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Oscillations of Fourier Coefficients of Modular Forms

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1. Introduction

Let f be a normalized Hecke eigenform of weight k for the full modular group and let $a(n)$ denote its n^{th} Fourier coefficient. Let us write, for each prime p,

$$a(p)=2p^{\left(\frac{k-1}{2}\right)}\cos\theta(p).$$

Since we know the truth of the Ramanujan-Petersson conjecture, it follows that the $\theta(p)$'s are real.

Inspired by the Sato-Tate conjecture for elliptic curves, Serre [14] conjectured that the $\theta(p)$'s are uniformly distributed in the interval $[0,\pi]$ with respect to the measure $\frac{1}{\pi}\sin^2\theta d\theta$. Following Serre, we shall refer to this as the Sato-Tate conjecture, there being no room for confusion.

It has been known for a long time that the truth of this conjecture implies much about the oscillatory behaviour of the Fourier coefficients. In particular, the following is implied by the Sato-Tate conjecture.

Theorem 1. *For any normalized Hecke eigenform,*

$$a(n)=\Omega_{\pm}\left(n^{\left(\frac{k-1}{2}\right)}\exp\left(\frac{c\log n}{\log\log n}\right)\right)$$

for some $c>0$.

Theorem 1 has a long history. Hardy [2] proved that

$$a(n)=\Omega\left(n^{\frac{k-1}{2}}\right)$$

and Rankin [11] showed

$$\limsup_{n\to\infty}\frac{|a(n)|}{n^{\left(\frac{k-1}{2}\right)}}=+\infty.$$

Then Joris [5] showed that for $\delta_k = 6/k^2$, we have

$$a(n) = \Omega\left(n^{\frac{k-1}{2}} \exp(c(\log n)^{\delta_k - \varepsilon})\right).$$

This was improved in [1] to $\delta_k = 1/k$. In the special case of $k = 12$, the authors in [1] showed

$$\tau(n) = \Omega\left(n^{\frac{11}{2}} \exp\left(c(\log n)^{\frac{2}{3} - \varepsilon}\right)\right)$$

for the Ramanujan τ-function. This was improved upon slightly in [8]. For an arbitrary cusp form, which is not necessarily an eigenfunction, we have:

Theorem 2. *Let*

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

be a cusp form of weight k for the full modular group. Then

$$a(n) = \Omega\left(n^{\frac{k-1}{2}} \exp\left(\frac{c \log n}{\log\log n}\right)\right)$$

for some $c_1 > 0$.

In view of Deligne's result, Theorems 1 and 2 reflect the best possible result, apart from the values of c and c_1.

Theorem 2 has an interesting application to Kloosterman sums. Define

$$S(n, m, c) = \sum_{\substack{a(c) \\ a\bar{a} \equiv 1(c)}} e^{\frac{2\pi i}{c}(na + m\bar{a})}.$$

Selberg [13] and Linnik [6] have independently conjectured that the sum, for $x \geqq (m, n)^{\frac{1}{2} + \varepsilon}$,

$$G(x) = \sum_{c \leqq x} \frac{S(n, m, c)}{c}$$

is $O(x^{\varepsilon})$. We deduce from Theorem 2:

Corollary. $G(x) = \Omega\left(\exp\left(\frac{c \log x}{\log\log x}\right)\right)$.

The main ingredient in the proof of Theorems 1 and 2 is a recent result of Shahidi [15] which we now describe.

Define for each integer $m \geqq 1$,

$$L_m(s) = \prod_p \prod_{j=0}^{m} \left(1 - \frac{e^{i\theta(p)(m-2j)}}{p^s}\right)^{-1}.$$

Clearly, $L_m(s)$ converges for $\mathrm{Re}\, s > 1$. It is conjectured that each $L_m(s)$ has an analytic continuation to the entire complex plane. Moreover, the Sato-Tate conjecture is true if and only if $L_m(1 + it) \neq 0$ for all real t and all $m \geqq 1$ (see Serre [14]).

The fact that $L_1(s)$ and $L_2(s)$ have an analytic continuation to the entire complex plane is classical and due to Hecke and Rankin, respectively. Shahidi [15, Theorems 4.1.1 and 4.1.2] has shown that $L_3(s)$, $L_4(s)$, and $L_5(s)$ have an analytic continuation to the entire complex plane and satisfy certain functional equations.

We will need the non-vanishing of $L_3(s)$ and $L_4(s)$ on the line $\operatorname{Re} s = 1$. Ogg [10] proved that if for each $r \leqq 2m$, $L_r(s)$ has an analytic continuation to $\operatorname{Re} s > \frac{1}{2} - \delta$ for some $\delta > 0$, then $L_m(1+it) \neq 0$. Murty [9] showed that it suffices to have analytic continuation up to $\operatorname{Re} s \geqq 1$ for the non-vanishing result to hold. As we do not have the analytic continuation of $L_r(s)$ for $r \geqq 6$, we cannot deduce the non-vanishing theorem from the results cited above. We therefore prove:

Theorem 3. *If $L_r(s)$ has an analytic continuation up to $\operatorname{Re} s \geqq \frac{1}{2}$, for $1 \leqq r \leqq 2m$, then $L_{2m-1}(1+it) \neq 0$. If $L_{2r}(s)$ has an analytic continuation up to $\operatorname{Re} s = 1$, for $1 \leqq r \leqq m$, then $L_{2m}(1+it) \neq 0$.*

Corollary. *$L_3(1+it) \neq 0$, $L_4(1+it) \neq 0$, for all $t \in \mathbb{R}$.*

Remark. This result is partially contained in Shahidi [15, Theorem 5.3]. Our proof is different. For our application, the fact that $L_4(1) \neq 0$ is crucial and this is not contained in [15].

At this stage, it seems relevant to ask the following: if it is known that $L_r(s)$ has an analytic continuation up to $\operatorname{Re} s = 1$ for $r \leqq R$, then what can be deduced about the oscillatory behaviour of the $a(n)$'s. An answer is supplied by the following.

Theorem 4. *Suppose that $L_r(s)$ has an analytic continuation up to $\operatorname{Re} s \geqq \frac{1}{2}$ for all $r \leqq 2m+2$. Then, each of the statements*

(i) *for any $\delta > 0$, $-\delta < 2\cos\theta(p) < \dfrac{2}{\delta(m+2)}$,*

(ii) *for any $\varepsilon > 0$,*

$$|2\cos\theta(p)| > \sqrt{\frac{4m+2}{m+2}} - \varepsilon,$$

(iii) *for any $\varepsilon > 0$, $2\cos\theta(p) > \beta_m - \varepsilon$, where*

$$\beta_m = \left\{ \frac{1}{4(m+2)} \binom{2m+2}{m+1} \right\}^{\frac{1}{2m+1}}$$

(with a corresponding result valid for negative values of $a(p)$), holds for some set of primes of positive density.

Setting $\delta = \sqrt{\dfrac{2}{m+2}}$ in (i), we deduce:

Corollary 1. *There is a positive density of primes p satisfying*

$$-\sqrt{\frac{2}{m+2}} < 2\cos\theta(p) < \sqrt{\frac{2}{m+2}}.$$

If in theorem (ii), we set $m=1$, we have:

Corollary 2. *For a positive density of primes p, we have*

$$|a(p)|>(\sqrt{2}-\varepsilon)p^{\frac{k-1}{2}}.$$

It is then an easy matter to deduce the Ω-theorem from this fact. The $\Omega_\pm$ theorem for the Fourier coefficients of an eigenform requires a further analysis of the sign changes of the $a(p)$'s.

Theorem 5. *Let*

$$f(z)=\sum_{n=1}^{\infty} a(n)e^{2\pi inz}$$

be an arbitrary cusp form belonging to any congruence subgroup. Then either $\operatorname{Re}a(p)$ *or* $\operatorname{Im}a(p)$ *changes sign infinitely often. Moreover, there exists a small positive number θ such that the number of sign changes for $p\leqq x$ is at least ax^θ for some $a>0$.*

If all the zeroes of $L_1(s)$ lying in the critical strip have real part $=\frac{1}{2}$, then the number of sign changes can be improved to $\gg x^{\frac{1}{2}-\varepsilon}$.

Finer theorems concerning the sign changes of the $a(p)$'s demand an examination of the real zeroes of $L_1(s)$. The following conjecture can be proved in many special cases.

Conjecture 1. *Let*

$$f(z)=\sum_{n=1}^{\infty} a(n)e^{2\pi inz}$$

be arbitrary cuspform with real coefficients and of weight k for the full modular group. Then,

(i) $\sum_{p\leqq x} a(p)p^{-(k-1)/2}=\Omega_\pm(x^{1/2}\log\log\log x/\log x)$,

ii) $a(n)=\Omega_\pm(n^{(k-1)/2}\exp(c\log n/\log\log n))$,

for some $c>0$.

The first part of the conjecture would follow if $L_1(s)$ has no real zeroes in $\frac{1}{2}<s\leqq 1$. The second part would follow if we knew that the $\theta(p)$'s are independently distributed for the various eigenforms. These possibilities are explored in the latter sections of the paper.

2. Non-Vanishing of $L_r(s)$

We first prove Theorem 3. We need the following lemma.

Lemma 1 (Murty [9]). *Let $f(s)$ be a function satisfying the following hypotheses:*

(a) *$f(s)$ is holomorphic in $\sigma>1$ and non-zero there,*

(b) *on the line $\sigma=1$, f is holomorphic except for a pole of order $e\geqq 0$ at $s=1$,*

(c) *for $\sigma>1$, $\log f(s)$ can be written as a Dirichlet series*

$$\sum_{n=1}^{\infty} b(n)n^{-s}$$

with $b(n)\geqq 0$.

Then, any zero of $f(s)$ *on* $\sigma=1$ *has order* $\leqq \frac{e}{2}$.

Proof of Theorem 3. We first show $L_{2m}(1+it)\neq 0$. Consider

$$f(s)=L_0 L_2 \ldots L_{2m}.$$

Then, in view of the identities

$$\tfrac{1}{2}+\cos\theta+\cos 2\theta+\ldots+\cos n\theta=\frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$$

and

$$\cos\theta+\cos 3\theta+\ldots+\cos(2n-1)\theta=\frac{\sin n\theta}{2\sin\theta}$$

we see that

$$\log L_r(s)=\sum_{n,p}\left(\frac{\sin(r+1)n\theta(p)}{\sin n\theta(p)}\right)\frac{1}{np^{ns}}.$$

Therefore, as

$$1+\frac{\sin 3\theta}{\sin\theta}+\frac{\sin 5\theta}{\sin\theta}+\ldots+\frac{\sin(2n-1)\theta}{\sin\theta}=\left(\frac{\sin n\theta}{\sin\theta}\right)^2,$$

we find that $\log f(s)$ is a Dirichlet series with non-negative coefficients. Moreover, the Euler product for $L_r(s)$ shows that $f(s)$ does not vanish in $\sigma>1$. An application of Lemma 1 with $e\leqq 1$ gives the result. Now consider

$$g(s)=(L_0 L_1 \ldots L_{2m-1})^2 L_{2m}.$$

An easy computation reveals,

$$\log g(s)=\sum_{n,p}\left((2m+1)+\sum_{j=0}^{2m-1} 2(j+1)\cos(2m-j)n\theta(p)\right)\frac{1}{np^{ns}}.$$

Since,

$$(2m+1)+\sum_{j=0}^{2m-1} 2(j+1)\cos(2m-j)\theta=\left(\frac{\sin(m+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}\right)^2,$$

$\log g(s)$ is a Dirichlet series with non-negative coefficients. If $L_{2m-1}(1+it)=0$, $t\neq 0$, then $g(s)$ has a zero of order $\geqq 2$. As $g(s)$ has a pole of order 2 at $s=1$, we get a contradiction by Lemma 1. If $L_{2m-1}(1)=0$, then $g(s)$ is regular. By the well-known theorem of Landau, $\log g(s)$ has a singularity at its abscissa of convergence. As $L_0(s)=\zeta(s)$ has zeroes in $\mathrm{Re}\, s\geqq\frac{1}{2}$, $g(s)$ has zeroes in this half plane. Therefore, the abscissa of convergence of $\log g(s)$ is $\sigma_0\geqq\frac{1}{2}$, and as $g(s)$ is analytic in $\mathrm{Re}\, s\geqq\frac{1}{2}$, σ_0 is a zero of $g(s)$. But then $g(\sigma)\geqq 1$ for $\sigma>\sigma_0$. We get a contradiction by letting $\sigma\to\sigma_0^+$. This completes the proof of the theorem.

3. Various Bounds for Fourier Coefficients

The non-vanishing of $L_r(s)$ on $\sigma=1$, allows us to deduce the following lemma which we need for the proof of Theorem 4.

Lemma 2. *Suppose $L_r(s)$ has an analytic continuation up to* $\mathrm{Re}\, s \geqq \frac{1}{2}$ *for all $r \leqq 2m+2$. Then*

(i) *for $r \leqq m+1$,*

$$\sum_{p \leqq x} (2\cos\theta(p))^{2r} = \frac{1}{r+1}\binom{2r}{r}(1+o(1))\frac{x}{\log x}$$

as $x \to \infty$ and

(ii) *for $r \leqq m$,*

$$\sum_{p \leqq x} (2\cos\theta(p))^{2r+1} = o(x/\log x)$$

as $x \to \infty$.

Proof. From Theorem 3, we know that $L_r(s)$ does not vanish on the line $\sigma=1$. Therefore, by the Wiener-Ikehara Tauberian theorem, we deduce for $1 \leqq r \leqq 2m+2$,

$$\sum_{p \leqq x} \frac{\sin(r+1)\theta(p)}{\sin\theta(p)} = o(x/\log x)$$

as $x \to \infty$, on using the identity

$$\frac{\sin(r+1)\theta}{\sin\theta} = \sum_{j=0}^{r} e^{i(r-2j)\theta}.$$

Writing $U_n(\cos\theta) = \dfrac{\sin(n+1)\theta}{\sin\theta}$ and $T_n(\cos\theta) = \cos n\theta$, for $n \geqq 1$, we find for $2 \leqq r \leqq 2m+2$,

$$\sum_{p \leqq x} T_r(\cos\theta(p)) = o(x/\log x),$$

because of the identity

$$2T_n(x) = U_n(x) - U_{n-2}(x).$$

Also,

$$\sum_{p \leqq x} T_2(\cos\theta(p)) = (-\tfrac{1}{2}+o(1))\frac{x}{\log x}$$

and

$$\sum_{p \leqq x} T_1(\cos\theta(p)) = o(x/\log x),$$

as $L_1(s)$ is regular and non-vanishing for $\mathrm{Re}\, s \geqq 1$.

If we define $T_0(x) = \frac{1}{2}$, the inverse relation for Chebychev polynomials gives (see Riordan [12, p. 54]),

$$(2\cos\theta)^r = 2\sum_{k=0}^{r'} \binom{r}{k} T_{r-2k}(\cos\theta),$$

where $r'=[r/2]$. Therefore,

$$\sum_{p\leq x}(2\cos\theta(p))^r=2\sum_{k=0}^{r'}\binom{r}{k}\sum_{p\leq x}T_{r-2k}(\cos\theta(p)).$$

By the above remarks, the inner sum is $o(x/\log x)$ unless $r-2k=2$ or 0. Hence (ii) is deduced without difficulty. In the other case, we find

$$\sum_{p\leq x}(2\cos\theta(p))^{2r}=\left(-\binom{2r}{r-1}+\binom{2r}{r}\right)(1+o(1))\frac{x}{\log x}$$

as $x\to\infty$. The term in brackets is easily seen to be $\frac{1}{r+1}\binom{2r}{r}$, as desired.

We will need the following combinatorial identities.

Lemma 3.

(i) $$\sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{2j}{j}\frac{1}{j+1}2^{-2j}=2^{-2r-1}\binom{2r+2}{r+1},$$

(ii) $$\sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{2j+2}{j+1}\frac{1}{j+2}2^{-2j}=2^{-2r}\frac{1}{r+2}\binom{2r+2}{r+1}.$$

Proof. (i) can be found in Riordan [12, p. 120], (ii) can be deduced easily from (i) in the following way. Denote the sum in (i) by $S(r)$. Considering $S(r+1)-S(r)$ and using the identity

$$\binom{r+1}{j}-\binom{r}{j}=\binom{r}{j-1},$$

(ii) is deduced easily after a change of variable and some simplifications.

Proof of Theorem 4. (i) Consider the polynomial

$$P_m(x)=(x^2-4)^m(x-\alpha)(x-\beta),$$

where α,β will be suitably chosen later. By Lemmas 2 and 3,

$$\frac{\log x}{x}\sum_{p\leq x}P_m(2\cos\theta(p))\sim(-1)^m\binom{2m+2}{m+1}\left(\frac{\alpha\beta}{2}+\frac{1}{m+2}\right).$$

Examining the graph of $P_m(x)$ and choosing α,β so that

$$\alpha\beta>-\frac{2}{m+2},$$

if m is even, and

$$\alpha\beta<-\frac{2}{m+2},$$

if m is odd, we set $\alpha=-\delta$ to get the desired result.

(ii) Consider $Q_m(x)=x^{2m}(x^2-\gamma)$ where γ shall soon be chosen. By Lemma 2,

$$\frac{\log x}{x}\sum_{p\leq x}Q_m(2\cos\theta(p))\sim\frac{1}{m+2}\binom{2m+2}{m+1}-\gamma\frac{1}{m+1}\binom{2m}{m}.$$

Examining the graph of $Q_m(x)$, we note that if

$$\gamma = \frac{m+1}{m+2}\binom{2m+2}{m+1}\binom{2m}{m}^{-1} = \frac{2(2m+1)}{m+2},$$

we obtain (ii).

(iii) Since

$$\sum_{p\leqq x} |2\cos\theta(p)|^{2m+1} \geqq \frac{1}{2}\sum_{p\leqq x} (2\cos\theta(p))^{2m+2}$$

we have by Lemma 2,

$$4\sum_{\substack{p\leqq x\\ a(p)>0}} (2\cos\theta(p))^{2m+1} \gtrsim \frac{1}{m+2}\binom{2m+2}{m+1}\frac{x}{\log x}$$

as $x\to\infty$. Thus for a positive proportion of the primes,

$$2\cos\theta(p) > \left\{\frac{1}{4(m+2)}\binom{2m+2}{m+1}\right\}^{\frac{1}{2m+1}} - \varepsilon.$$

This completes the proof of the theorem.

Remark. We note that $\beta_m \to 2$, as $m\to\infty$. It is an interesting question to ask for the optimal value of β_m under the hypothesis of the theorem. By considering various polynomials, one can show

$$2\cos\theta(p) > 1.14 - \varepsilon$$

for a positive density of primes, with the present state of knowledge. V. Kumar Murty has informed me that if $L_6(s)$ has an analytic continuation up to $\mathrm{Re}\, s \geqq \frac{1}{2}$, then one can show

$$2\cos\theta(p) > 1.33 - \varepsilon$$

for a positive density of the primes.

4. Sign Changes of $a(p)$

The classical method of analytic number theory gives a zero-free region of the type

$$\sigma > 1 - A/\log(|t|+2), \qquad A > 0,$$

for $L_1(s)$ and $L_2(s)$. This fact enables us to deduce:

Lemma 4. *Let*

$$f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$$

by a Hecke eigenform of weight k for the full modular group. Then, there is a $v>0$, such that if $h=x^{\theta}$, $v<\theta<1$, we have

(i) $$\sum_{x\leqq p\leqq x+h} a(p)p^{-(k-1)/2}=o(h),$$

(ii) $$\sum_{x\leqq p\leqq x+h} a(p)^2p^{-(k-1)}\gg h.$$

Proof. (ii) is contained in Moreno [7], and (i) is easily obtained from methods of that paper.

Let g be another normalised Hecke eigenform of weight k for the full modular group. Let g have Fourier expansion

$$g(z)=\sum_{n=1}^{\infty} b(n)e^{2\pi inz}.$$

Then, Rankin showed that the series

$$L(s,f,g)=\sum_{n=1}^{\infty} a(n)b(n)n^{-s-k+1}$$

has an analytic continuation to $\operatorname{Re}s\geqq\frac{1}{2}$ and satisfies a functional equation.

For any eigenform with Fourier expansion

$$h(z)=\sum_{n=1}^{\infty} c(n)e^{2\pi inz},$$

define $L_2(s,h)$ as the $L_2(s)$ defined in Sect. 1. The next lemma concerns the non-vanishing of $L(s,f,g)$ on $\sigma=1$.

Lemma 5. *For f, g orthogonal, normalised Hecke eigenforms of weight k,*

(i) $L(1+it,f,g)\neq 0$,

(ii) $L(s,f,g)\neq 0$ *for* $\sigma>1-C/\log(|t|+2)$, *for some* $C>0$,

(iii) *there is a v such that if $h=x^{\theta}$, $v<\theta<1$, we have*

$$\sum_{x\leqq p\leqq x+h} a(p)b(p)p^{-(k-1)}=\underline{o}(h).$$

Proof. We shall prove (ii) from which (i) follows. (iii) is then easily deduced from the methods of Moreno [7].

Consider

$$D(s)=\zeta^2(s)L_2(s,f)L(s+it,f,g)L(s-it,f,g)L_2(s,g).$$

Then, an easy verification shows that

$$\log D(s)=\sum_{n,p} C(n,p)n^{-1}p^{-ns},$$

where $C(n,p)=8(\cos n\theta(p))(\cos n\phi(p))(\cos nt\log p)+2\cos 2n\theta(p)+2\cos 2n\phi(p)+4$.

In view of the identity

$$C(n,p)=4|\cos\theta(p)-p^{it}\cos\phi(p)|^2$$

we see that $\log D(s)$ is a Dirichlet series with non-negative coefficients. Therefore, for $\sigma > 1$

$$D(\sigma) = \zeta^2(\sigma) L_2(\sigma, f) L(\sigma + it, f, g) L(\sigma - it, f, g) L_2(\sigma, g) \geqq 1$$

taking logarithmic derivatives and using the fact that $L_2(s, f)$, $L(s, f, g)$, $L_2(s, g)$ are regular in $\sigma \geqq \frac{1}{2}$, we find by the standard method that $L(s, f, g)$ is zero-free in $\sigma > 1 - \frac{C_1}{\log(|t|+2)}$, for some $C_1 > 0$. This completes the proof.

We now prove Theorem 5.

Proof of Theorem 5. For suitable complex numbers e_i, $i = 1, \ldots, d$, where $d = \dim S_k$ (the space of cusp forms of weight k for the full modular group), we can write

$$a(p) = \sum_{i=1}^{d} e_i a_i(p),$$

where

$$f_i(z) = \sum_{n=1}^{\infty} a_i(n) e^{2\pi i n z}$$

is a normalised Hecke eigenform.

Suppose that neither $\mathrm{Re}\, a(p)$ nor $\mathrm{Im}\, a(p)$ changes sign in $x \leqq p \leqq x + h$, for $h = x^\theta$, where θ satisfies the conditions of Lemmas 4 and 5(iii). Put $a_p = a(p) p^{-(k-1)/2}$. Then,

$$\left| \sum_{x \leqq p \leqq x+h} \mathrm{Re}\, a_p \right| = \sum_{x \leqq p \leqq x+h} |\mathrm{Re}\, a_p|$$

and

$$\left| \sum_{x \leqq p \leqq x+h} \mathrm{Im}\, a_p \right| = \sum_{x \leqq p \leqq x+h} |\mathrm{Im}\, a_p| .$$

Therefore,

$$\sum_{x \leqq p \leqq x+h} (|\mathrm{Re}\, a_p| + |\mathrm{Im}\, a_p|) = \underline{o}(x^\theta) . \tag{1}$$

On the other hand, by the estimate of Deligne, the left hand side of (1) is

$$\gg \sum_{x \leqq p \leqq x+h} |a_p|^2 = \sum_{1 \leqq i, j \leqq d} e_i \bar{e}_j \sum_{x \leqq p \leqq x+h} a_i(p) a_j(p) p^{-(k-1)} \gg h \equiv x^\theta ,$$

by Lemma 4(ii). This is a contradiction to (1). Therefore, at least one of $\mathrm{Re}\, a(p)$ or $\mathrm{Im}\, a(p)$ changes sign in the interval $x \leqq p \leqq x + h$. This completes the proof of Theorem 5.

5. Main Theorems

By Corollary 2 of Theorem 4, we have

$$|a(p)| > (\sqrt{2} - \varepsilon) p^{\frac{k-1}{2}}$$

for a positive proportion of the primes. So we must have

$$a(p) > (\sqrt{2}-\varepsilon)p^{\frac{k-1}{2}} \tag{1}$$

or

$$a(p) < (-\sqrt{2}-\varepsilon)p^{\frac{k-1}{2}} \tag{2}$$

for a positive proportion of the primes. Without loss, suppose that (1) is true.

Let

$$n = \prod_{p \leqq x}{}^{*} p,$$

where the product is over those primes satisfying (1).

Then

$$a(n) > n^{\frac{k-1}{2}} (\sqrt{2}-\varepsilon)^{\delta x/\log x}$$

for some $\delta > 0$. As $\log n \sim \delta x$ we get

$$a(n) > n^{\frac{k-1}{2}} \exp\left(\frac{C \log n}{\log\log n}\right)$$

for some $C > 0$ and an infinity of n. If q is a prime such that $a(q) < 0$ [which exists either by Theorem 4(iii) or Theorem 5], the n's constructed above are coprime to q and hence

$$-a_{qn} = (-a_q)a_n \gg n^{\frac{k-1}{2}} \exp\left(\frac{C \log n}{\log\log n}\right),$$

which completes the proof of Theorem 1.

Proof of Theorem 2. Let $f_1, \ldots, f_r$ be a basis of eigenforms. Then there are complex numbers $C_1, \ldots, C_r$ such that

$$f(z) = \sum_{i=1}^{r} C_i f_i(z) .$$

Let

$$f_j(z) = \sum_{n=1}^{\infty} a(n, j) e^{2\pi i n z} .$$

The matrix $(a(j, i))$ for $j \leqq r$, $i \leqq r$ is a non-singular matrix; for if not we can find $d_1, \ldots, d_r$ not all zero such that

$$d_1 f_1 + \ldots + d_r f_r = f_0$$

has a zero of order $\geqq r+1$ at $i\infty$. The dimension formula shows that $f_0 \equiv 0$. But then, this contradicts the independence of the f_j's. Hence, we can find $m_1, \ldots, m_r$ so that

$$\sum_{j=1}^{r} a(j, i) m_j = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 . \end{cases}$$

Now consider

$$g(z)=\sum_{i=1}^{r} m_i T_i(f),$$

where T_i is the i^{th} Hecke operator. Then

$$\begin{aligned} g(z) &= \sum_{i=1}^{r} m_i \sum_{j=1}^{r} C_j\ a(i,j) f_j(z) \\ &= \sum_{j=1}^{r} C_j f_j(z) \sum_{i=1}^{r} a(i,j) m_i = C_1 f_1(z). \end{aligned}$$

On the other hand, the n^{th} Fourier coefficient of $g(z)$ is

$$= \sum_{i=1}^{r} m_i \sum_{d|(n,i)} a(ni/d^2) d^{k-1}$$

by noting what the Hecke operators do to the Fourier coefficients. Now suppose, that for all n sufficiently large,

$$|a(n)| < \varepsilon n^{\frac{k-1}{2}} \exp\left(\frac{C \log n}{\log\log n}\right).$$

Then, if $d(i)$ denotes the number of divisors of i,

$$\begin{aligned} |C_1 a(n,1)| &\leqq \sum_{i=1}^{r} |m_i| i^{k-1} \varepsilon d(i) n^{\frac{k-1}{2}} \exp\left(\frac{C \log n}{\log\log n}\right) \\ &\ll \varepsilon n^{\frac{k-1}{2}} \exp\left(\frac{C \log n}{\log\log n}\right) \end{aligned}$$

which contradicts our previous theorem for suitable C and ε.

Remark. Conjecture 1 is certainly true whenever $k=12, 16, 18, 20, 22, 26$ (see Sect. 6). In case $k=24$, one can compute m_1 and m_2 above and find that $m_1>0$, $m_2>0$. This enables one to show Conjecture 1(ii) by the above method, in this case also.

We can now prove the corollary on Kloosterman sums.

Proof of Corollary. It is well-know that the Poincaré series

$$G_t(z)=\tfrac{1}{2} \sum_{(c,d)=1} (cz+d)^{-k} e^{2\pi i t(az+b)/(cz+d)}$$

for $1 \leqq t \leqq r$, where r is the dimension of the space of cusp forms of weight k, span this space. Moreover, the n^{th} Fourier coefficient of $G_t(z)$ is

$$(nt)^{\frac{k-1}{2}} \left\{\delta_{tn} + \pi \sum_{c=1}^{\infty} \frac{S(t,n,c)}{c} \cdot J_{k-1}\left(\frac{4\pi \sqrt{nt}}{c}\right)\right\},$$

where $S(t,n,c)$ is the Kloosterman sum defined previously and J_k denotes the usual Bessel function. Set

$$H(x)=\sum_{c \leqq x} S(n,m,c).$$

Now suppose the corollary is false. Then, given $\varepsilon>0$,

$$|G(x)| \leqq \varepsilon \exp\left(\frac{C\log x}{\log\log x}\right)$$

for all x sufficiently large and by partial summation,

$$|H(x)| \leqq \varepsilon x \exp\left(\frac{C\log x}{\log\log x}\right).$$

It is easy to see that

$$\begin{aligned}
&\sum_{c>\sqrt{n}} \frac{S(n,t,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right) \\
&= \sum_{c>\sqrt{n}} G(c)\left(J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c+1}\right) - J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right)\right) \\
&\ll \sqrt{n} \sum_{c>\sqrt{n}} \frac{G(c)}{c^2} J'_{k-1}(\xi_c),
\end{aligned}$$

where $\xi_c \in \left(\frac{4\pi\sqrt{nt}}{c+1}, \frac{4\pi\sqrt{nt}}{c}\right)$. As $J'_{k-1}(x) \ll x^{-1/2}$, we find that the above sum is

$$\ll \varepsilon n^{-1/4} \int_{\sqrt{n}}^{\infty} \frac{e^{C\log t/\log\log t}}{t^{3/2}}\, dt.$$

An integration by parts reveals that it is

$$\ll \varepsilon \exp\left(\frac{C\log n}{\log\log n}\right).$$

As there are no cusp forms of weight 10, it follows from the above that

$$\sum_{c \leqq \sqrt{n}} \frac{S(n,t,c)}{c} J_9\left(\frac{4\pi\sqrt{nt}}{c}\right) \ll \varepsilon \exp\left(\frac{C\log n}{\log\log n}\right).$$

Now

$$\frac{1}{\sqrt{n}} \sum_{c \leqq \sqrt{n}} S(n,t,c) J_{10}\left(\frac{4\pi\sqrt{nt}}{c}\right) \ll \sum_{c \leqq \sqrt{n}} \frac{H(c)}{c^2} J'_{10}(\eta_c)$$

for some $\eta_c \in \left(\frac{4\pi\sqrt{nt}}{c+1}, \frac{4\pi\sqrt{nt}}{c}\right)$. By our supposition, this sum is

$$\ll n^{-1/4} \sum_{c \leqq \sqrt{n}} \frac{H(c)}{c^{3/2}} \ll \varepsilon \exp\left(\frac{C_1 \log n}{\log\log n}\right)$$

as $J'_{k-1}(x) \ll x^{-1/2}$. But then, in view of the identity.

$$\frac{20 J_{10}(x)}{x} = J_9(x) + J_{11}(x),$$

we find,

$$\sum_{c \leqq n} \frac{S(n,t,c)}{c} J_{11}\left(\frac{4\pi\sqrt{nt}}{c}\right) \ll \varepsilon \exp\left(\frac{C_1 \log n}{\log\log n}\right)$$

for some constant $C_1 > 0$. This contradicts Theorem 2, which completes the proof of the corollary.

6. Real Zeroes and Oscillations

By standard techniques of analytic number theory, it is possible to show the following (see Ingham [3, 4]).

Theorem 6. *Let f be a normalized eigenform of weight k, with Fourier expansion*

$$\sum_{n=1}^{\infty} a(n) e^{2\pi i n z}.$$

Suppose $L_1(s)$ has no real zero in the interval $\frac{1}{2} < s \leqq 1$. Then

$$\sum_{p \leqq x} a(p) p^{-(k-1)/2} = \Omega_{\pm}(x^{1/2} \log\log\log x / \log x).$$

Remark. Even if $L_1(\frac{1}{2}) = 0$, it is of no consequence to the Ω result because the contribution from the zero is $O(x^{1/2}/\log x)$. It is easily seen that whenever $k \equiv 2 \pmod 4$, $L_1(\frac{1}{2}) = 0$, for instance.

One can show by means of the formula

$$v_\infty + v_i/2 + v_p/3 + \sum_{P \neq i, p} v_P = k/12, \tag{3}$$

where v_p denotes the order of the zero of f at P, i and p denote $\sqrt{-1}$ and $e^{2\pi i/3}$, respectively, that $L_1(s)$ has no real zero in $\frac{1}{2} < s \leqq 1$ for $k < 24$ and $k = 26$. For $k = 24$, it is possible to show the existence of at least one eigenfunction satisfying the hypothesis. We briefly indicate how this is done. Consider

$$(2\pi)^{-(s+\frac{1}{2}k-\frac{1}{2})} \Gamma(s + \tfrac{1}{2}k - \tfrac{1}{2}) L_1(s) = \int_1^{\infty} f(iy) y^{(k-1)/2} (y^s + i^k y^{1-s}) \frac{dy}{y}. \tag{4}$$

It is then easy to show that the integral is non-negative for the specified values of k using (3). For example, if $k = 22$,

$$v_i/2 + v_p/3 + \sum_{P \neq i, p} v_P = 5/6,$$

which clearly implies $f(iy) \geqq 0$. For $k = 26$, f is up to a constant multiple equal to $\Delta E_4^2 E_6$ where Δ denotes Ramanujan,s cusp form, E_4, E_6 denote usual Eisenstein series. As we know the location of the zeroes of E_4 and E_6, the desired result follows in this case also. Moreover, (4) can be used to show that

$$(2\pi)^{-s} \Gamma(s) L_1\left(s - \frac{k-1}{2}\right)$$

is a monotone increasing function for $s \geqq \frac{1}{2}$, just by expanding the integral about $s = \frac{1}{2}$. In fact, it seems reasonable to make the following conjecture.

Conjecture 2. *If f is a normalized Hecke eigenform, then*

$$(2\pi)^{-s}\Gamma(s)L_1\left(s-\frac{k-1}{2}\right)$$

is a monotone increasing function of s for $s \geqq \frac{1}{2}$.

Indeed, if $L_1(s)$ satisfies the analog of the Riemann hypothesis, the conjecture is true (see Stark and Zagier [16]).

7. Concluding Remarks

We discuss Conjecture 1(ii) as the first part of Conjecture 1 has been dealt with in the previous section.

The truth of the conjecture would follow if we postulate that the angles $\theta(p,i)$ for $i=1,\ldots,r$ corresponding to the basis of normalized Hecke eigenforms $f_1,\ldots,f_r$, respectively, are independently distributed, with respect to the Sato-Tate measure. This can be cast, in the familiar way, in terms of L-series. For $\underline{m}=(m_1,\ldots,m_r)$, let

$$f(p,\underline{m})=\sum_{a=1}^{r}(m_a-2j_a)\theta(p,a)$$

and consider

$$L_{\underline{m}}(s)=\prod_p\prod_{j_1=0}^{m_1}\cdots\prod_{j_r=0}^{m_r}\left(1-\frac{e^{if(p,\underline{m})}}{p^s}\right)^{-1}.$$

For $\underline{m}=(m,0,\ldots,0)$ we retrieve $L_m(s)$ as

$$L_{\underline{m}}(s)=(L_m(s))^r$$

Clearly, $L_m(s)$ defines an analytic function for $\operatorname{Re}s>1$.

Conjecture 3. *Each $L_{\underline{m}}(s)$ has an analytic continuation to the entire complex plane and $L_{\underline{m}}(1+it)\neq 0$, for $m_1+\ldots+m_r\geqq 1$.*

It may be remarked that if $m_1+\ldots+m_r=2$, then $L_{\underline{m}}(s)$ is essentially the familiar convolution of L-series. Lemma 5 gives the non-vanishing of these L-series on $\sigma=1$.

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