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The Pair Correlation of Zeros of Functions in the Selberg Class

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1 Introduction

The Riemann hypothesis (RH for short) asserts that all the nontrivial zeros ρ of $\zeta(s)$ satisfy $\operatorname{Re}\rho = 1/2$. Montgomery [7] conjectures that, in addition to satisfying RH, these zeros satisfy his pair correlation conjecture. A consequence of this is the following. Assuming RH, let us write $1/2 + i\gamma$ for a typical nontrivial zero of $\zeta(s)$. If $0 \notin [\alpha, \beta]$ and $T \to \infty$, one expects

$$\#\left\{0 < \gamma, \gamma' < T\colon\ \alpha \le \frac{(\gamma-\gamma')\log T}{2\pi} \le \beta\right\} \sim \left(\frac{T}{2\pi}\log T\right)\int_\alpha^\beta \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du.$$

This is consistent with the pair correlation function of eigenvalues of a random Hermitian operator.

Selberg [11] defines a general class $\mathcal{S}$ of Dirichlet series that admit analytic continuation, functional equation, and Euler product; see below for the definition of $\mathcal{S}$. We refer to Conrey and Ghosh [3], Murty [9], and Kaczorowski and Perelli [6] for the basic properties of $\mathcal{S}$. For the class $\mathcal{S}$, Selberg conjectures an analogue of RH, which we denote by GRH. A function $F \in \mathcal{S}$ is called primitive if $F = F_1F_2$ with $F_1, F_2 \in \mathcal{S}$ implies that $F_1 = 1$ or $F_2 = 1$. One knows by the work of Bochner [1], Selberg [11], and Conrey and Ghosh [3] that every element of $\mathcal{S}$ can be written as a product of primitive functions in $\mathcal{S}$. Conrey and Ghosh [3] prove that Selberg orthonormality conjecture implies that such factorization is unique. In [9], Murty shows that Selberg's orthonormality conjecture implies both the Artin conjecture on the holomorphy of nonabelian L-functions, and the Langlands reciprocity conjecture for solvable extensions of $\mathbb{Q}$. Thus, Selberg's conjectures have important consequences central to number theory.

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The purpose of this paper is to show that the celebrated conjectures of Selberg essentially follow from a pair correlation conjecture for the Selberg class. In fact, we formulate the pair correlation conjecture for two primitive functions of the Selberg class and derive consequences from such behavior. In particular, we show that the pair correlation conjecture implies the unique factorization conjecture.

The Selberg class $\mathcal{S}$ is defined by the following axioms.

(i) (Dirichlet series). Every $F \in \mathcal{S}$ is a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_F(n) n^{-s},$$

absolutely convergent for $\mathrm{Re}\,(s) = \sigma > 1$.

(ii) (Analytic continuation). There exists an integer $m \geq 0$ such that $(s-1)^m F(s)$ is an entire function of finite order.

(iii) (Functional equation). $F \in \mathcal{S}$ satisfies a functional equation of type

$$\Phi(s) = \omega \overline{\Phi}(1-s),$$

where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

with $Q > 0$, $\lambda_j > 0$, $\mathrm{Re}\,\mu_j \geq 0$, and $|\omega| = 1$. Here $\overline{f}(s) = \overline{f(\overline{s})}$.

(iv) (Ramanujan hypothesis). For every $\varepsilon > 0$, $a_F(n) \ll n^{\varepsilon}$.

(v) (Euler product). $F \in \mathcal{S}$ satisfies

$$\log F(s) = \sum_{n=1}^{\infty} b_F(n) n^{-s},$$

where $b_F(n) = 0$ unless $n = p^m$ with $m \geq 1$, and $b_F(n) \ll n^{\theta}$ for some $\theta < 1/2$.

We use the following notation. $d_F = 2\sum_{j=1}^{r} \lambda_j$ is the degree of $F(s)$, $m_F \geq 0$ is the order of the pole at $s = 1$,

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \Lambda_F(n) n^{-s}, \qquad \Lambda_F(n) = b_F(n) \log n,$$

and

$$\psi_{F,G}(x) = \sum_{n \leq x} \Lambda_F(n) \overline{\Lambda_G(n)}.$$

We explicitly remark that all constants in the O and $\ll$-symbols may depend, even if not explicitly written, on the data of the functions of $\mathcal{S}$ involved.

Our first result deals with an upper bound for $\psi_{F,G}(x)$. The trivial upper bound for $\psi_{F,G}(x)$ is $O(x^{1+\varepsilon})$, for every $F, G \in \mathcal{S}$. A better result can be obtained under GRH.

Theorem 1. Let $F \in \mathcal{S}$ and assume GRH. Then

$$\psi_{F,F}(x) \ll x \log^2 x. \qquad \square$$

Clearly, by Cauchy-Schwarz inequality, the same bound holds for $\psi_{F,G}(x)$ as well, for every $F, G \in \mathcal{S}$, under GRH.

Turning to the pair correlation of zeros of functions in $\mathcal{S}$, we define

$$\mathcal{F}_{F,G}(\alpha) = \frac{\pi}{d_F T \log T} \sum_{-T \le \gamma_F, \gamma_G \le T} T^{i\alpha d_F(\gamma_F - \gamma_G)} w(\gamma_F - \gamma_G),$$

where $w(u) = 4/(4+u^2)$. Observe that our definition of $\mathcal{F}_{F,G}(\alpha)$ is not symmetric in F and G; in a way, we fix F and play G against F. Although not strictly necessary, this lack of symmetry is convenient in the proof of Theorem 2 below.

The pair correlation conjecture, denoted by *PC*, is as follows. Assume GRH and let $F, G \in \mathcal{S}$ be primitive. Then, as $T \to \infty$,

$$\mathcal{F}_{F,G}(\alpha) = \begin{cases} \delta_{F,G}|\alpha| + d_G T^{-2|\alpha| d_F} \log T (1 + o(1)) + o(1) & \text{if } |\alpha| \le 1, \\ \delta_{F,G} + o(1) & \text{if } |\alpha| \ge 1 \end{cases}$$

uniformly for α in any bounded interval. Here $\delta_{F,G} = 1$ if $F = G$ and $\delta_{F,G} = 0$ otherwise. We refer to Rudnick and Sarnak [10] for several results on pair correlation conjectures for *GL*(2) L-functions.

We recall that the trivial zeros of a function $F \in \mathcal{S}$ are those coming from the poles of the Γ-factors in its functional equation. We also recall that the zeros of two functions $F, G \in \mathcal{S}$ are called strongly distinct if they are placed at different points. With the above notation, we have the following theorem.

Theorem 2. Let $F, G \in \mathcal{S}$ be primitive functions. Then we have the following.

(i) All but $o(T \log T)$ nontrivial zeros up to T of F(s) are simple under PC.

(ii) All but $o(T \log T)$ nontrivial zeros up to T of F(s) and G(s) are strongly distinct under PC, if $F \neq G$.

(iii) If the asymptotic formula of PC holds for some $\alpha_0 > 0$, then $\mathcal{S}$ has unique factorization. $\square$

We finally remark that (ii) of Theorem 2 should be compared with Theorem 1 of Bombieri and Perelli [2], where the pair correlation conjecture is not assumed.

2 The Landau-Gonek formula and proof of Theorem 1

We first recall some standard analytic properties of the functions $F \in \mathcal{S}$, which are easily proved by classical techniques. First of all, using the functional equation, Stirling's formula and the Phragmén-Lindelöf theorem, we see that $F(s)$ is of finite order in any right half-plane. Moreover, writing

$$N_F(T) = \#\{\rho = \beta + i\gamma : F(\rho) = 0,\ 0 \le \beta \le 1 \text{ and } |\gamma| \le T\},$$

an application of the argument principle shows that

$$N_F(T) = \frac{d_F}{\pi} T \log T + c_F T + O(\log T) \tag{1}$$

with a suitable constant c_F. Finally, given $|t| \gg 1$, there exists $\tau \in [t, t+1]$ such that

$$\frac{F'}{F}(\sigma + i\tau) \ll \log^2 |t| \tag{2}$$

uniformly for $-2 \le \sigma \le 3$.

Our first aim in this section is to extend to $\mathcal{S}$ the uniform version of Landau's formula obtained by Gonek [5]. We remark that, for simplicity, we do not try to get a sharp form of the error term.

Proposition 1. Let $F \in \mathcal{S}$. Then for $T > 1$ and $n \in \mathbb{N}$, $n > 1$, we have

$$\sum_{-T \le \gamma \le T} n^\rho = -\frac{T}{\pi} \Lambda_F(n) + O(n^{3/2} \log^2 T),$$

where $\rho = \beta + i\gamma$ runs over the nontrivial zeros of $F(s)$. □

Proof. We follow the argument in Theorem 1 of [5], so we only give a sketch of the proof. Let $n \in \mathbb{N}$, $n > 1$, $T > 1$, and let R denote the rectangle, oriented counterclockwise, with vertices $(3/2) - iT$, $3/2 + iT$, $-(1/2) + iT$, and $-(1/2) - iT$. Clearly,

$$\frac{1}{2\pi i} \int_R \frac{F'}{F}(s) n^s ds = \sum_\rho n^\rho - m_F n, \tag{3}$$

where ρ runs over the zeros of $F(s)$ inside the rectangle R. We denote by $I_1, \dots, I_4$ the four parts of the integral in (3) relative to the sides of R, starting with the right vertical one and proceeding counterclockwise. Here we assume that T is chosen in a way such that (2) holds. Moreover, we may assume that there are no trivial zeros of $F(s)$ on the left vertical side.

We have

$$I_1 = -\frac{1}{2\pi}\int_{-T}^{T}\sum_{m=1}^{\infty}\Lambda_F(m)\left(\frac{n}{m}\right)^{3/2+it}dt = -\frac{T}{\pi}\Lambda_F(n) + O\left(\sum_{\substack{m=1\\ m\neq n}}^{\infty}|\Lambda_F(m)|\left(\frac{n}{m}\right)^{3/2}\frac{1}{|\log\frac{n}{m}|}\right),$$

the series being convergent by the hypothesis $b_F(n) \ll n^{\theta}$ for some $\theta < 1/2$. Splitting the series into the ranges $m \le n/2$, $n/2 < m < 2n$, $m \ge 2n$, and denoting their contribution by $\sum_1$, $\sum_2$, and $\sum_3$, we see that

$$\sum\nolimits_1 + \sum\nolimits_3 \ll n^{3/2}$$

trivially, while

$$\sum\nolimits_2 \ll \max_{n/2<m<2n}|\Lambda_F(m)|\sum_{n/2<m<2n}\left|\frac{n}{n-m}\right| \ll n^{3/2}.$$

Hence

$$I_1 = -\frac{T}{\pi}\Lambda_F(n) + O(n^{3/2}). \tag{4}$$

From (2), we see immediately that

$$I_2, I_4 \ll n^{3/2}\log^2 T. \tag{5}$$

Finally, in order to estimate I_3, we use the functional equation of $F(s)$ to relate $(F'/F)(-(1/2)+it)$ to $(\overline{F}'/\overline{F})((3/2)-it)$. The Γ-factors give rise to a term of the form $c_1 \log c_2(|t|+2) + O(1/(|t|+2))$, and after integration, we easily get that

$$I_3 \ll n^{3/2}\log^2 T. \tag{6}$$

Hence from (3)–(6), we have

$$\sum_{\rho} n^{\rho} = m_F n - \frac{T}{\pi}\Lambda_F(n) + O(n^{3/2}\log^2 T).$$

But the contribution of the trivial zeros is $O(1)$, and Proposition 1 follows immediately.

Now we are ready for the proof of Theorem 1. Assuming GRH and denoting $1/2+i\gamma$ by ρ, the generic nontrivial zero of $F(s)$, for $T > 1$ from Proposition 1, we get

$$|\Lambda_F(n)|^2 = -\frac{\pi}{T}\sum_{-T\le\gamma\le T}\Lambda_F(n)n^{(1/2)-i\gamma} + O\left(\frac{n^2\log^2 T}{T}\right);$$

and hence, multiplying each side by $\log(x/n)$ and summing over $n \le x$, we obtain

$$\sum_{n\le x} |\Lambda_F(n)|^2 \log\frac{x}{n} \ll \frac{1}{T}\left|\sum_{-T\le\gamma\le T}\sum_{n\le x}\Lambda_F(n) n^{(1/2)-i\gamma}\log\frac{x}{n}\right| + \frac{x^3\log x\log^2 T}{T}. \tag{7}$$

Denoting by $V(\gamma)$ the inner sum in the double sum in the right-hand side of (7), by Perron's formula, we get

$$V(\gamma) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{F'}{F}(s-1/2+i\gamma)\frac{x^s}{s^2}\,ds.$$

Shifting the line of integration to $\sigma = 3/4$, by a standard argument, we obtain

$$\begin{aligned} V(\gamma) &= m_F\frac{x^{(3/2)-i\gamma}}{(3/2-i\gamma)^2} - \sum_\rho \frac{x^{\rho+(1/2)-i\gamma}}{(\rho+1/2-i\gamma)^2} + O(x^{3/4}\log(|\gamma|+2)) \\ &\ll \frac{x^{3/2}}{|\gamma|^2+1} + x\sum_{\gamma}\frac{1}{|\gamma-\gamma|^2+1} + x^{3/4}\log(|\gamma|+2), \end{aligned} \tag{8}$$

since $(F'/F)(1/4+it) \ll \log(|t|+2)$ under GRH. Moreover, from (1), we have

$$\sum_{\gamma}\frac{1}{|\gamma-\gamma|^2+1} \ll \log(|\gamma|+2). \tag{9}$$

From (7)–(9), we get

$$\sum_{n\le x}|\Lambda_F(n)|^2\log\frac{x}{n} \ll x\log^2 T + \frac{x^3\log x\log^2 T}{T},$$

and hence, choosing $T = x^3$, we obtain

$$\sum_{n\le x}|\Lambda_F(n)|^2\log\frac{x}{n} \ll x\log^2 x. \tag{10}$$

We remove the factor $\log(x/n)$ in (10) in a standard way, thus getting from (10) that

$$\sum_{x/2<n\le x}|\Lambda_F(n)|^2 \ll x\log^2 x. \tag{11}$$

Hence Theorem 1 follows from (11) by a standard dichotomy argument.

3 The explicit formula method and pair correlation

Here we extend the theorem in [7] to Selberg's class. Since we follow closely the argument in [7], we only give a sketch of the proof.

It is easy to see that in the case of $F \in \mathcal{S}$, the last formula on p. 185 of [7] becomes, for $x > 1$ and $x \neq p^m$,

$$\sum_{n\le x}\Lambda_F(n)n^{-s} = -\frac{F'}{F}(s) + m_F\frac{x^{1-s}}{1-s} - \sum_{\rho}\frac{x^{\rho-s}}{\rho-s} + \sum_{j=1}^{r}\sum_{n=0}^{\infty}\frac{x^{-((n+\mu_j)/\lambda_j)-s}}{\frac{n+\mu_j}{\lambda_j}+s} - m_F\frac{x^{-s}}{s}, \tag{12}$$

provided $s \neq 1$ if $m_F \neq 0$ and s is not a zero of $F(s)$; here ρ runs over the nontrivial zeros of $F(s)$. This follows from an application of Perron's formula and standard bounds for $F'/F(s)$, like (2). Moreover, if $m_F \neq 0$, the term $-m_F(x^{-s}/s)$ must be canceled by some of the terms with $n = 0$ in the double sum in (12).

Assuming GRH, (12) becomes

$$\sum_{\gamma}\frac{x^{i\gamma}}{(\sigma-1/2)+i(t-\gamma)} = x^{\sigma-(1/2)+it}\left\{\frac{F'}{F}(s) + \sum_{n\le x}\Lambda_F(n)n^{-s} - m_F\frac{x^{1-s}}{1-s}\right.$$
$$\left. - \sum_{j=1}^{r}\sum_{n=0}^{\infty}\frac{x^{-((n+\mu_j)/\lambda_j)-s}}{\frac{n+\mu_j}{\lambda_j}+s} + m_F\frac{x^{-s}}{s}\right\}. \tag{13}$$

We compute (13) for $s = \sigma + it$ and $s' = 1 - \sigma + it$, with $\sigma > 1$, and then subtract. The left-hand side becomes

$$(2\sigma-1)\sum_{\gamma}\frac{x^{i\gamma}}{(\sigma-1/2)^2+(t-\gamma)^2}, \tag{14}$$

while the right-hand side is

$$-x^{-(1/2)}\left\{\sum_{n>x}\Lambda_F(n)\left(\frac{x}{n}\right)^{\sigma+it} + \frac{F'}{F}(1-\sigma+it)x^{1-\sigma+it} + \sum_{n\le x}\Lambda_F(n)\left(\frac{x}{n}\right)^{1-\sigma+it}\right.$$
$$- m_F\frac{(2\sigma-1)x}{(\sigma-it)(\sigma-1+it)} + \sum_{j=1}^{r}\sum_{n=0}^{\infty}\frac{(2\sigma-1)x^{-((n+\mu_j)/\lambda_j)}}{(\sigma-1-it-\frac{n+\mu_j}{\lambda_j})(\sigma+it+\frac{n+\mu_j}{\lambda_j})}$$
$$\left. -m_F\frac{(2\sigma-1)}{(\sigma-1-it)(\sigma+it)}\right\}. \tag{15}$$

By the functional equation, we get

$$\frac{F'}{F}(1-\sigma+it) = -2\log Q - \frac{\overline{F}'}{\overline{F}}(\sigma-it) - \sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}(\lambda_j(1-\sigma+it)+\mu_j)$$
$$- \sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}(\lambda_j(\sigma-it)+\overline{\mu}_j),$$

and hence by Stirling's formula, see, e.g., chapter 10 of Davenport [4], writing $\tau = |t| + 2$, we obtain that

$$\frac{F'}{F}(1-\sigma+it) = -2\log Q - \frac{\overline{F}'}{\overline{F}}(\sigma-it) - 2\sum_{j=1}^{r}\lambda_j \log \lambda_j \tau + O\left(\frac{1}{\tau}\right)$$
$$= -d_F \log \tau + O_\sigma(1).$$

Hence, by an easy computation, (15) becomes

$$-x^{-(1/2)}\left(\sum_{n\le x}\Lambda_F(n)\left(\frac{x}{n}\right)^{1-\sigma+it} + \sum_{n>x}\Lambda_F(n)\left(\frac{x}{n}\right)^{\sigma+it}\right)$$
$$+ x^{(1/2)-\sigma+it}(d_F \log\tau + O_\sigma(1)) + O_\sigma(x^{1/2}\tau^{-2}) + O_\sigma(x^{-(1/2)}\tau^{-1}). \quad (16)$$

Choosing $\sigma = 3/2$, from (13), (14), and (16) we get

$$2\sum_{\gamma}\frac{x^{i\gamma}}{1+(t-\gamma)^2} = -x^{-(1/2)}\left(\sum_{n\le x}\Lambda_F(n)\left(\frac{x}{n}\right)^{-(1/2)+it} + \sum_{n>x}\Lambda_F(n)\left(\frac{x}{n}\right)^{(3/2)+it}\right)$$
$$+ x^{-1+it}d_F\log\tau + O(x^{-1}) + O(x^{1/2}\tau^{-2}) + O(x^{-(1/2)}\tau^{-1}), \quad (17)$$

which, with obvious notation, we write as

$$L_F(x,t) = R_F(x,t).$$

Given $F, G \in \mathcal{S}$ satisfying GRH, we denote by γ_F and γ_G the imaginary part of the nontrivial zeros of $F(s)$ and $G(s)$. We have

$$\int_{-T}^{T} L_F(x,t)\overline{L_G(x,t)}\,dt = 4\sum_{\gamma_F,\gamma_G} x^{i(\gamma_F-\gamma_G)}\int_{-T}^{T}\frac{dt}{(1+(t-\gamma_F)^2)(1+(t-\gamma_G)^2)}. \quad (18)$$

Arguing as in Section 3 of [7] and using standard estimates on the number of zeros of $F(s)$ and $G(s)$ (see (1)), we get

$$\sum_{\substack{\gamma_F,\gamma_G\\ \gamma_F\notin[-T,T]}}\int_{-T}^{T}\frac{dt}{(1+(t-\gamma_F)^2)(1+(t-\gamma_G)^2)} \ll \log^3 T, \quad (19)$$

and similarly, this is true for the sum with $\gamma_G \notin [-T,T]$. Moreover,

$$\sum_{-T\le\gamma_F,\gamma_G\le T}\int_{T}^{\infty}\frac{dt}{(1+(t-\gamma_F)^2)(1+(t-\gamma_G)^2)} \ll \log^2 T, \quad (20)$$

and similarly, this is true for the integral from $-\infty$ to $-T$. Hence from (19) and (20), the right-hand side of (18) is

$$= 4 \sum_{-T\le\gamma_F,\gamma_G\le T} x^{i(\gamma_F-\gamma_G)} \int_{-\infty}^{\infty} \frac{dt}{(1+(t-\gamma_F)^2)(1+(t-\gamma_G)^2)} + O(\log^3 T), \tag{21}$$

and from the calculus of residues, the integral in (21) equals

$$\frac{\pi}{2}\frac{4}{4+(\gamma_F-\gamma_G)^2} = \frac{\pi}{2}w(\gamma_F-\gamma_G).$$

Hence we have obtained that

$$\int_{-T}^{T} L_F(x,t)\overline{L_G(x,t)}\,dt = 2\pi \sum_{-T\le\gamma_F,\gamma_G\le T} x^{i(\gamma_F-\gamma_G)} w(\gamma_F-\gamma_G) + O(\log^3 T) \tag{22}$$

uniformly in x.

Now, with obvious notation, we write $R_F(x,t)$ as

$$R_F(x,t) = A_F + B_F + C_F,$$

and similarly for $R_G(x,t)$, and compute the integral of the cross products. Writing

$$\Lambda_F(n,x) = \begin{cases} \Lambda_F(n)\left(\frac{n}{x}\right)^{1/2} & n \le x, \\ \Lambda_F(n)\left(\frac{x}{n}\right)^{3/2} & n > x \end{cases} \tag{23}$$

and using Montgomery and Vaughan's [8] large sieve in the form

$$\int_{-T}^{T} \left(\sum_n a_n n^{-it}\right)\overline{\left(\sum_n b_n n^{-it}\right)}\,dt = 2T\sum_n a_n\overline{b}_n + O\left(\left(\sum_n n|a_n|^2\right)^{1/2}\left(\sum_n n|b_n|^2\right)^{1/2}\right),$$

we get

$$\int_{-T}^{T} A_F\overline{A_G}\,dt = \frac{2T}{x}\sum_{n=1}^{\infty}\Lambda_F(n,x)\overline{\Lambda_G(n,x)} + O\left(\frac{1}{x}\left(\sum_n n|\Lambda_F(n,x)|^2\right)^{1/2}\left(\sum_n n|\Lambda_G(n,x)|^2\right)^{1/2}\right).$$

By partial summation, from Theorem 1, we get

$$\sum_n n|\Lambda_F(n,x)|^2 \ll x^2\log^2 x$$

and hence

$$\int_{-T}^{T} A_F\overline{A_G}\,dt = \frac{2T}{x}\sum_{n=1}^{\infty}\Lambda_F(n,x)\overline{\Lambda_G(n,x)} + O(x\log^2 x), \tag{24}$$

the sum being clearly convergent. Similarly,

$$\int_{-T}^{T} |A_F|^2\,dt \ll T\log x + x\log^2 x. \tag{25}$$

It is easy to see that

$$\int_{-T}^{T} B_F\overline{B_G}\,dt = 2d_F d_G\frac{T\log^2 T}{x^2} + O\left(\frac{T\log T}{x^2}\right) \tag{26}$$

and

$$\int_{-T}^{T} |B_F|^2\,dt \ll \frac{T\log^2 T}{x^2}. \tag{27}$$

Moreover, we have trivially that

$$\int_{-T}^{T} C_F\overline{C_G}\,dt \ll \frac{T}{x^2} + x \tag{28}$$

and

$$\int_{-T}^{T} |C_F|^2\,dt \ll \frac{T}{x^2} + x. \tag{29}$$

From (24)–(29), we see that, for $T \geq x$,

$$\int_{-T}^{T} R_F(x,t)\overline{R_G(x,t)}\,dt = \frac{2T}{x}\sum_{n=1}^{\infty}\Lambda_F(n,x)\overline{\Lambda_G(n,x)} + 2d_F\,d_G\frac{T\log^2 T}{x^2} + O\left(x\log^2 x + \frac{T\log T\log^{1/2} x}{x} + \left(\frac{T}{x}\right)^{1/2}\log T\log x\right),$$

and hence from (17) and (22), we get

$$2\pi \sum_{-T\le\gamma_F,\gamma_G\le T} x^{i(\gamma_F-\gamma_G)} w(\gamma_F-\gamma_G)$$
$$= \frac{2T}{x}\sum_{n=1}^{\infty} \Lambda_F(n,x)\overline{\Lambda_G(n,x)} + 2d_F d_G \frac{T\log^2 T}{x^2}$$
$$+ O\left(x\log^2 x + \frac{T\log T\log^{1/2} x}{x} + \left(\frac{T}{x}\right)^{1/2}\log T\log x + \log^3 T\right) \tag{30}$$

uniformly for $T \ge x > 1$.

Given $\varepsilon > 0$, write $x = T^{\alpha d_F}$ and recall that $d_F \ge 1$ by Theorem 3.1 of [3]. Hence, recalling that

$$\mathcal{F}_{F,G}(\alpha) = \mathcal{F}_{F,G}(\alpha, T) = \frac{\pi}{d_F T\log T}\sum_{-T\le\gamma_F,\gamma_G\le T} T^{i\alpha d_F(\gamma_F-\gamma_G)} w(\gamma_F-\gamma_G),$$

from (30) we see that for $0 \le \alpha \le (1-\varepsilon)/d_F$,

$$\mathcal{F}_{F,G}(\alpha) = \frac{1}{d_F x\log T}\sum_{n=1}^{\infty}\Lambda_F(n,x)\overline{\Lambda_G(n,x)} + d_G T^{-2\alpha d_F}\log T$$
$$+ O(\alpha^{1/2}T^{-\alpha d_F}\log^{1/2} T) + O\left(\frac{1}{\log T}\right).$$

Writing $K = T^{-2\alpha d_F}\log T$, we easily see that

$$K + O(\sqrt{\alpha K}) = \begin{cases} o(1) & \text{if } \alpha \ge \frac{3}{4}\frac{\log\log T}{d_F\log T}, \\ K(1+o(1)) & \text{if } \alpha \le \frac{3}{4}\frac{\log\log T}{d_F\log T}, \end{cases}$$

and hence, using the notation in (23), we finally get the following proposition.

Proposition 2. Assume GRH, and let $F, G \in \mathcal{S}$, $\varepsilon > 0$, and $x = T^{\alpha d_F}$. Then, uniformly for $0 \le \alpha \le (1-\varepsilon)/d_F$, as $T \to \infty$, we have

$$\mathcal{F}_{F,G}(\alpha) = \frac{1}{d_F x\log T}\sum_{n=1}^{\infty}\Lambda_F(n,x)\overline{\Lambda_G(n,x)} + d_G T^{-2\alpha d_F}\log T(1+o(1)) + o(1).$$ □

4 The pair correlation conjecture and proof of Theorem 2

By partial summation, we get

$$\sum_{n=1}^{\infty}\Lambda_F(n,x)\overline{\Lambda_G(n,x)} = -\frac{1}{x}\int_1^x \psi_{F,G}(t)dt + 3x^3\int_x^{\infty}\frac{\psi_{F,G}(t)}{t^4}dt, \tag{31}$$

the second integral being trivially convergent. Based on Selberg's conjectures, we may expect that, given two primitive functions $F, G \in \mathcal{S}$,

$$\psi_{F,G}(x) = \begin{cases} (1+o(1))x\log x & \text{if } F = G, \\ o(x\log x) & \text{if } F \neq G. \end{cases} \tag{32}$$

More generally, given any two functions $F, G \in \mathcal{S}$, we may expect that

$$\psi_{F,G}(x) = (m_{F\otimes\overline{G}} + o(1))x\log x, \tag{33}$$

where $m_{F\otimes\overline{G}}$ is a nonnegative integer such that $m_{F\otimes\overline{F}} \geq 1$ and $m_{F\otimes\overline{G}}$ is consistent with (32); i.e.,

$$m_{F\otimes\overline{G}} = \begin{cases} 1 & F = G, \\ 0 & F \neq G \end{cases}$$

for primitive F and G. Hence from (31) and (33), we see that

$$\sum_{n=1}^{\infty} \Lambda_F(n,x)\overline{\Lambda_G(n,x)} = (m_{F\otimes\overline{G}} + o(1))x\log x, \tag{34}$$

and from (34) and Proposition 2, we obtain

$$\mathcal{F}_{F,G}(\alpha) = \alpha m_{F\otimes\overline{G}} + d_G T^{-2\alpha d_F}(1+o(1)) + o(1) \tag{35}$$

under the assumption (33) and the hypotheses of Proposition 2.

Hence, based on (35), we state the pair correlation conjecture *PC* in the case of primitive $F, G \in \mathcal{S}$. Assuming GRH,

$$\mathcal{F}_{F,G}(\alpha) = \begin{cases} \delta_{F,G}|\alpha| + d_G T^{-2|\alpha|d_F}\log T(1+o(1)) + o(1) & \text{if } |\alpha| \leq 1, \\ \delta_{F,G} + o(1) & \text{if } |\alpha| \geq 1 \end{cases}$$

as $T \to \infty$, uniformly for α in any bounded interval. Observe that

$$\mathcal{F}_{F,G}(-\alpha) = \overline{\mathcal{F}_{F,G}(\alpha)}, \tag{36}$$

so the asymptotic behavior of $\mathcal{F}_{F,G}(\alpha)$ is known over $\mathbb{R}$, under *PC*.

Now we derive some consequences from *PC*, of a type similar to that in Montgomery [7]. By (3) of [7], we have

$$\sum_{-T\leq\gamma_F,\gamma_G\leq T} r\left((\gamma_F-\gamma_G)\frac{d_F\log T}{2\pi}\right) w(\gamma_F-\gamma_G) = \frac{d_F}{\pi} T\log T \int_{-\infty}^{+\infty} \mathcal{F}_{F,G}(\alpha)\hat{r}(\alpha)\,d\alpha \tag{37}$$

for suitable kernels $r(u)$, where the Fourier transform $\hat{r}(\alpha)$ is defined as

$$\hat{r}(\alpha) = \int_{-\infty}^{+\infty} r(u)e(-\alpha u)du, \quad e(x) = e^{2\pi i x}.$$

Choosing $r_\delta(u) = (\sin \pi\delta u/\pi\delta u)^2$ with any $\delta > 0$, it is well known that

$$\hat{r}_\delta(\alpha) = \begin{cases} \frac{1}{\delta^2}(\delta - |\alpha|) & \text{if } |\alpha| \le \delta, \\ 0 & \text{if } |\alpha| \ge \delta. \end{cases}$$

Hence by partial integration, we get that

$$\int_{-\infty}^{+\infty} aT^{-2|\alpha|b}(\log T)\hat{r}_\delta(\alpha)d\alpha = \frac{1}{\delta}\frac{a}{b} + O\left(\frac{1}{\delta^2 \log T}\right) \tag{38}$$

for any fixed constants $a, b > 0$, while it is immediate to see that

$$\int_{-\infty}^{+\infty} f(\alpha)\hat{r}_\delta(\alpha)d\alpha = \begin{cases} \frac{1}{3}\delta & \text{if } \delta \le 1, \\ 1 - \frac{1}{\delta} + \frac{1}{3\delta^2} & \text{if } \delta \ge 1, \end{cases} \tag{39}$$

where

$$f(\alpha) = \begin{cases} |\alpha| & \text{if } |\alpha| \le 1, \\ 1 & \text{if } |\alpha| \ge 1. \end{cases}$$

For a given $\delta > 0$, we denote by $PC(\delta)$ the pair correlation hypothesis PC in the restricted range $|\alpha| \le \delta$. Hence, from $PC(\delta)$ and (36)–(39), we obtain that, for a primitive F,

$$\sum_{-T \le \gamma_F, \gamma'_F \le T} \left(\frac{\sin \frac{\delta}{2}(\gamma_F - \gamma'_F)d_F \log T}{\frac{\delta}{2}(\gamma_F - \gamma'_F)d_F \log T}\right)^2 w(\gamma_F - \gamma'_F) \sim \left(\frac{1}{\delta} + \frac{\delta}{3}\right)\frac{d_F}{\pi} T \log T$$

if $\delta \le 1$, and

$$\sum_{-T \le \gamma_F, \gamma'_F \le T} \left(\frac{\sin \frac{\delta}{2}(\gamma_F - \gamma'_F)d_F \log T}{\frac{\delta}{2}(\gamma_F - \gamma'_F)d_F \log T}\right)^2 w(\gamma_F - \gamma'_F) \sim \left(1 + \frac{1}{3\delta^2}\right)\frac{d_F}{\pi} T \log T \tag{40}$$

if $\delta \ge 1$, while for primitive F and G, $F \ne G$, we have

$$\sum_{-T \le \gamma_F, \gamma_G \le T} \left(\frac{\sin \frac{\delta}{2}(\gamma_F - \gamma_G)d_F \log T}{\frac{\delta}{2}(\gamma_F - \gamma_G)\, d_F \log T}\right)^2 w(\gamma_F - \gamma_G) \sim \left(\frac{1}{\delta}\frac{d_G}{d_F}\right)\frac{d_F}{\pi} T \log T. \tag{41}$$

It is now easy to prove the first two assertions of Theorem 2. In fact, choosing δ arbitrarily large, from (40) we have

$$\sum_{-T \le \gamma_F \le T} m\left(\frac{1}{2} + i\gamma_F\right) \le (1 + \varepsilon)\frac{d_F}{\pi} T \log T,$$

where $m(1/2+i\gamma_F)$ denotes the multiplicity of the zero $(1/2)+i\gamma_F$ and $\varepsilon>0$ is arbitrarily small, and hence (i) of Theorem 2 follows in view of (1). Similarly, from (41), we obtain

$$\sum_{\substack{-T\le\gamma_F,\gamma_G\le T\\ \gamma_F=\gamma_G}} 1 = o(T\log T),$$

and (ii) follows.

In order to prove the last assertion of Theorem 2, we assume that the asymptotic formula of *PC* holds for some $\alpha_0>0$. Suppose that there is no unique factorization in $\mathcal{S}$. Then there exists $F\in\mathcal{S}$ such that

$$\prod_{j=1}^{k} F_j(s)^{e_j} = F(s) = \prod_{i=1}^{h} G_i(s)^{f_i} \tag{42}$$

with $e_j, f_i\ge 1$ and $F_j(s), G_i(s)$ primitive and distinct. We consider the quantity

$$\mathcal{F}(\alpha)=\frac{\pi}{d_{F_1}T\log T}\sum_{-T\le\gamma_{F_1},\gamma_F\le T} T^{i\alpha d_{F_1}(\gamma_{F_1}-\gamma_F)}w(\gamma_{F_1}-\gamma_F)$$

and compute $\mathcal{F}(\alpha)$ in two different ways by (42). Using the left-hand side of (42), we get

$$\begin{aligned}\mathcal{F}(\alpha)&=\frac{\pi}{d_{F_1}T\log T}\sum_{j=1}^{k} e_j\sum_{-T\le\gamma_{F_1},\gamma_{F_j}\le T} T^{i\alpha d_{F_1}(\gamma_{F_1}-\gamma_{F_j})}w(\gamma_{F_1}-\gamma_{F_j})\\ &=e_1\frac{\pi}{d_{F_1}T\log T}\sum_{-T\le\gamma_{F_1},\gamma'_{F_1}\le T} T^{i\alpha d_{F_1}(\gamma_{F_1}-\gamma'_{F_1})}w(\gamma_{F_1}-\gamma'_{F_1})\\ &\quad+\sum_{j=2}^{k} e_j\frac{\pi}{d_{F_1}T\log T}\sum_{-T\le\gamma_{F_1},\gamma_{F_j}\le T} T^{i\alpha d_{F_1}(\gamma_{F_1}-\gamma_{F_j})}w(\gamma_{F_1}-\gamma_{F_j})\\ &=e_1\mathcal{F}_{F_1,F_1}(\alpha)+\sum_{j=2}^{k} e_j\mathcal{F}_{F_1,F_j}(\alpha).\end{aligned} \tag{43}$$

Analogously, using the right-hand side of (42), we get

$$\mathcal{F}(\alpha)=\sum_{i=1}^{h} f_i\mathcal{F}_{F_1,G_i}(\alpha). \tag{44}$$

Choosing $\alpha=\alpha_0$ in (43) and (44), by our hypothesis, we get

$$e_1\alpha_0+o(1)=\mathcal{F}(\alpha_0)=o(1)$$

as $T\to\infty$, a contradiction. Theorem 2 is therefore proved. ∎

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