Indian J. Pure Appl. Math., **45**(5): 691-706, October 2014

RAMANUJAN - FOURIER SERIES AND A THEOREM OF INGHAM

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(*Received* 8 *January* 2014; *after final revision* 14 *July* 2014;
accepted 21 *July* 2014)

Key words :

1 Introduction

In 1918, Ramanujan [10] introduced an exponential sum (now called Ramanujan sum) and indicated how one can use this sum to study arithmetical functions. More precisely, we define the Ramanujan sum as:

$$c_r(n) = \sum_{(a,r)=1} e(an/r), \qquad e(t) := e^{2\pi i t},$$

where the sum is over a complete set of $\phi(r)$ coprime residue classes (mod r). Basic properties of Ramanujan sums can be found in [4] and [7] as well as in [10]. In his paper, Ramanujan expressed a variety of arithmetical functions in the form of an infinite series:

$$\sum_{r=1}^{\infty} a_r c_r(n).$$

[1]Research of the second author was partially supported by a Simons Fellowship and an NSERC Discovery grant.

For example, for $\Re(s) > 0$, he showed

$$\frac{\sigma_s(n)}{n^s} = \zeta(s+1) \sum_{r=1}^{\infty} \frac{c_r(n)}{r^{s+1}},$$

where

$$\sigma_s(n) = \sum_{d|n} d^s.$$

The existence of such series (now called Ramanujan - Fourier series) for a given arithmetical function as well as their convergence properties has been the object of some study, most notably recorded in the monographs [11] and [12] (see also the survey [8]). It seems fair to say that a comprehensive theory has not yet been developed that embraces all of the results in this area, especially those dealing with conditionally convergent series, though there are are some notable papers like [6] in this context.

For example, Hardy [3] proved that $\frac{\phi(n)}{n}\Lambda(n)$ has a conditionally convergent series:

$$\frac{\phi(n)}{n}\Lambda(n) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n), \tag{1}$$

where $\Lambda(n)$ is the von Mangoldt function defined as $\log p$ when n is a power of a prime p and zero otherwise. Our present paper was motivated by a heuristic result presented in [2] in which the authors noted that one can derive the Hardy-Littlewood conjecture regarding twin primes using Ramanujan - Fourier series. Indeed, proceeding heuristically, we have using (1),

$$\sum_{n \le N} \frac{\phi(n)}{n}\Lambda(n) \frac{\phi(n+h)}{n+h}\Lambda(n+h) = \sum_{r,s} \frac{\mu(r)}{\phi(r)} \frac{\mu(s)}{\phi(s)} \sum_{n \le N} c_r(n) c_s(n+h).$$

We now utilise an orthogonality property for Ramanujan sums (first noticed by Carmichael [1]):

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \le N} c_r(n) c_s(n+h) = \begin{cases} c_r(h) & \text{if } r = s \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

This leads to the heuristic result

$$\sum_{n \le N} \Lambda(n)\Lambda(n+h) \sim N \sum_{r=1}^{\infty} \frac{\mu^2(r)}{\phi(r)} c_r(h)$$

which agrees with the Hardy-Littlewood conjecture (which they made based on a heuristic derived from the more complicated circle method).

As is well-known, the Ramanujan - Fourier series of a function (if it exists) need not be unique. Given a function f and the existence of a Ramanujan - Fourier series,

$$f(n) = \sum_{q=1}^{\infty} \widehat{f}(q) c_q(n),$$

Carmichael's orthogonality relation (2) allows one to associate a set of "natural coefficients" (also called Ramanujan - Fourier coefficients):

$$\widehat{f}(r)\phi(r) = \lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} f(n) c_r(n),$$

provided these limits exist. It is convenient to introduce the notation

$$M(f) :== \lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} f(n),$$

so that the Ramanujan - Fourier coefficients can be written as $M(fc_r)/\phi(r)$.

Our goal in this paper is to study more generally, the question of when such series can be utilised to derive asymptotic formulas for sums of the form

$$\sum_{n\leq N} f(n) g(n+h).$$

We also consider the conjugate problem of the asymptotics for

$$\sum_{n\leq N} f(n) g(N-n).$$

For instance, such asymptotics for $f = g = \Lambda$ would give a conjectural formula for the Goldbach conjecture which again agrees with the Hardy-Littlewood conjecture.

With some care, it turns out that one can obtain sufficiently general results. As special cases of our theorem, we recover certain formulas of Ingham [5] obtained by him using different methods. Our methods do not apply to the Goldbach problem or the twin prime problem because the corresponding Ramanujan - Fourier series are not absolutely convergent.

2 Preliminary lemmas

Lemma 1.

$$\sum_{n\leq N} c_r(n) c_s(n) \leq d(r) d(s) (r,s) N,$$

where (r,s) denotes the gcd of r and s.

Proof. We have

$$\sum_{n\le N} c_r(n)c_s(n) = \sum_{n\le N}\sum_{d|r,d|n}\sum_{e|s,e|n} \mu(r/d)\mu(s/e)de\,.$$

We interchange the summations to get

$$\sum_{d|r,e|s} \mu(r/d)\mu(s/e)de \sum_{n\le N; d|n, e|n} 1.$$

The innermost sum is equal to

$$\left[\frac{N}{[d,e]}\right],$$

where $[d,e]$ denotes the lcm of d and e. Thus, the sum in question is

$$\le N\sum_{d|r,e|s}(d,e),$$

using the familiar formula that $de = (d,e)[d,e]$. We can write this sum as

$$N\sum_{d|r,e|s}\sum_{\delta|(d,e)}\phi(\delta) = N\sum_{\delta|(r,s)}\phi(\delta)d(r/\delta)d(s/\delta)$$

and this is less than $Nd(r)d(s)(r,s)$, as stated. □

We also need the finer:

Lemma 2.

$$\sum_{n\le N} c_r(n)c_s(n+h) = \delta_{r,s}Nc_r(h) + O(rs\log rs).$$

Proof. Let

$$S = \sum_{n\le N} c_r(n)c_s(n+h) = \sum_{(a,r)=1}\sum_{(b,s)=1} e\left(\frac{bh}{s}\right)\sum_{n\le N} e\left(\left(\frac{a}{r}+\frac{b}{s}\right)n\right). \tag{3}$$

$\frac{a}{r}+\frac{b}{s}\in\mathbb{Z}$ if and only if $r=s$ and $a=r-b$. In this case,

$$S = Nc_r(h)\,. \tag{4}$$

If $\frac{a}{r}+\frac{b}{s}\notin\mathbb{Z}$, then $e\left(\frac{as+br}{rs}\right)$ is a root of unity which is not equal to 1. Let us write $N=(rs)Q+R$, where $0\le R<rs$. Then the inner sum is

$$\begin{aligned}\sum_{n\le N} e\left(\left(\frac{a}{r}+\frac{b}{s}\right)n\right) &= \sum_{n\le (rs)Q} e\left(\left(\frac{a}{r}+\frac{b}{s}\right)n\right) + \sum_{(rs)Q<n\le N} e\left(\left(\frac{a}{r}+\frac{b}{s}\right)n\right)\\ &= \sum_{(rs)Q<n\le N} e\left(\left(\frac{a}{r}+\frac{b}{s}\right)n\right)\end{aligned} \tag{5}$$

which is in absolute value

$$\leq \frac{1}{||\frac{as+br}{rs}||}, \tag{6}$$

where $||x||$ indicates the distance between x and the nearest integer and we have used the identity $\displaystyle\sum_{k=1}^{rs} e\left(\frac{kl}{rs}\right) = 0$ if $l \neq 0$. When $(r,s) = 1$, $as + br$ run through all the coprime residue classes modulo rs and hence the sum

$$|S| \leq \sum_{(a,r)=1} \sum_{(b,s)=1} \frac{1}{||\frac{as+br}{rs}||} \leq \sum_{(k,rs)=1} \frac{rs}{k} \ll rs \log rs\,. \tag{7}$$

If $d = (r,s) > 1$, then we write $r = dr_1$ and $s = ds_1$ so that $(r_1, s_1) = 1$. Then,

$$\frac{a}{s} + \frac{b}{r} = \frac{as_1 + br_1}{[r,s]}$$

and

$$||(as+br)/rs|| = ||(as_1+br_1)/[r,s]||.$$

As a ranges over coprime residues (mod r) and b ranges over coprime residues (mod s), the expression $as_1 + br_1$ ranges over residue classes mod $[r,s]$ and these residue classes are distinct (mod $r_1 s_1$). Thus, a single class (mod $[r,s]$) is repeated at most d times so that

$$\begin{aligned} |S| &\ll \sum_{(a,r)=1} \sum_{(b,s)=1} \frac{1}{||\frac{as_1+br_1}{dr_1 s_1}||} \\ &\ll d[r,s] \log d[r,s] \\ &= O(rs \log rs)\,. \end{aligned} \tag{8}$$

This completes the proof. □

Lemma 3.

$$|\sum_{n \leq N} c_r(n) c_s(n+h)| \leq N^{1/2}(N+|h|)^{1/2}(rs)^{1/2} d(r) d(s).$$

Remark. *This result is essentially best possible since in the case of $h = 0$ and $r = s = N$, we have*

$$\sum_{n \leq r} |c_r(n)|^2 = r\phi(r) \tag{9}$$

Proof. We apply the Cauchy-Schwarz inequality and estimate

$$\sum_{n\le N}|c_r(n)|^2=\sum_{n\le N}\sum_{d|(n,r)}\sum_{e|(n,r)}\mu(d/r)\mu(e/r)de=\sum_{d|r,e|r}\mu(r/d)\mu(r/e)de\left[\frac{N}{[d,e]}\right],$$

from which we deduce that the sum in question is

$$\le N\sum_{d|r,e|r}(d,e)=N\sum_{d|r,e|r}\sum_{\delta|(d,e)}\phi(\delta)=N\sum_{\delta|r}\phi(\delta)d(r/\delta)^2\le Nrd(r)^2.$$

The corresponding estimate for $c_s(n+h)$ is similar and this completes the proof. □

3 A Parseval type formula

These results now allow us to proceed as before and prove the following theorem which is analogous to Parseval's formula.

Theorem 4. *Suppose that f and g are two arithmetical functions with absolutely convergent Ramanujan - Fourier series:*

$$f(n)=\sum_{r=1}^{\infty}\widehat{f}(r)c_r(n),\qquad g(n)=\sum_{s=1}^{\infty}\widehat{g}(s)c_s(n),$$

respectively. Suppose that

$$\sum_{r,s}|\widehat{f}(r)\widehat{g}(s)|(r,s)d(r)d(s)<\infty.$$

Then, as N tends to infinity,

$$\sum_{n\le N}f(n)g(n)\sim N\sum_{r=1}^{\infty}\widehat{f}(r)\widehat{g}(r)\phi(r).$$

Proof. We have

$$\sum_{n\le N}f(n)g(n)=\sum_{r,s}\widehat{f}(r)\widehat{g}(s)\sum_{n\le N}c_r(n)c_s(n),$$

the interchange of summations being admissible since the series are absolutely convergent. We split the outer sum over r,s into two parts. The first part is over $rs<U$

with U to be chosen later tending to infinity. The second part is over $rs > U$. In the first part, we use Lemma 2 with $h = 0$ and get

$$N \sum_{r^2<U} \widehat{f}(r)\widehat{g}(r)\phi(r) + \sum_{rs<U} |\widehat{f}(r)\widehat{g}(s)| O(rs \log rs).$$

The error is easily seen to be $O(U \log U)$ since $rs < U$ and the series over r, s is absolutely convergent. For the second part, we use the estimate of Lemma 1 to get that it is

$$\ll N \sum_{rs>U} |\widehat{f}(r)||\widehat{g}(s)|(r,s)d(r)d(s) = o(N)$$

by our hypothesis. Indeed, our hypothesis implies

$$\sum_{r=1}^{\infty} |\widehat{f}(r)\widehat{g}(r)| r d(r)^2 < \infty$$

so that

$$\sum_{r^2>U} \widehat{f}(r)\widehat{g}(r)\phi(r) = o(1),$$

which completes the proof. □

4 A shifted Parseval type formula

We use Lemma 3 to prove:

Theorem 5. *Suppose f and g are two arithmetical functions with absolutely convergent Ramanujan - Fourier series*

$$f(n) = \sum_{r=1}^{\infty} \widehat{f}(r)c_r(n), \quad g(n) = \sum_{s=1}^{\infty} \widehat{g}(s)c_s(n).$$

Suppose further that

$$\sum_{r,s} |\widehat{f}(r)\widehat{g}(s)|(rs)^{1/2}d(r)d(s) < \infty.$$

Then, as N tends to infinity,

$$\sum_{n\le N} f(n)g(n+h) \sim N \sum_{r=1}^{\infty} \widehat{f}(r)\widehat{g}(r)c_r(h).$$

Proof. As before, we study

$$\sum_{r,s} \widehat{f}(r)\widehat{g}(s) \sum_{n\le N} c_r(n)c_s(n+h).$$

As before, we split the sum according as $rs < U$ and $rs > U$. On the first part, we apply Lemma 2, and see that the innermost sum is

$$\delta_{r,s} N c_r(h) + O(rs \log rs).$$

Inserting this into the first part, we deduce the main term for a suitable choice of U tending to infinity. For the second part, we apply Lemma 3 and deduce that it is $o(N)$. This completes the proof. □

5 Ramanujan - Fourier series of shifted functions

Given an arithmetical function f, we let $f_h(n) := f(n+h)$. We would like to apply the previous theorem with $g = g_h$ so that we can obtain an asymptotic formula for

$$\sum_{n\le N} f(n)g(n+h).$$

However, this leads to the natural question: if f has a Ramanujan - Fourier series, then does f_h also have one? If so, are the Ramanujan - Fourier coefficients given by $\widehat{f}(r)c_r(h)/\phi(r)$? This is not clear and it forces us to consider the following variations of our earlier lemmas.

Lemma 6.

$$\sum_{n\le N} c_r(n)c_s(n+h) \le N\phi(s)d(r).$$

Proof. The sum in question is

$$\sum_{n\le N} \sum_{d|r, d|n} \mu(r/d)d \sum_{(b,s)=1} e(b(n+h)/s),$$

which is

$$= \sum_{(b,s)=1} e(bh/s) \sum_{d|r} \mu(r/d)d \sum_{n\le N; d|n} e(bn/s).$$

The innermost sum is $\le N/d$ and inserting this estimate gives the result. □

Reversing the roles of r and s gives

Lemma 7. *For $h \leq N$,*

$$\sum_{n \leq N} c_r(n) c_s(n+h) \leq 2N\phi(r)d(s).$$

This allows us to prove the following.

Theorem 8. *Suppose that f has an absolutely convergent Ramanujan - Fourier series with coefficients $\widehat{f}(q)$ satisfying*

$$\sum_{q=1}^{\infty} |\widehat{f}(q)| d(q) < \infty.$$

Then, f_h has an absolutely convergent Ramanujan - Fourier series with coefficients $\widehat{f}(r)c_r(h)/\phi(r)$.

Remark. *It is not clear if the Ramanujan - Fourier series of f_h also converges to f_h.*

Proof. We evaluate the means $M(f_h c_r)$ and show that it is equal to $\widehat{f}(r)c_r(h)$. Indeed,

$$\sum_{n \leq N} f(n+h)c_r(n) = \sum_{q=1}^{\infty} \widehat{f}(q) \sum_{n \leq N} c_q(n+h)c_r(n),$$

the interchange of summation being allowed since the Ramanujan - Fourier series is absolutely convergent. We split the outer sum into two parts $q \leq U$ and $q > U$, with U to be chosen later. Let us look at the first sum: by the previous results, the inner sum is

$$\delta_{q,r} N c_r(h) + O(rq \log rq),$$

so that the first sum is

$$N\widehat{f}(r)c_r(h) + O(rU \log rU),$$

since by assumption

$$\sum_{q=1}^{\infty} |\widehat{f}(q)| < \infty.$$

For $q > U$, we use the penultimate estimate to get

$$\sum_{q > U} \widehat{f}(q) N\phi(r)d(q) = o(N),$$

since our assumption is that

$$\sum_{q=1}^{\infty} |\widehat{f}(q)| d(q) < \infty.$$

Choosing $U \log U$ as $o(N)$ completes the proof. □

6 The conjugate problem

It is clear that our method can be applied to study the conjugate problem related to the asymptotics of the sums

$$\sum_{n<M} f(n)g(N-n).$$

Often such questions are studied with $M = N$, but our method is versatile to deal with this more general case. Based on the analysis of the previous sections, it is clear how one should proceed. Hence, we will give in this section only the main steps and leave the details to the reader.

Lemma 9. *We have:*

(a) $\sum_{n<M} c_r(n)c_s(N-n) = \delta_{r,s}Mc_r(N) + O(rs\log rs)$

(b) $|\sum_{n\le M} c_r(n)c_s(N-n)| \le M^{1/2}(M+N)^{1/2}(rs)^{1/2}d(r)d(s)$.

With this lemma in place, it is easy to show:

Theorem 10. *Suppose f and g are two arithmetical functions with absolutely convergent Ramanujan - Fourier series*

$$f(n) = \sum_{r=1}^{\infty} \widehat{f}(r)c_r(n), \quad g(n) = \sum_{s=1}^{\infty} \widehat{g}(s)c_s(n).$$

Suppose further that

$$\sum_{r,s} |\widehat{f}(r)\widehat{g}(s)|(rs)^{1/2}d(r)d(s) < \infty.$$

Then, for $M < N$,

$$\sum_{n<M} f(n)g(N-n) = M\sum_{r=1}^{\infty} \widehat{f}(r)\widehat{g}(r)c_r(N) + o(M)$$

as M tends to infinity.

7 A theorem of Ingham

In 1927, Ingham [5] proved by an elementary method that for a non-zero integer h,

$$\sum_{n\le N} d(n)d(n+h) \sim \frac{6}{\pi^2}\sigma_{-1}(h)N(\log N)^2$$

and

$$\sum_{n<N} d(n)d(N-n) \sim \frac{6}{\pi^2}\sigma_1(N)(\log N)^2$$

as N tends to infinity. At the end of this paper, he stated that his method can also be applied to show that

$$\sum_{n\le N} \sigma_\alpha(n)\sigma_\beta(n+h) \sim \frac{1}{\alpha+\beta+1}\frac{\zeta(\alpha+1)\zeta(\beta+1)}{\zeta(\alpha+\beta+2)}\sigma_{-\alpha-\beta-1}(h)N^{\alpha+\beta+1}, \tag{10}$$

and

$$\sum_{n<N} \sigma_\alpha(n)\sigma_\beta(N-n) \sim \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\frac{\zeta(\alpha+1)\zeta(\beta+1)}{\zeta(\alpha+\beta+2)}\sigma_{\alpha+\beta+1}(N). \tag{11}$$

In the special case that α, β are positive odd integers, the second formula appears in the work of Ramanujan [9] where more precise results are obtained using the theory of modular forms. Some of these results of Ingham can be deduced from our framework. We record these in the following.

Corollary 11. *For $\alpha, \beta > 1/2$, we have for any integer h,*

$$\sum_{n\le N} \frac{\sigma_\alpha(n)}{n^\alpha}\frac{\sigma_\beta(n+h)}{(n+h)^\beta} \sim N\zeta(\alpha+1)\zeta(\beta+1)\sum_{r=1}^{\infty}\frac{c_r(h)}{r^{\alpha+\beta+2}},$$

as N tends to infinity. In particular,

$$\sum_{n\le N} \frac{\sigma_\alpha(n)\sigma_\beta(n)}{n^{\alpha+\beta}} \sim N\frac{\zeta(\alpha+1)\zeta(\beta+1)\zeta(\alpha+\beta+1)}{\zeta(\alpha+\beta+2)}.$$

For $h \neq 0$, the sum is asymptotic to

$$N\frac{\zeta(\alpha+1)\zeta(\beta+1)}{\zeta(\alpha+\beta+2)}\sigma_{-\alpha-\beta-1}(h),$$

as N tends to infinity.

Proof. In [9], Ramanujan showed that

$$\sigma_s(n) = n^s\zeta(s+1)\sum_{r=1}^{\infty}\frac{c_r(n)}{r^{s+1}} \tag{12}$$

for $s > 0$. Let us take $f(n) = \sigma_\alpha(n)/n^\alpha$ and $g(n) = \sigma_\beta(n)/n^\beta$ in Theorem 5 and verify that the conditions of the theorem are satisfied. We need to check that

$$\sum_{r,s} r^{-\alpha-1}s^{-\beta-1}(rs)^{1/2}d(r)d(s) < \infty .$$

Since the divisor function satisfies $d(r) = O(r^\epsilon)$ for any $\epsilon > 0$, we see that the series converges absolutely for $\alpha, \beta > 1/2$. Thus we get, for $\alpha\ \beta > \frac{1}{2}$ and $h \neq 0$,

$$\begin{aligned}\sum_{n\le N}\frac{\sigma_\alpha(n)}{n^\alpha}\frac{\sigma_\beta(n+h)}{(n+h)^\beta} &\sim N\zeta(\alpha+1)\zeta(\beta+1)\sum_{r=1}^{\infty}\frac{c_r(h)}{r^{\alpha+\beta+2}}\\ &= N\frac{\zeta(\alpha+1)\zeta(\beta+1)}{\zeta(\alpha+\beta+2)}\frac{\sigma_{\alpha+\beta+1}(h)}{h^{\alpha+\beta+1}}\\ &= N\frac{\zeta(\alpha+1)\zeta(\beta+1)}{\zeta(\alpha+\beta+2)}\sigma_{-(\alpha+\beta+1)}(h)\,. \end{aligned} \tag{13}$$

Using partial summation one gets Ingham's result (10) for α and $\beta > \frac{1}{2}$. The case $h = 0$ is easily deduced using the identity

$$\sum_{r=1}^{\infty}\frac{\phi(r)}{r^s} = \frac{\zeta(s-1)}{\zeta(s)} .$$

□

In the same paper [5], Ingham stated the following result without proof. He said that the proof will appear elsewhere but as far as we are aware, he never published it. We obtain it as a corollary of our theorem.

Corollary 12.

$$\sum_{n\le N}\phi(n)\ \phi(n+h) \sim A_h\frac{N^3}{3}\,, \tag{14}$$

where

$$A_h = \prod_p\left(1-\frac{2}{p^2}\right)\prod_{p|h}\frac{p^3-2p+1}{p(p^2-2)}\,. \tag{15}$$

Proof. Ramanujan [9] showed that

$$\phi(n) = \frac{6}{\pi^2}n\sum_{r=1}^{\infty}\frac{\mu(r)}{\phi_2(r)}c_r(n)\,, \tag{16}$$

where $\phi_2(r) = r^2\prod_{p|r}\left(1-\frac{1}{p^2}\right)$.

Let us take $f(n) = g(n) = \dfrac{\phi(n)}{n}$ in Theorem 5. Since

$$\phi_2(r) = r^2 \prod_{p|r} \left(1 - \frac{1}{p^2}\right) \geq r^2 \prod_{p} \left(1 - \frac{1}{p^2}\right) = \frac{6r^2}{\pi^2},$$

it is easily seen that the conditions of the theorem are satisfied. Thus we have

$$\begin{aligned}
\sum_{n \leq N} \frac{\phi(n)}{n} \frac{\phi(n+h)}{n+h} &\sim N \left(\frac{6}{\pi^2}\right)^2 \sum_{r} \frac{\mu^2(r)}{\phi_2(r)^2} c_r(h) \\
&= N \left(\frac{6}{\pi^2}\right)^2 \prod_{p} \left(1 + \frac{\mu^2(p)}{\phi_2(p)^2} c_p(h)\right) \\
&= N \prod_{p} \left(1 - \frac{1}{p^2}\right)^2 \prod_{p \nmid h} \left(1 - \frac{1}{(p^2-1)^2}\right) \prod_{p|h} \left(1 + \frac{p-1}{(p^2-1)^2}\right)
\end{aligned}$$

It is straightforward to show that the constant factor is A_h and using partial summation, we get (14). $\square$

Using Theorem 10, we can derive the analogous theorem for Ingham's second result (11). Indeed, by Theorem 10, we have:

Theorem 13. *For $\alpha, \beta > 1/2$, we have for any positive integers $M < N$,*

(a) As M tends to infinity,

$$\sum_{n<M} \frac{\sigma_\alpha(n)}{n^\alpha} \frac{\sigma_\beta(N-n)}{(N-n)^\beta} \sim \frac{M\zeta(\alpha+1)\zeta(\beta+1)}{\zeta(\alpha+\beta+2)} \frac{\sigma_{\alpha+\beta+1}(N)}{N^{\alpha+\beta+1}};$$

(b) As N tends to infinity,

$$\sum_{n<N} \sigma_\alpha(n)\sigma_\beta(N-n) \sim \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \frac{\zeta(\alpha+1)\zeta(\beta+1)}{\zeta(\alpha+\beta+2)} \sigma_{\alpha+\beta+1}(N).$$

Proof. The first result is an immediate consequence of Theorem 10 applied to $f(n) = \sigma_\alpha(n)/n^\alpha$ and $g(n) = \sigma_\beta(n)/n^\beta$ as in the proof of Corollary 11. Indeed, to apply the theorem, we need to check the same convergence as we did in the proof of Corollary 11. To deduce the second result, we apply partial summation. Since the derivation is not totally routine, we give some details. Indeed, using the first result and applying partial summation, we have as N tends to infinity,

$$\sum_{n<N} \sigma_\alpha(n)\sigma_\beta(N-n) \sim \frac{\zeta(\alpha+1)\zeta(\beta+1)}{\zeta(\alpha+\beta+2)} \frac{\sigma_{\alpha+\beta+1}(N)}{N^{\alpha+\beta+1}} \int_1^N t^\alpha (N-t)^\beta dt.$$

The last integral is easily transformed into the beta function and we find

$$\int_1^N t^\alpha (N-t)^\beta dt = N^{\alpha+\beta+1}\int_{1/N}^1 x^\alpha (1-x)^\beta dx \sim N^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$$

from which Ingham's result is easily deduced. □

In the same way one can also obtain the formula

$$\sum_{n=1}^{N-1} \phi(n)\phi(N-n) \sim \frac{1}{6}A_N N^3$$

stated (without proof) by Ingham [5].

We remark that the method has considerable flexiblity and we can also handle various sums of "mixed" functions.

8 Concluding remarks

We are unable to deduce the full theorem of Ingham by our method simply because we cannot ascertain that the shifted function $\sigma_\beta(n+h)/(n+h)^\beta$ satisfies the hypotheses of Theorem 4. In other words, we need to know that this shifted function has an absolutely convergent Ramanujan - Fourier series with coeffcients given by

$$\zeta(\beta+1)c_r(h)/\phi(r)r^{\beta+1}.$$

If we knew this, then Theorem 4 implies Ingham's result for all $\alpha, \beta > 0$. To see this, we need to verify the condition of Theorem 4, namely that

$$\sum_{r,s}\frac{1}{r^{\alpha+1}}\frac{1}{s^{\beta+1}}(r,s)d(r)d(s) < \infty.$$

But this is easily seen by noting that the sum is

$$\sum_{r,s}\frac{d(r)}{r^{\alpha+1}}\frac{d(s)}{s^{\beta+1}}\sum_{t|(r,s)}\phi(t) \le \sum_{t=1}^{\infty}\phi(t)\left(\sum_{t|r}\frac{d(r)}{r^{\alpha+1}}\right)\left(\sum_{t|s}\frac{d(s)}{s^{\beta+1}}\right).$$

Using the fact that $d(ab) \le d(a)d(b)$, we find the sum is

$$\ll \sum_{t=1}^{\infty}\frac{d(t)^2\phi(t)}{t^{\alpha+\beta+2}} < \infty$$

for $\alpha, \beta > 0$.

Acknowledgements. We would like to thank Sanoli Gun and Purusottam Rath for some useful discussions. We also thank the referee for helpful remarks on an earlier version of this paper.

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