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General Section

On the normal number of prime factors of sums of Fourier coefficients of eigenforms ☆

M. Ram Murty [a,*], V. Kumar Murty [b], Sudhir Pujahari [c]

[a] *Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, K7L 3N6, Canada*
[b] *Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 2E4, Canada*
[c] *Department of Mathematics, University of Hong Kong, Hong Kong*

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A B S T R A C T

We study the normal number of prime factors of $a_f(p)+a_g(p)$ with p prime and f, g distinct Hecke eigenforms of weight two. Assuming a quasi-generalized Riemann hypothesis, we show that the normal order is $\log\log p$. We also obtain an estimate for the number of primes p for which $a_f(p) + a_g(p) = 0$.

1. Introduction

More than a hundred years ago, Hardy and Ramanujan [3] proved in 1917 that a random number n "usually" has $\log\log n$ distinct prime factors. To be precise, they showed that if $\omega(n)$ denotes the number of distinct prime factors of n, then for any $\epsilon > 0$, the number of $n \leq x$ that satisfy the inequality

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* Corresponding author.
E-mail addresses: murty@queensu.ca (M. Murty), murty@math.toronto.edu (V. Murty), spujahari@gmail.com (S. Pujahari).

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$$|\omega(n) - \log\log n| > (\log\log n)^{1/2+\epsilon}$$

is $o(x)$ as x tends to infinity. In 1934, Turán [16] gave a simpler proof of their theorem which ignited the rapid development of probabilistic number theory. We refer the reader to the monograph of Kubilius [4] for an account of this subject.

In 1984, the first two authors [8] initiated a study of the normal number of prime factors of the Ramanujan τ-function $\tau(p)$ evaluated at prime arguments. More generally, they studied $\omega(a_f(p))$ where $a_f(p)$ denotes the p-th Fourier coefficient of a normalized Hecke eigenform f of weight greater than or equal to two. Adopting the method of Turán and combining this with information supplied by the ℓ-adic representation $\rho_{\ell,f}$ attached to f by Deligne [1], they proved that

$$\sum_{\substack{p\le x,\\ a_f(p)\neq 0}} (\omega(a_f(p)) - \log\log p)^2 = O(\pi(x)\log\log x),$$

provided $a_f(p) \in \mathbb{Z}$ for all primes p and subject to a quasi-generalized Riemann hypothesis for certain Dedekind zeta functions. Here, $\pi(x)$ is the number of primes p up to x. This latter constraint of a quasi-generalized Riemann hypothesis is necessitated by the use of the effective Chebotarev density theorem due to Lagarias and Odlyzko [5] and refined by Serre [15]. The "quasi" generalized Riemann hypothesis refers to the assertion that the Dedekind zeta function $\zeta_K(s)$ attached to an algebraic number field K has no zeroes in the region $\mathrm{Re}(s) > 1 - \delta$ for some fixed δ with $1/2 < \delta < 1$. The application of the Chebotarev density theorem is enabled by Deligne's theorem that

$$\mathrm{tr}\,\rho_{\ell,f}(\sigma_p) = a_f(p)$$

for ℓ prime coprime to p and the level of f, and σ_p is the Frobenius automorphism attached to p in $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Deligne, in fact, established a more general result of which this is a corollary. For f a normalized Hecke eigenform of weight k on $\Gamma_0(N)$ with rational integer coefficients, he showed the existence of a representation ρ_f such that

$$\rho_f : G_{\mathbb{Q}} \to GL_2(\widehat{\mathbb{Z}}), \qquad \text{with } \mathrm{tr}\,\rho_f(\sigma_p) = a_f(p), \quad \det\rho_f(\sigma_p) = p^{k-1}.$$

Throughout this paper, we will assume our normalized eigenforms have rational integral Fourier coefficients. As initiated and developed by Serre, the existence of this representation, combined with knowledge of its image provided by the work of Momose [7], Ribet [14], and Loeffler [6] allows for an arithmetic study of the Fourier coefficients via the Chebotarev density theorem. For instance, if ℓ is a prime not dividing the level of f and a is a residue class $(\mathrm{mod}\,\ell)$, then the set of primes p for which

$$a_f(p) \equiv a (\mathrm{mod}\,\ell)$$

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has a (Chebotarev) density $\delta(\ell, a)$. This was the underlying philosophy of the arithmetical investigations leading to the normal order result of the first two authors in [8]. In a later paper [9], they extended these results and established an analogue of the Erdős-Kac theorem for the number of prime factors of Fourier coefficients of normalized Hecke eigenforms with integer coefficients.

Building on the work of Ribet [14] and Momose [7], Loeffler [6] has considered the image of the product representation associated to two or more eigenforms. In particular, Theorem 3.4.1 of [6] shows the following. If f and g are non-CM type normalized eigenforms of weights k_f and $k_g \geq 2$, then either the image of

$$\rho_f \times \rho_g : G_{\mathbb{Q}} \to GL_2(\widehat{\mathbb{Z}}) \times GL_2(\widehat{\mathbb{Z}})$$

is open or $k_f = k_g$ and f is a Galois conjugate of a twist of g (in other words, there exists an automorphism σ and a finite order Dirichlet character χ so that $f^\sigma = g \otimes \chi$). When the image is open, we will say "f and g are independent." This has the following arithmetic implication. If f and g are independent, then there is a constant $c(f, g)$ such that the density of primes p satisfying

$$a_f(p) \equiv a \pmod{\ell_1}, \qquad a_g(p) \equiv b \pmod{\ell_2}$$

is given by $\delta(\ell_1, a)\delta(\ell_2, b)$ where ℓ_1, ℓ_2 are primes (not necessarily distinct) greater than $c(f, g)$ and the density of primes with $a_f(p) \equiv a \pmod{\ell_1}$ is $\delta(\ell_1, a)$ and the density of primes with $a_g(p) \equiv b \pmod{\ell_2}$ is $\delta(\ell_2, b)$. We will use this result to study the number of prime factors of the sum $a_f(p) + a_g(p)$. In particular, we show that the set of primes p for which ℓ divides $a_f(p) + a_g(p)$ has a Chebotarev density $\delta(\ell)$. From the analysis of the relevant group theory, it will transpire that for f and g independent,

$$\delta(\ell) = \frac{1}{\ell} + O\left(\frac{1}{\ell^2}\right).$$

Thus, these algebraic theorems fit neatly into a probabilistic model that the sum of the Fourier coefficients $a_f(p) + a_g(p)$ can be viewed as a sum of two "independent" random variables so that one can make predictions using ideas from probabilistic number theory. These predictions can then be proved using tools from analytic and algebraic number theory with the effective Chebotarev density theorem being at the forefront of such tools. This paper is meant to initiate such a program of investigation.

Our main theorem is:

Theorem 1. *Suppose f and g are two independent normalized Hecke eigenforms, not of CM type, with integer coefficients, and of weight 2. Assuming a quasi-generalized Riemann hypothesis, we have*

$$\sum_{\substack{p \leq x, \\ a_f(p)+a_g(p) \neq 0}} \left(\omega(a_f(p) + a_g(p)) - \log\log x\right)^2 = O(\pi(x) \log\log x),$$

where the sum is over primes $p \leq x$.

In deriving these results, we will also need to count the number of primes $p \leq x$ for which $a_f(p) + a_g(p) = 0$ and this is a problem of independent interest. This question is treated in Section 4. In the next section, we derive a very general theorem applicable in a wide range of situations. It is a variation on the setting discussed in [8].

2. Some general theorems

Throughout, p will denote a prime number. Suppose we have a sequence of integers a_p with polynomial growth: for some fixed $A > 0$,

$$a_p = O(p^A). \tag{1}$$

We also assume that for some $\eta > 0$

$$\#\{p \leq x : a_p = 0\} = O(x^{1-\eta}). \tag{2}$$

For ℓ prime, define

$$\pi(x, \ell) := |\{p \leq x : a_p \equiv 0 \pmod{\ell}\}|$$

and suppose that there exists a $\delta(\ell)$ such that

$$\sum_{\ell < x^\eta} |\pi(x, \ell) - \delta(\ell)\pi(x)| \ll \frac{x}{\log x}. \tag{3}$$

Assume further that

$$\sum_{\ell} \left| \delta(\ell) - \frac{1}{\ell} \right| = O(1). \tag{4}$$

For a fixed z, let $\omega_z(n)$ denote the number of prime factors of n which are less than z. With $p \leq x$ and $z = x^\eta$, we have $\omega(a_p) = \omega_z(a_p) + O(1)$, by virtue of (1). It is now clear that

$$\sideset{}{'}\sum_{p \leq x} \omega(a_p) = \sideset{}{'}\sum_{p \leq x} \omega_z(a_p) + O(\pi(x)) = \sum_{\ell < x^\eta} \pi(x, \ell) + O(\pi(x)),$$

where the dash on the sum (here and henceforth) indicates that we sum over those primes p for which $a_p \neq 0$. Using (3), we find

$$\sideset{}{'}\sum_{p \leq x} \omega(a_p) = \sum_{\ell < x^\eta} \delta(\ell)\pi(x) + O(\pi(x)).$$

Applying (4), we get

$$\sideset{}{'}\sum_{p\le x}\omega(a_p)=\pi(x)\sum_{\ell<x^\eta}\frac{1}{\ell}+O(\pi(x)).$$

Since

$$\sum_{\ell<x^\eta}\frac{1}{\ell}=\log\log x+O(1),\tag{5}$$

this proves:

Theorem 2. *Assume that a sequence of integers a_p satisfies (1), (3) and (4). Then*

$$\sideset{}{'}\sum_{p\le x}\omega(a_p)=\pi(x)\log\log x+O(\pi(x)).$$

This theorem suggests that "on average", the number of prime factors of a_p is $\log\log p$. Using our axiomatic approach, we can move towards a second moment calculation if we further hypothesize the following. For distinct primes ℓ_1 and ℓ_2, let

$$\pi(x,\ell_1\ell_2):=|\{p\le x : a_p\equiv 0 \pmod{\ell_1\ell_2}\}|.$$

Assume that for some η_1 satisfying $0<\eta_1\le\eta/2$, we have

$$\sum_{\ell_1,\ell_2<x^{\eta_1},\ell_1\ne\ell_2}\left|\pi(x,\ell_1\ell_2)-\delta(\ell_1)\delta(\ell_2)\pi(x)\right|\ll\frac{x}{\log x}.\tag{6}$$

Letting $u=x^{\eta_1}$, we see that for non-zero a_p, the polynomial growth condition (1) implies $\omega(a_p)=\omega_u(a_p)+O(1)$ so that

$$\sideset{}{'}\sum_{p\le x}\omega(a_p)^2=\sideset{}{'}\sum_{p\le x}(\omega_u(a_p)+O(1))^2=\sideset{}{'}\sum_{p\le x}\omega_u(a_p)^2+\sideset{}{'}\sum_{p\le x}O(\omega_u(a_p))+O(\pi(x)).$$

By Theorem 2, we see that the last sum is $O(\pi(x)\log\log x)$ so that

$$\sideset{}{'}\sum_{p\le x}\omega(a_p)^2=\sideset{}{'}\sum_{p\le x}\omega_u(a_p)^2+O(\pi(x)\log\log x).$$

We immediately see that

$$\sideset{}{'}\sum_{p\le x}\omega_u(a_p)^2=\sum_{\ell_1,\ell_2\le u}\pi(x,[\ell_1,\ell_2])+O(\pi(x)),$$

where $[\ell_1,\ell_2]$ denotes the least common multiple of ℓ_1 and ℓ_2. When $\ell_1=\ell_2$, the sum is $O(\pi(x)\log\log x)$ by our earlier calculation. Thus,

$$\sum_{p\le x}{}' \omega_u(a_p)^2 = \sum_{\ell_1,\ell_2\le u,\ell_1\ne\ell_2} \pi(x,\ell_1\ell_2) + O(\pi(x)\log\log x).$$

Injecting (6) in the right hand side shows that the sum is

$$\sum_{\ell_1,\ell_2\le u,\ell_1\ne\ell_2} \delta(\ell_1)\delta(\ell_2)\pi(x) + O(\pi(x)\log\log x).$$

Dropping the condition that $\ell_1 \ne \ell_2$ in the first sum and noting the error introduced, we find that the sum is

$$\pi(x)\left(\sum_{\ell\le u}\delta(\ell)\right)^2 + O(\pi(x)\log\log x).$$

Using (5) gives the final estimate which we record in the theorem below.

Theorem 3. *Assume that the sequence of integers a_p satisfies (1), (3), (4) and (6). Then,*

$$\sum_{p\le x}{}' \omega(a_p)^2 = \pi(x)(\log\log x)^2 + O(\pi(x)\log\log x).$$

Using the technique of Chebycheff's inequality in probability theory, we are now in a position to deduce the normal number of prime factors of a_p in the following:

Corollary 4. *Assume that the sequence of integers a_p satisfies (1), (3), (4) and (6). Fix $\epsilon > 0$. The number of primes $p \le x$ for which $a_p \ne 0$ and*

$$|\omega(a_p) - \log\log x| > (\log\log x)^{1/2+\epsilon}$$

is $O(\pi(x)/(\log\log x)^{2\epsilon})$.

Proof. Indeed, we combine all of our estimates and deduce

$$\sum_{p\le x}{}' (\omega(a_p) - \log\log x)^2 = O(\pi(x)\log\log x).$$

From this, the number of primes p for which $|\omega(a_p) - \log\log x| > (\log\log x)^{1/2+\epsilon}$ is $O(\pi(x)/(\log\log x)^{2\epsilon})$. □

In other words, under the general axiomatization encoded in (1), (3), (4) and (6), we have a normal order theorem. In the next section, we will apply this to study the normal number of prime factors of sums of Fourier coefficients of Hecke eigenforms with integer coefficients.

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3. Preliminary results

We record in this section various preliminary results that will be used in the course of our discussion in later sections. Let $\mathbb{F}_\ell$ denote the finite field of ℓ elements. We will need to study the group

$$G_\ell := \{(u, u') \in GL_2(\mathbb{F}_\ell) \times GL_2(\mathbb{F}_\ell) : \det u = \det u'\}.$$

Our first goal is to determine the size of G_ℓ. This is easily done by noting that the map

$$GL_2(\mathbb{F}_\ell) \times GL_2(\mathbb{F}_\ell) \to \mathbb{F}_\ell^*$$

given by

$$(u, u') \mapsto \frac{\det u}{\det u'},$$

is a surjective map whose kernel is precisely G_ℓ. Since $|GL_2(\mathbb{F}_\ell)| = (\ell^2 - 1)(\ell^2 - \ell)$, we deduce the following.

Lemma 5.

$$|G_\ell| = \ell^2(\ell + 1)^2(\ell - 1)^3 = \ell^7 + O(\ell^6).$$

We will need to study the size of the following subset of G_ℓ:

$$C_\ell := \{(u, u') \in GL_2(\mathbb{F}_\ell) \times GL_2(\mathbb{F}_\ell) : \operatorname{tr} u + \operatorname{tr} u' = 0, \quad \det u = \det u'\}.$$

To this end, we need to study the conjugacy classes of $GL_2(\mathbb{F}_\ell)$. The structure of these classes (along with the conventional names assigned to each type of class) is summarized in Table 1 (see for example, p. 68 of [2]).

We will say that the central, parabolic, hyperbolic and elliptic conjugacy classes are of type 1, 2, 3 and 4 respectively. It is also useful to recall that the matrices

$$\begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} \text{ and } \begin{pmatrix} d & 0 \\ 0 & b \end{pmatrix}$$

are conjugate in $GL_2(\mathbb{F}_\ell)$ as are the matrices

$$\begin{pmatrix} c & b\epsilon \\ b & c \end{pmatrix} \text{ and } \begin{pmatrix} c & -b\epsilon \\ -b & c \end{pmatrix}.$$

This observation accounts for the factor 1/2 in the column in Table 1 giving the number of such classes.

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Table 1
Conjugacy classes of $GL_2(\mathbb{F}_\ell)$.

conjugacy class rep.	# of such classes	# of elements in each class
$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $a \neq 0$ (central)	$\ell - 1$	1
$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, $a \neq 0$ (parabolic)	$\ell - 1$	$\ell^2 - 1$
$\begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix}$ with $b \neq d$, (hyperbolic)	$\frac{1}{2}(\ell-1)(\ell-2)$	$\ell^2 + \ell$
$\begin{pmatrix} c & b\epsilon \\ b & c \end{pmatrix}$ with $\left(\frac{\epsilon}{\ell}\right) = -1$ and $b \neq 0$, (elliptic)	$\frac{1}{2}\ell(\ell-1)$	$\ell^2 - \ell$

We introduce the 4×4 matrix C with entry c_{ij} defined as follows. Given a class U of type i, c_{ij} is the number of classes U' of type j such that $U \times U' \subseteq C_\ell$. The structure of this matrix is summarized in the lemma below.

Lemma 6.

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. As the matrix is symmetric, there is not too much to check. The diagonal entries are clear (and easy to verify). Indeed, if u is of type i, and $(u, u') \in C_\ell$ with u' of type i, then the Jordan form of u' (up to conjugacy) is uniquely determined. Thus, the diagonal entries of C are all 1. If (u, u') belongs to C_ℓ with u of type 1 and u' of type 2, we see that

$$u = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \implies u' = \begin{pmatrix} -a & 1 \\ 0 & -a \end{pmatrix}.$$

Thus $c_{12} = c_{21} = 1$. Let us look at c_{13}. Suppose we have

$$u = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad u' = \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix}$$

and $(u, u') \in C_\ell$. Then $2a = b + d$ and $a^2 = bd$ which implies that $b = d$, a contradiction. Similarly, $c_{23} = 0$. To determine c_{14} suppose

$$u = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad u' = \begin{pmatrix} c & b\epsilon \\ b & c \end{pmatrix}$$

and $(u, u') \in C_\ell$. Then $c = -a$ and $a^2 = c^2 - \epsilon b^2$ implies that $b = 0$, a contradiction. Similarly $c_{24} = 0$. Finally, we need to check c_{34}. To this end, suppose

$$u = \begin{pmatrix} b' & 0 \\ 0 & d \end{pmatrix}, \quad u' = \begin{pmatrix} c & b\epsilon \\ b & c \end{pmatrix}$$

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and $(u, u') \in C_\ell$. Then, $b' + d = -2c$ and $b'd = c^2 - \epsilon b^2$. This implies $4\epsilon b^2 = (b' - d)^2$, which is a contradiction since ϵ is not a square. This proves $c_{34} = 0$. $\square$

It is now easy to determine the size of C_ℓ using Lemma 6. Indeed, letting n_i be the size of any conjugacy class of type i and m_i the number of classes of type i, we have

$$\sum_{i,j} c_{ij} m_i n_i n_j$$

as the size of C_ℓ. The relevant information is found in Table 1 and we see the main contribution comes from c_{33} and c_{44}. We record the final result in the following.

Lemma 7.

$$|C_\ell| = \ell^6 + O(\ell^5).$$

Proof. The contributions from c_{33} and c_{44} are

$$\frac{1}{2}(\ell - 1)(\ell - 2)(\ell^2 + \ell)^2 + \frac{1}{2}\ell(\ell - 1)(\ell^2 - \ell)^2 = \ell^6 + O(\ell^5).$$

The contributions from c_{11} and c_{12} are

$$(\ell - 1) + (\ell^2 - 1)^2(\ell - 1) = O(\ell^5).$$

The two off-diagonal terms c_{12} and c_{21} contribute

$$2(\ell - 1)(\ell^2 - 1).$$

The final tally is therefore $\ell^6 + O(\ell^5)$. $\square$

We combine these results and record for future reference the following.

Lemma 8.

$$\delta(\ell) := \frac{|C_\ell|}{|G_\ell|} = \frac{1}{\ell} + O\left(\frac{1}{\ell^2}\right).$$

Proof. Combining Lemmas 7 and 5, we have

$$\delta(\ell) = \frac{\ell^6 + O(\ell^5)}{\ell^7 + O(\ell^6)} = \frac{1}{\ell} + O\left(\frac{1}{\ell^2}\right). \quad \square$$

The following versions of the effective Chebotarev density theorem derived in [5], [15] and [11] will be used at various places in this paper and so we record it below for ease of reference.

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Proposition 9. *Let L/K be a finite Galois extension of number fields with group G. Let C be a subset of G stable under conjugation. Denote by n_K the degree $[K : \mathbb{Q}]$, d_K the discriminant of K and $\pi_C(x) = \pi_C(x, L/K)$ the number of prime ideals $\mathfrak{p}$ of K unramified in L, with norm less than or equal to x satisfying $\sigma_\mathfrak{p} \in C$ (where $\sigma_\mathfrak{p}$ is the Frobenius automorphism of $\mathfrak{p}$ attached to L/K). Let $P(L/K)$ be the set of rational primes p for which there is a prime ideal $\mathfrak{p}$ in K which ramifies in L and $\mathfrak{p}|p$. Set*

$$M(L/K) := [L : K]|d_K|^{1/n_K} \prod_{p \in P(L/K)} p.$$

Assuming the Riemann hypothesis holds for the Dedekind zeta function $\zeta_L(s)$, we have

(a) *We have*

$$\pi_C(x) = \frac{|C|}{|G|}\pi(x) + O\left(|C|x^{1/2} n_K \log(M(L/K)x)\right).$$

(b) *If in addition all the Artin L-series attached to irreducible representations of G are holomorphic at $s \neq 1$, then*

$$\pi_C(x) = \frac{|C|}{|G|}\pi(x) + O\left(|C|^{1/2} x^{1/2} n_K \log(M(L/K)x)\right).$$

The quantity $M(L/K)$ that appears in the above estimates is an expression involving ramified primes. There are several ways of estimating it. For example, when L/K is Galois, using the estimates provided on page 259 of [11] or on page 129 of [15], we have

$$\log M(L/K) \ll \log[L : \mathbb{Q}] + \sum_{p \in P(L/K)} \log p$$

where the implied constant is absolute. We will also be implicitly using this result in various estimations below.

Some remarks are in order. Lagarias and Odlyzko [5] first obtained a version of the effective Chebotarev density theorem assuming the generalized Riemann hypothesis. This was further refined by Serre [15] and the version in (a) is due to him. The improvement (b) arising from the additional assumption of the Artin holomorphy conjecture was derived by the authors in [11] and the form we have written down is on page 266 of this paper. In particular, if L/K is abelian, the Artin holomorphy conjecture holds and so the improved version in (b) is valid on the assumption of the generalized Riemann hypothesis alone. We will use this remark below.

Finally in this section, we recall two important observations which are used in the next section. These are recorded in Proposition 8 of [15] and allow us, under some hypotheses, to replace $\mathrm{Gal}(L/K)$ with a subquotient.

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They are also given in Proposition 3.9 and Proposition 3.12 of [11] in a form that is more suitable for estimation of error terms. For the first observation, let H be a subgroup of $G = \mathrm{Gal}(L/K)$ and let C be a conjugacy class of G having non-trivial intersection with H. Let C_H be a conjugacy class of H in this intersection, and let K' denote the subfield of L fixed by H. Note that $C \cap H$ might consist of several conjugacy classes and our calculation is independent of which one is chosen for C_H. Then, let us set

$$\lambda \ = \ \frac{|C|/|G|}{|C_H|/|H|},$$

and note that as $|C| \leq |C_H| \cdot [G : H]$, we have $\lambda \leq 1$. Then, we have (see [11] and [13])

$$\pi_C(x, L/K) \ = \ \lambda \pi_{C_H}(x, L/K') \ + \ O(n_{K'} x^{1/2} \log M(L/K)x). \tag{7}$$

Combining this with Proposition 9(b) above, we deduce that assuming the GRH and Artin's conjecture for L/K, we have

$$\pi_C(x, L/K) \ = \frac{|C|}{|G|}\pi(x) \ + \ O(|C_H|^{1/2} x^{1/2} n_{K'} \log(M(L/K)x)). \tag{8}$$

Recall that the $M(L/K)$ term is as in Proposition 9.

It is worthwhile making explicit a version of (7) in which C is replaced by a union D of conjugacy classes. Let us suppose that each conjugacy class C in D has non-trivial intersection with H and as above, let C_H denote a conjugacy class in $C \cap H$ (noting again that there may be more than one, and that our calculations are independent of the choice of C_H). Then [11], pp. 266-267 shows that

$$\pi_D(x, L/K) \ = \ \sum_{C \subseteq D} \frac{|C|}{|G|} \frac{|H|}{|C_H|} \pi_{C_H}(x, L/K') + O\left(\left(\max \frac{|C|}{|C_H|} \right) \left(n_K x^{\frac{1}{2}} + \frac{1}{|G|} \log d_L \right) \right). \tag{9}$$

For the second observation, let H be a normal subgroup and suppose that C is a conjugacy class of G. Then, the image $\overline{C}$ of C in G/H is a conjugacy class of G/H. In general, the converse is false: the pull-back C of a conjugacy class $\overline{C}$ in G/H need not be stable under conjugation in G. For this, we need to impose the condition $HC \subseteq C$. Note that this implies that C is a union of H-cosets and so $|C| \ = \ |\overline{C}| \cdot |H|$, or equivalently, $|\overline{C}|/|G/H| \ = \ |C|/|G|$. With this condition in place, let N be the fixed field of H. Then, we have

$$\pi_C(x, L/K) \ = \ \pi_{\overline{C}}(x, N/K) + \ O(n_K x^{1/2} \log M(L/K)x). \tag{10}$$

This holds good even if C is a union of conjugacy class satisfying $HC \subseteq C$.

Combining this with Proposition 9(b) above, we deduce that assuming the GRH and Artin's conjecture for N/K, we have

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$$\pi_C(x, L/K) = \frac{|C|}{|G|}\pi(x) + O(|\overline{C}|^{1/2}x^{1/2}n_K \log M(L/K)x). \tag{11}$$

As noted earlier, if L/K is abelian, then the assumption of the Artin conjecture can be dispensed with. We will utilize these remarks in the next section.

4. Applications to modular forms of weight two

Let us now consider the case of two normalized Hecke eigenforms f and g of weights k_1 and k_2 and levels N_f, N_g respectively, with integer coefficients and without complex multiplication. Let us further assume that f and g are independent in the sense of Loeffler [6], Ribet [14] (in the case of level 1) and Momose [7] (for higher level). From their work, we see that the image of the map

$$\rho_f \times \rho_g : G_{\mathbb{Q}} \to GL_2(\widehat{\mathbb{Z}}) \times GL_2(\widehat{\mathbb{Z}})$$

is open. Since

$$\widehat{\mathbb{Z}} = \prod_\ell \mathbb{Z}_\ell,$$

this means that for ℓ sufficiently large, we have a map

$$\rho_{\ell,f} \times \rho_{\ell,g} : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_\ell) \times GL_2(\mathbb{Z}_\ell)$$

whose image equals

$$\{(u, u') : \det u = v^{k_1-1}, \det u' = v^{k_2-1} \text{ for some } v \in \mathbb{Z}_\ell^*\}.$$

Recall that if σ_p denotes the Frobenius automorphism attached to p in $G_{\mathbb{Q}}$, then

$$a_f(p) = \operatorname{tr} \rho_{\ell,f}(\sigma_p), \quad a_g(p) = \operatorname{tr} \rho_{\ell,g}(\sigma_p),$$

where $a_f(p)$ and $a_g(p)$ are the p-th Fourier coefficients of f and g respectively.

In the context above, all this discussion implies the existence of Galois extensions $K_{\ell,f,g}$ such that $\mathrm{Gal}(K_{\ell,f,g}/\mathbb{Q})$ is isomorphic to $G_\ell(k_1, k_2)$ which is

$$\{(u, u') \in GL_2(\mathbb{F}_\ell) \times GL_2(\mathbb{F}_\ell) : \det u = v^{k_1-1}, \det u' = v^{k_2-1} \text{ for some } v \in \mathbb{F}_\ell^*\}.$$

In addition, the ramified primes of the field $K_{\ell,f,g}$ are contained in the primes dividing $\ell N_f N_g$ by the work of [6]. This also follows from results of Deligne [1] that are highlighted on page 176 of [15] because the product representation $\rho_{\ell,f} \times \rho_{\ell,g}$ is unramified outside of $\ell N_f N_g$.

In this section, we will study the case $k_1 = k_2 = 2$. Then $G_\ell(2, 2)$ is the group G_ℓ discussed in the earlier section. We want to apply the discussion of the previous sections

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to study the normal number of prime factors of $a_f(p) + a_g(p)$. But before we do this, we need to eliminate those primes p for which $a_f(p) + a_g(p) = 0$. To this end, let us study how often $a_f(p) + a_g(p) = 0$, which is a problem of independent interest. This problem can be approached in various ways. One way is to apply the effective Chebotarev density theorem [5]. Applying unconditional results gives weak results that are not sufficient for our application. In this context, we refer the reader to [13] for a discussion of such unconditional results.

What we need is the conditional effective Chebotarev density theorem which assumes both the generalized Riemann hypothesis and the Artin holomorphy conjecture as in [11]. One can also derive variations of the effective Chebotarev density theorem assuming only a quasi-generalized Riemann hypothesis. We do not pursue it here but proceed assuming both the Artin holomorphy conjecture and the generalized Riemann hypothesis. Under such assumptions, we have that the number of primes $p \le x$ for which

$$a_f(p) + a_g(p) \equiv 0 (\operatorname{mod} \ell)$$

is

$$S_\ell(x) := \frac{|C_\ell|}{|G_\ell|}\pi(x) + O(|C_\ell|^{1/2} x^{1/2} \log \ell N x), \tag{12}$$

where C_ℓ is the set of elements $(u, u') \in G_\ell$ such that $\operatorname{tr} u + \operatorname{tr} u' \equiv 0 (\operatorname{mod} \ell)$. Here, the implied constant is absolute and $N = \max(N_f, N_g)$.

Clearly, the number of primes $p \le x$ for which $a_f(p) + a_g(p) = 0$ is bounded by $S_\ell(x)$ for any value of ℓ. The idea then is to choose a suitable value of ℓ to minimize the estimate. But in order to apply (12) to derive an estimate, we need to have some idea of the size of G_ℓ and C_ℓ. This was done in the previous section. Injecting those results here, we have

$$S_\ell(x) = \delta(\ell)\pi(x) + O(\ell^3 x^{1/2} \log \ell N x),$$

where

$$\delta(\ell) = \frac{1}{\ell} + O\left(\frac{1}{\ell^2}\right). \tag{13}$$

To apply our formalism, we need to eliminate from this count the number of primes $p \le x$ for which $a_f(p) + a_g(p) = 0$. For this purpose, we can use (13) to get an estimate. Indeed, choosing a prime ℓ in the interval

$$\left[\frac{x^{1/8}}{(\log x)^{1/4}(\log Nx)^{1/4}}, \frac{2x^{1/8}}{(\log x)^{1/4}(\log Nx)^{1/4}}\right]$$

which we can do by Bertrand's postulate, we derive an estimate of

$$\ll \; x^{7/8}\left(\frac{\log Nx}{(\log x)^3}\right)^{1/4}$$

for the number of such primes. The implied constant is absolute.

However, this estimate has the defect that it assumes both the generalized Riemann hypothesis *and* the Artin holomorphy conjecture. Using a variation of an argument first introduced in [11], we will show in the next theorem that the use of the Artin holomorphy conjecture can be eliminated. In fact, by making use of various abelian extensions, we will derive the following improvement.

Theorem 10. *Suppose that f and g are independent normalized Hecke eigenforms of weight 2 with integer coefficients and assume the generalized Riemann hypothesis for Dedekind zeta functions. Then, the number of primes $p \leq x$ for which*

$$a_f(p) + a_g(p) = 0$$

is

$$\ll \; x^{6/7}\left(\frac{(\log Nx)^2}{(\log x)^5}\right)^{1/7}$$

where the implied constant is absolute and effective. In particular, for $x \gg_N 1$, this is

$$\ll \; x^{6/7}.$$

Proof. We modify a technique used on pages 271-273 of [11]. Denote by $V(x)$ the number of primes $p \leq x$ for which $a_f(p) + a_g(p) = 0$. The goal is to show, assuming the Riemann Hypothesis for Dedekind zeta functions, that we have

$$V(x) \; \ll \; x^{6/7}\left(\frac{(\log Nx)^2}{(\log x)^5}\right)^{1/7}$$

For a prime ℓ, set $\pi_\ell(x)$ to be the number of primes $p \leq x$ for which

$$a_f(p) + a_g(p) \; \equiv \; 0 \bmod \ell.$$

Also set M_p (resp. N_p) to be the field $\mathbb{Q}(\sqrt{a_f(p)^2 - 4p})$ (resp. $\mathbb{Q}(\sqrt{a_g(p)^2 - 4p})$). These fields are imaginary quadratic over $\mathbb{Q}$ by Hasse's inequality.

Following the line argument on pages 269-270 of [11], we consider the sum

$$S(y,x) \; = \sum_{\substack{p \leq x \\ a_f(p)+a_g(p)=0}} \; \sum_{\substack{y \leq \ell \leq 2y \\ \ell \text{ splits in } M_p \text{ and } N_p}} 1$$

with y an auxiliary parameter to be chosen. Since a positive density (in fact $\geq 1/4$) of primes ℓ satisfy the splitting conditions in the inner sum, we see that

$$S(y,x) \gg \pi(y)V(x). \tag{14}$$

On the other hand, interchanging the summation, we see that

$$S(y,x) = \sum_{y\le \ell\le 2y} \sum_{\substack{p\le x\\ a_f(p)+a_g(p)=0\\ \ell \text{ splits in } M_p \text{ and } N_p}} 1.$$

If we weaken the condition in the inner sum to $a_f(p) + a_g(p) \equiv\ 0 \bmod \ell$, then we see that

$$S(y,x)\ \le \sum_{y\le \ell\le 2y} \sum_{\substack{p\le x\\ a_f(p)+a_g(p)\equiv 0 \bmod \ell\\ \ell \text{ splits in } M_p \text{ and } N_p}} 1. \tag{15}$$

In particular, putting (14) and (15) together, we deduce that

$$V(x)\ \ll\ \max_{y\le \ell\le 2y} \sum_{\substack{p\le x\\ a_f(p)+a_g(p)\equiv 0 \bmod \ell\\ \ell \text{ splits in } M_p \text{ and } N_p}} 1. \tag{16}$$

Now let us consider the inner sum in some detail. The fact that ℓ splits in M_p and N_p means that the characteristic polynomial of the Frobenius at p factors (mod ℓ). This means that the Frobenius at p in

$$G_\ell\ =\ G_\ell(2,2)$$

is contained in $C_\ell \cap B_\ell$, where B_ℓ is the product subgroup

$$B_\ell := \{(u,u') \in B(\mathbb{F}_\ell) \times B(\mathbb{F}_\ell) : \det(u) = v, \det(u') = v,\ v \in \mathbb{F}_\ell^\times\}$$

where $B(\mathbb{F}_\ell)$ denotes the Borel subgroup of $GL_2(\mathbb{F}_\ell)$ consisting of upper triangular matrices.

Note that $B(\mathbb{F}_\ell)$ has size $\ell(\ell-1)^2$. It is easy to see (using the argument we had used above to determine the size of G_ℓ) that B_ℓ is a subgroup of G_ℓ of size $\ell^2(\ell-1)^3$. The idea now is to invoke (9) to move the problem to K/L where L is the fixed field of B_ℓ.

The conjugacy set C_ℓ is the union of $\asymp \ell^2$ conjugacy classes of G_ℓ, each of size $\asymp \ell^4$. Each conjugacy class in $C_\ell \cap B_\ell$ has size $\asymp \ell^2$. Therefore, if we apply (9) for the extension $K/\mathbb{Q}$ and $D = C_\ell$, then the error term is $O(\ell^2 x^{1/2}\log \ell)$. Moreover, the main term is

$$\sum_{C\subseteq C_\ell} \frac{|C|}{|G_\ell|}\frac{|B_\ell|}{|\Xi_C|}\pi_{\Xi_C}(x, K/L)$$

where Ξ_C is a conjugacy class in $C \cap B_\ell$. Noting that

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$$\frac{|C|}{|G_\ell|}\frac{|B_\ell|}{|\Xi_C|} \ll 1$$

we may majorize the above by

$$\sum_{C\subseteq C_\ell} \pi_{\Xi_C}(x, K/L) = \pi_{\Xi}(x, K/L)$$

where Ξ is the union of the Ξ_C. Thus,

$$\pi_{C_\ell}(x, K/\mathbb{Q}) \ll \pi_{\Xi}(x, K/L) + \ell^2 x^{1/2}\log \ell N.$$

Note that $|\Xi| \asymp \ell^4$.

Denote by Z_ℓ the subgroup of B_ℓ consisting of the scalar matrices $(\lambda I, \lambda I)$ with $\lambda \in \mathbb{F}_\ell^\times$ and denote by U_ℓ the subgroup of B_ℓ consisting of unipotent matrices. Both Z_ℓ and U_ℓ are normal subgroups of B_ℓ and we may consider the normal subgroup $H_\ell = Z_\ell U_\ell$. We see that H_ℓ has order $\ell^2(\ell-1) \asymp \ell^3$. Let J be its fixed field. Then, J/L is abelian with Galois group B_ℓ/H_ℓ which has size $(\ell-1)^2$.

Since $H_\ell\Xi \subseteq \Xi$, we may invoke (11), and in fact, count the primes of L whose Frobenius in J lies in the image $\overline{\Xi}$ of Ξ in $\mathrm{Gal}(J/L)$. The condition $H_\ell\Xi \subseteq \Xi$ implies that Ξ is a union of H_ℓ cosets and so,

$$|\overline{\Xi}| = |\Xi|/|H_\ell| \asymp \ell.$$

Applying the Chebotarev density theorem (Proposition 9) to the abelian extension J/L, we see that the number of primes $\mathfrak{p}$ of L with $\mathrm{Norm}(\mathfrak{p}) \le x$ which are unramified in J and for which and $\sigma_{\mathfrak{p}} \in \overline{\Xi}$ is

$$\ll \frac{\ell}{(\ell-1)^2}\pi(x) + O(\ell^2\ell^{1/2}x^{1/2}\log \ell N x). \tag{17}$$

This estimates the size of the inner sum in (16). With this estimate in hand, we choose

$$y = \frac{x^{1/7}}{(\log x)^{2/7}(\log Nx)^{2/7}}.$$

Then for any prime $\ell \in [y, 2y]$, we see that (17) is

$$\ll \pi(x)/y \tag{18}$$

Inserting this into (16), the proof is complete. □

If we only assume a quasi-generalized Riemann hypothesis, we deduce an estimate of the form $O(x^\theta)$ with some $\theta < 1$ for the number of primes p for which $a_f(p) + a_g(p) = 0$. This is actually sufficient for our purposes. Much weaker conditions can be formulated and these are discussed at the end of the paper.

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It is also worth remarking that the estimate of Theorem 10 is also valid for any counting of the number of primes $p \leq x$ such that $ra_f(p) + sa_g(p) = 0$ for any fixed integers r and s. In particular, we have an estimate for the number of primes $p \leq x$ for which $a_f(p) = a_g(p)$.

5. Proof of Theorem 1

We are now in a position to apply our axiomatic treatment. We let $\delta(\ell) = |C_\ell|/|G_\ell|$, then by our calculation above $\delta(\ell) = 1/\ell + O(1/\ell^2)$. Let $\pi(x, \ell)$ be the number of primes $p \leq x$ for which $a_f(p)+a_g(p)$ is non-zero and divisible by ℓ. From the preceding discussion, it is clear that if we assume a quasi-generalized Riemann hypothesis, we have for some $\eta > 0$, that

$$\sum_{\ell < x^\eta} |\pi(x, \ell) - \delta(\ell)\pi(x)| \ll \frac{x}{\log x}.$$

Thus all the hypotheses of Theorem 2 are satisfied and we can deduce:

Theorem 11. *Let f and g be independent, normalized Hecke eigenforms of weight 2 which are not of CM type and with integer Fourier coefficients. Assuming a quasi-generalized Riemann hypothesis, we have*

$$\sum_{p \leq x}{}' \omega(a_f(p) + a_g(p)) = \pi(x) \log\log x + O\left(\frac{x}{\log x}\right),$$

where the dash on the summation is over primes p such that $a_f(p) + a_g(p) \neq 0$. Unconditional results for this cognate problem were also studied in some earlier works of the authors, namely [10] and [12].

This theorem gives us the knowledge of the average number of prime factors of $a_f(p)+ a_g(p)$. To establish the normal order, we need to verify that conditions of Theorem 3 are satisfied. This means that we need to study for distinct primes ℓ_1, ℓ_2, the quantity $\pi(x, \ell_1\ell_2)$ which is the number of primes $p \leq x$ for which $\ell_1\ell_2$ divides $a_f(p) + a_g(p)$. From our earlier discussion, this is straightforward since by the theorem of Loeffler [6], Ribet [14] and Momose [7], the image of the representation

$$\rho_f \times \rho_g : G_{\mathbb{Q}} \to GL_2(\widehat{\mathbb{Z}}) \times GL_2(\widehat{\mathbb{Z}})$$

is open for f and g independent and of non-CM type. An application of the Chebotarev density theorem (assuming GRH and the Artin holomorphy conjecture) gives analogously as before,

$$\pi(x, \ell_1\ell_2) = \delta(\ell_1)\delta(\ell_2)\pi(x) + O((\ell_1\ell_2)^3 x^{1/2} \log N\ell_1\ell_2 x).$$

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The key point is that the Galois group to consider now is simply $G_{\ell_1} \times G_{\ell_2}$. Indeed, we apply Proposition 9 to the Galois extension $K_{\ell,f,g}$ as in Section 4. If we remove from this set the primes $p \leq x$ for which $a_f(p) + a_g(p) = 0$ and denote the number remaining by $\pi^*(x, \ell_1\ell_2)$, then we have (assuming a quasi-GRH) that for some $\eta_1 > 0$

$$\sum_{\ell_1,\ell_2,<x^{\eta_1},\ell_1\neq\ell_2} |\pi^*(x,\ell_1\ell_2) - \delta(\ell_1)\delta(\ell_2)\pi(x)| \ll \frac{x}{\log x}. \tag{19}$$

Thus, from our axiomatic formulation, we immediately deduce Theorem 1 for weight two modular forms.

6. Concluding remarks

Our assumptions regarding the integrality of Fourier coefficients of eigenforms of weight 2 means that we have modular forms attached to elliptic curves. Thus, our theorem can be viewed as giving the normal order of the number of prime factors of $a_p(E_1) + a_p(E_2)$ for two non-isogenous elliptic curves E_1 and E_2. (Here, $a_p(E)$ has the usual meaning: $p + 1 - a_p(E)$ is the number of points of $E \pmod p$.)

One can easily extend our results to the study of the normal number of prime factors of $P(a_f(p), a_g(p))$ where $P(x, y) \in \mathbb{Z}[x, y]$. This will introduce a minor form of complexity that can be treated by our methods. We chose not to do it here for the sake of elucidating the essential idea of our method.

As noted several times, the assumption of the quasi-generalized Riemann hypothesis can be weakened considerably. Some of these relaxations have been alluded to in our earlier work [8]. However, even these relaxed conditions are out of reach of our current knowledge of the analytic theory of Artin L-functions. Still, one can deduce unconditional theorems that are applicable in a wide range of problems by following the method of [10]. Using recent advances proving the automorphy of symmetric power L-functions, it should be possible to improve the results of [10] even further to derive estimates similar to those that arise in the classical prime number theorem.

One can generalize this work to higher weights. In this context, the problem branches into two different problems, namely the case when f and g have the same weight and the case when they have different weights. One can expect similar results in these cases also.

Another line of research is to move towards a theorem of Erdős-Kac type. At least in the case f and g both have weight two, it should be possible to derive such a theorem using some of the results obtained in this paper. We plan to return to these questions later.

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