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-t^2/2}\, dt$$

and our previous remarks show that the same is true for $\nu(g(n))$.

For Theorems 2, it was already noted in Murty and Murty (1984) that Fourier coefficients of normalized Hecke eigenforms of $\Gamma_0(N)$ satisfy (H_0), (H_1) and (H_2) via the associated l-adic representation. We will not repeat this here. Theorem 2 now follows from Theorem 1 after this observation.

6. Concluding Remarks

If we let

$$P(x, \alpha) = \text{card}\left(p \leqslant x : \frac{\nu(g(p)) - \log\log p}{(\log\log p)^{1/2}} \leqslant \alpha\right)$$

then utilising the methods of this paper, one can show that for each fixed k,

$$\sum_{p \leqslant x} (\nu(g(p)) - \log\log p)^k = (c_k + o(1)) \frac{x}{\log x} (\log\log x)^{k/2}$$

where

$$c_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^k e^{-t^2/2}\, dt.$$

The case $g(p) = p - 1$ was treated by Halberstam (1955). It follows from Tchebycheff's method of moments that

$$\lim_{x \to \infty} \frac{P(x, \alpha)}{x/\log x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-t^2/2}\, dt.$$

This of course yields a corresponding result for the Fourier coefficients of normalized eigenforms. In fact, since

$$\text{card}(p \leqslant x : a_p = 0) = o(x/\log x)$$

by a result of Serre (1982), we have :

Theorem 3—Let f be a normalized Hecke eigenform on $\Gamma_0(N)$ with rational integer coefficients and without complex multiplication. Then

$$\text{card}\left(p \leqslant x : a_p \neq 0 \text{ and } \frac{\nu(a_p) - \log\log p}{\sqrt{\log\log p}} \leqslant \alpha\right) = (c + o(1))\, \pi(x)$$

where

$$c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-t^2/2}\, dt.$$

In the special case of $g(n) = \phi(n)$ in Theorem 1, we recover the theorem of Erdös and Pomerance (1984). In fact, our method has been inspired by their work. The hypothesis (H_1) is verified to hold by invoking the Bombieri-Vinogradov theorem.

The hypothesis (H_1) is actually too strong an assumption, as was already pointed out in Murty and Murty (1984). A suitable analogue of the classical Rodosski and Tatuzawa would suffice for our purposes. It would be a fruitful investigation to find out if such an analogue exists for Artin L-series.

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