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ON THE ESTIMATION OF EIGENVALUES OF HECKE OPERATORS

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Dedicated to the Memory of E.G. Straus and R.A. Smith

1. Introduction. Let $\mathbf{A}$ denote the adele ring of the rational numbers $\mathbf{Q}$ and suppose that we have a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A})$. (For the terminology and the details, we refer the reader to Gelbart [3] and Jacquet and Langlands [5]). Langlands [9] has described how one can attach an L-function to π. To describe this construction briefly, one can associate to π, a family of local representations π_p for each prime p of $\mathbf{Q}$. This family is uniquely determined by π such that

(i) π_p is irreducible for every p,

(ii) for all but finitely many primes p, π_p is unramified (that is, the restriction of π_p to $\mathrm{GL}_2(\mathbf{Z}_p)$ contains the identity representation exactly once),

(iii) π can be factored as the restricted infinite tensor product $\pi = \bigotimes_p \pi_p$.

Let S denote the set of primes p for which π_p is ramified. For $p \notin S$, it is known that π_p corresponds canonically to a semisimple conjugacy class σ_p in $\mathrm{GL}_2(\mathbf{C})$, where σ_p contains a matrix of the form

$$\begin{vmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{vmatrix}.$$

If r denotes any finite dimensional complex representation of $\mathrm{GL}_2(\mathbf{C})$, one can attach an L-series $L(s, \pi, r)$ as follows.

$$L(s, \pi, r) = \prod_p L(s, \pi_p, r),$$

where

$$L(s, \pi_p, r) = \det(1 - r(\sigma_p)p^{-s})^{-1}$$

whenever π_p is unramified. If π_p is ramified, and r is standard, we refer to J-L [5] for the definition of $L(s, \pi_p, r)$. It is known [9] that each $L(s, \pi, r)$

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converges in some half-plane and we conjecture that it has an analytic continuation and satisfies a suitable functional equation. We also have the Ramanujan-Petersson conjecture

$$|\alpha_p| = |\beta_p| = 1.$$

If r is ρ_2, the standard representation, we shall simply write $L(s, \pi)$ for $L(s, \pi, \rho_2)$. In fact, $L(s, \pi)$ has classical origins. $L(s, \pi)$ is the Mellin transform of either a cusp form or a Maass wave form on $\Gamma_0(N)$, for some natural number N.
More precisely, let

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

be a cusp form on $\Gamma_0(N)$ of integral weight k and Nebentypus $\chi \pmod N$. Suppose further that f is a normalised eigenfunction for all the Hecke operators. Then there exists a cuspidal automorphic representation π_f of $GL_2(\mathbf{A})$ such that

$$L(s, \pi_f) = (2\pi)^{-(s+k-1/2)}\, \Gamma\left(s + \frac{k-1}{2}\right) \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where $a_n = n^{-(k-1)/2}\, a(n)$. Moreover, for this π_f, we have

$$\alpha_p \beta_p = \chi(p) \text{ and } \alpha_p + \beta_p = a_p.$$

The Ramanujan-Petersson conjecture in this case of holomorphic forms was settled by Deligne [2].

Now, let f be a real analytic modular form on $\Gamma_0(N)$ with character $\chi \pmod N$. That is, f satisfies $f((az + b)(cz + d)) = \chi(d) f(z)$. A Maass form (of weight 0) is an eigenfunction of the non-Euclidean Laplacian,

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

which also satisfies polynomial growth conditions at each cusp of $\Gamma_0(N)\backslash \mathfrak{h}$, where $\mathfrak{h}$ denotes the upper half plane. Let f, satisfying $\Delta f = \lambda f$, be an eigenfunction of the Hecke operators. Such an f necessarily has a Fourier expansion of the form

$$f(z) = \sum_{n \neq 0} a_n \sqrt{y}\, K_{ir}(4\pi |n| y) e^{2\pi i n x},$$

where $K_s(u) = \int_0^\infty e^{-y/2(u+1/u)}\, u^{s-1}\, du$, and $\lambda = 1/4 + r^2$ is the eigenvalue. The Ramanujan-Petersson conjecture for Maass forms then reads

$$|a_n| \leqq |a_1| \cdot d(n),$$

where $d(n)$ is the number of divisors of n. This conjecture is still open.

As a consequence of recent developments. Serre has observed that one can show

$$a_n = O(n^{1/5}d(n)).$$

In connection with the eigenvalue λ, one knows that r is purely real, or purely imaginary. We have the conjecture (Selberg)

$$\lambda \geqq \frac{1}{4}.$$

Selberg showed **[13]** that $\lambda \geqq 3/15$. In **[9]**, Langlands viewed the Selberg conjecture as the Ramanujan conjecture at the infinite prime and showed how one can deduce both of these conjectures if we had an analytic continuation of $L(s, \pi, r_m)$, where r_m denotes the m-th symmetric power of the standard embedding ρ_2 of $\mathrm{GL}_2(\mathbf{C})$ into $\mathrm{GL}_2(\mathbf{C})$. We describe Langlands conjectural approach below.

2. Upper bounds for eigenvalues. If $m = 2$, 3 or 4, it is known that $L(s, \pi, r_m)$ has a meromorphic continuation to the entire complex plane and satisfies a suitable functional equation. The case $m = 2$ (for which $L(s, \pi, r_2)$ is in fact proved to be entire) is due to Shimura **[15]**, when π is associated to a holomorphic modular form, and the general case for $m = 2$ is due to Gelbart and Jacquet **[4]**. Shahidi **[14]** recently settled the cases of $m = 3$ and 4. These results are valid over any field and not just over $\mathbf{Q}$.

The Euler factors of $L(s, \pi, r_m)$ are easy to describe. We have, for unramified π_p,

$$L(s, \pi_p, r_m) = \prod_{j=0}^{m} \left(1 - \frac{\alpha_p^{m-j}\,\beta_p^j}{p^s}\right)^{-1}.$$

From the analytic properties of $L(s, \pi, r_m)$, we would like to infer the size of the α_p.

The philosophy behind this idea can be revealed by the following simple observation. Suppose $a_n \geqq 0$ and $f(s) = \sum_{n=1}^{\infty} a_n/n^s$ respresents an analytic function for $\mathrm{Re}\, s > 1$. Then, it is almost immediate that $a_n = O(n^{1+\varepsilon})$. To see this, we first note that

$$e^{-1/x} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)x^s\, ds.$$

Therefore

$$\sum_{n=1}^{\infty} a_n e^{-n/x} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)f(s)x^s\, ds.$$

Moving the line of integration to $\sigma = 1 + \varepsilon$, we obtain $\sum_{n=1}^{\infty} a_n e^{-n/x} =$

$O(x^{1+\varepsilon})$, as the Γ-function decays exponentially as $t \to \infty$ in any half-plane and $f(s)$ is bounded for $\sigma \geqq 1 + \varepsilon$. It follows that $a_n = O(n^{1+\varepsilon})$.

We remark that this result cannot be substantially improved. Indeed, consider

$$\zeta(ks - k + 1) = \sum_{n=1}^{\infty} \frac{n^{k-1}}{n^{ks}}.$$

This series converges for Re $s > 1$ and $a_n = \Omega(n^{1-1/k})$ As k can be arbitrarily large, we find that there exist series with $a_n \geqq 0$ and $a_n = \Omega(n^{1-\varepsilon})$.

Now, suppose $L(s, \pi, r_m \otimes \tilde{r}_m)$ is analytic for Re $s > 1$, where $\tilde{r}_m$ denotes the contragredient representation of r_m. Then it follows from the above that

$$\left| \sum_{j=0}^{m} \alpha_p^{m-j} \beta_p^j \right|^2 = O(p^{1+\varepsilon}).$$

Since $|\alpha_p \beta_p| = 1$, we must have one of $|\alpha_p|$ or $|\beta_p|$, bounded. Suppose without loss, $\beta_p = O(1)$. Then, we see that

$$\alpha_p = O(p^{1/2m+\varepsilon})$$

follows from the preceding discussion. In an analogous fashion, considering $L(s, \pi_p, r_m \otimes \tilde{r}_m)$ when $p = \infty$, we find that $|\text{Re } r| \leqq 1/m$ as $L(s, \pi, r_m \otimes \tilde{r}_m)$ is analytic for Re $s > 1$. Letting $m \to \infty$, we deduce both the Ramanujan-Petersson conjecture and the Selberg conjecture. These considerations are due to Langlands **[9]**. Analogous ideas were utilised by Deligne **[2]** in his proof of the Ramanujan-Petersson conjecture for the case of holomorphic forms. In that case, the p-Euler factor can be identified as an L-function of a certain algebraic variety over the rational function field over the finite field of p-elements. The necessary analytical properties are deduced by the etale cohomology theory of such varieties.

It is a result of Jacquet, Piatetskii-Shapiro and Shalika (see H. Jacquet, From GL_2 to GL_n, 1975 U.S.-Japan Seminar on Number Theory, Ann Arbor) that $L(s, \pi, r_2 \otimes \tilde{r}_2)$ has an analytic continuation to the entire complex plane except for a simple pole at $s = 1$ arising from $\zeta(s)$. By our earlier remarks, it follows that $a_p = O(p^{1/4+\varepsilon})$ and that $\lambda \geqq 3/16$.

Utilising a general lemma on Dirichlet series with functional equations, one can improve the above exponent of 1/4 to 1/5, as was noticed by Serre. Such a lemma was first proved by Landau [7] and generalized by Chandrasekharan and Narasimhan **[1]**. We state a specialization of their theorem for future reference.

LEMMA 1. *Let $a_n \geqq 0$ and set $f(s) = \sum_{n=1}^{\infty} a_n/n^s$. Suppose $f(s)$ is convergent in some half-plane and that it has analytic continuation, except for a simple pole at $s = 1$ of order k, to the entire complex plane and it satisfies a functional equation*

$$c^s \Delta(s) f(s) = c^{1-s} \Delta(1-s) f(1-s)$$

where c is a certain positive constant and $\Delta(s) = \prod_{i=1}^N \Gamma(\alpha_i s + \beta_i)$. *Then*

$$\sum_{n \leqq x} a_n = x P_{k-1}(\log x) + O(x^{(2A-1)(2A+1)} \log^{k-1} x),$$

where $A = \sum_{i=1}^N \alpha_i$, *and* P_{k-1} *is a polynomial of degree* $k - 1$.

REMARK. If, instead of $a_n \geqq 0$, we assume that $\sum_{x \leqq n \leqq x'} |a_n| = O(x^\theta)$, then the conclusion of the lemma still holds with an error term of

$$O(x^{(2A-1)(2A+1)} \log^{k-1} x + x^\theta).$$

(See [1] or[7] for a proof.)

It is known that if we set $\Gamma(s, \pi, r_m)$ to be the product If the Γ-factors appearing in $L(s, \pi, r_m)$, then

$$\Gamma(s, \pi, r_m) = \prod_{j=0}^m \pi^{-(s-\lambda_j)/2} \Gamma\left(\frac{s - \lambda_j}{2}\right),$$

where λ_j are certain real numbers. If π corresponds to a Maass wave form with eigenvalue $\lambda = 1/4 + r^2$, then $\lambda_j = i(m - 2j)r$ in the above product. With this definition of the Γ-factors, we have the functional equation

$$L(s, \pi, r_m) = L(1 - s, \pi, \tilde{r}_m),$$

where $\tilde{r}_m$ is the contragredient of r_m, for $m = 2, 3$ and 4.

Applying the above lemma to the Dirichlet series $f(s) = L(s, \pi, r_2 \otimes \tilde{r}_2)$, we find that

$$|a_p| \leqq p^{1/5} + p^{-1/5}$$

since $f(s)$ has a simple pole at $s = 1$.

We can ask if there is a corresponding improvement at the infinite prime. That is, can we obtain a better lower bound for λ_1?

To this end, let us define

$$L_m(s) = \prod_{p \neq \infty} L(s, \pi_p, r_m)$$

and write

$$L_2(s) L_4(s) = \sum_1^\infty \frac{a_n}{n^s},$$

for Re $s > 1$. We have the following

THEOREM 1. *The Dirichlet series* $\sum_1^\infty a_n/n^s$ *converges for* Re $s > 9/11$ *and therefore the function*

$$g(s) = L_2(s) L_4(s)$$

has an analytic continuation for Re $s > 9/11$. *If* $g(s)$ *has no real zeroes for* $s > 9/11$, *then we have* $\lambda_1 \geqq 403/1936 = .208\ldots$.

REMARK. If, in addition, we assume the Ramanujan-Petersson conjecture, we obtain $\lambda_1 > .21$.

PROOF. We have that if

$$f(s) = \zeta(s)g(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

then $a_n \geqq 0$ and

$$\sum_{n \leqq x} a_n = cx + O(x^{4/5}),$$

by our earlier remarks. We would like to apply Lemma 1 to $g(s)$ and obtain information concerning partial sums of the coefficients. In view of the remark following Lemma 1, we need to obtain an estimate for

$$\sum_{x<n<x+x^{4/5}} |c_n|.$$

Since $c_n = \sum_{d|n} \mu(d)a_{n/d}$, where μ is the familar Möbius function, we have the majorization

$$|c_n| \leqq \sum_{d|n} a_d.$$

Applying Lemma 1 to the series $\zeta(s)f(s)$, we find that

$$\sum_{x<n<x+x^{4/5}} |c_n| = O(x^{9/11}).$$

Therefore, applying Lemma 1 to $g(s)$ yields

$$\sum_{n \leqq x} c_n = O(x^{9/11}).$$

This establishes that the Dirichlet series defined by $g(s)$ converges for Re $s > 9/11$ and hence defines an analytic function there. If now, we suppose that $g(s)$ has no real zeroes in this half-plane, then

$$\frac{L(s, \pi, r_2 \otimes \bar{r}_2)}{\zeta(s)g(s)} = \Gamma(s, \pi, r_2 \otimes \bar{r}_2)$$

is analytic for Re $s > 9/11$. Therefore, the Γ-factors are analytic there. It follows that

$$|\mathrm{Re}\, r| \leqq \frac{9}{11},$$

and the corresponding lower bound for λ_1 follows from this.

3. Ω-theorems for Fourier coefficients. We begin with the following lemma.

LEMMA 3. *Suppose that*

(1) $f(s)$ *is an analytic function for* $\operatorname{Re} s \geqq 1$, *except for a simple pole at* $s = 1$;

(2) $\log f(s) = \sum_{n=1}^{\infty} c_n/n^s$ *with* $c_n \geqq 0$.

Then $f(1 + it) \neq 0$ *for any real* t.

PROOF. Suppose $f(1 + it) = 0$. Then we consider the classical combination

$$g(s) = f^3(s)f^4(s + it)f(s + 2it),$$

and note that $g(s)$ has a zero at $s = 1$. Then, as usual, $\log g(s)$ is a Dirichlet series with non-negative coefficients and so $\log g(\sigma) \geqq 0$, for $\sigma > 1$. Thus, $|g(\sigma)| \geqq 1$, for $\sigma > 1$. Letting $\sigma \to 1^+$ gives a contradiction.

We apply the lemma to the function $f(s) = \zeta(s)L_2(s)L_4(s)$.
Since $L(s, \pi, r_2 \otimes \tilde{r}_2) = \Gamma(s, \pi, r_2 \otimes \tilde{r}_2)f(s)$, we find that $f(s)$ has a simple pole at $s = 1$ and is analytic for $\operatorname{Re} s \geqq 1$, $s \neq 1$. By the lemma, it does not vanish on the line $\sigma = 1$. The Wiener-Ikehara Tauberian theorem implies

THEOREM 2. $\sum_{p \leqq x} |a_p|^4 \log p = (2 + o(1))x$, *as* $x \to \infty$.

PROOF. From the Tauberian theorem, we have

$$\sum_{p \leqq x} |\alpha_p^2 + \alpha_p\beta_p + \beta_p^2|^2 = (1 + o(1))x/\log x.$$

Now, $a_p = \alpha_p + \beta_p$ and so

$$\begin{aligned} |a_p|^4 &= |\alpha_p^2 + 2\alpha_p\beta_p + \beta_p^2|^2 \\ &= |\alpha_p^2 + \alpha_p\beta_p + \beta_p^2|^2 + 1 + 2\operatorname{Re}(\bar{\chi}(p)(\alpha_p^2 + \beta_p^2 + \alpha_p\beta_p)). \end{aligned}$$

Now, $L(s, \pi, r_2 \otimes \bar{\chi})$ and $L(s, \pi, \tilde{r}_2 \otimes \chi)$ do not vanish on the line $\sigma = 1$ since $\zeta(s)L(s, \pi, r_2 \otimes \bar{\chi})$ has non-negative coefficients in its defining Dirichlet series and an easy application of Lemma 3 gives the non-vanishing result. It therefore follows, by the Tauberian theorem, that

$$\sum_{p \leqq x} \operatorname{Re}(\bar{\chi}(p)\,(\alpha_p^2 + \alpha_p\beta_p + \beta_p^2)) = o(x/\log x)$$

as $x \to \infty$. Combining this with our earlier derivation gives us the desired result.

COROLLARY 1. *For a positive proportion of the primes*, $|a_p| \geqq 1.189\ldots$

COROLLARY 2. $a_n = \Omega(\exp(c \log n)(/\log\log n))$ *for some constant* $c > 0$.

PROOF. As a_n is multiplicative, we set N to be the product of the primes $p \leqq x$ such that $|a_p| \geqq 1.189$. Then, by a familiar argument,

$$|a_N| \geqq \exp\left(\frac{c \log N}{\log \log N}\right),$$

for some $c > 0$, which proves the result.

REMARKS. A version of Theorem 2 was proved for holomlrphic modular forms in [11], without Nebentypus. The non-vanishing of $L_4(s)$ on $\sigma = 1$ and the Ω-result for the Fourier coefficients was established there only in this case also.

Corollary 2 has an interesting application to modular forms of half integral weight.

COROLLARY 3. *Let $g(z) = \sum_1^\infty c_n e^{2\pi i n z}$ be a modular form of weight $k + 1/2$ for $\Gamma_0(4N)$ with character χ and k an integer $\geqq 3$. Then*

$$c_n = \Omega(n^{k/2-1/4} \exp(c \log n/\log \log n)),$$

for some $c > 0$.

PROOF. Shimura and Niwa have constructed a lifting of cusp forms of weight $k + 1/2$ on $\Gamma_0(4N)$, with character χ, to cusp forms of weight $2k$ on $\Gamma_0(2N)$, with character χ^2, for each squarefree integer N. If g is an eigenfunction of $T_{k+1/2}(p^2)$, then defining, for every D,

$$c_D a_n = \sum_{d|n} d^{k-1} c_{n^2 D/d^2}\left(\frac{D}{d}\right)$$

we find $f(z) = \sum_1^\infty a_n e^{2\pi i n z}$ is a modular form of weight $2k$. If the stated Ω-result is false, then we would get

$$|c_n| \leqq \varepsilon\, n^{k/2-1/4} \exp(c \log n/\log \log n)$$

for all n sufficiently large. Substituting into the above formula yields

$$|a_n| = \varepsilon\, n^{k-1/2} \exp(c \log n/\log \log n) \sum_{d|n} d^{-1/2}.$$

The last sum is $O(\exp(c(\log n)^{1/2}))$ and, therefore, the above estimate contradicts Corollary 2. This completes the proof.

4. The Selberg-Linnik conjecture. Let us define the Kloostermann sum

$$S(n, m, c) = \sum_{\substack{a \,(\mathrm{mod}\, c) \\ a\bar{a} \equiv 1\,(c)}} e^{2\pi i/c\,(ma+n\bar{a})}.$$

The estimate

$$|S(n, m, c)| \leqq d(c) c^{1/2} (m, n, c)^{1/2}$$

is well-known and due to Weil [17]. Many problems in additive number theory reduce to estimating sums of the type

$$G(x) = \sum_{c \leqq x} \frac{S(m, n, c)}{c}.$$

One expects considerable cancellation to occur in the sum. Selberg [13] and Linnik [10] independently conjectured that for $x \geqq (m, n)^{1/2+\varepsilon}$,

$$\sum_{c \leqq x} \frac{S(m, n, c)}{c} = O(x^{\varepsilon}).$$

Our Ω-result for the Fourier coefficients gives us that $G(x) = \Omega(\exp(c \log x)/(\log \log x))$ for some $c > 0$. This was proved in [11].

Selberg's original motivation for this conjecture was that it could yield a proof of the Ramanujan-Petersson conjecture in the holomorphic case corresponding to the full modular group. With this in view, we prove the following result, which was certainly known to Selberg.

THEOREM 3. *Suppose that*

$$f(z) = \sum_{x=1}^{\infty} a(n) e^{2\pi i n z}$$

is a modular form of weight k for the full modular group. If $G(x) = O(x^{\beta+\varepsilon})$, *then*

$$a(n) = O(n^{k-1+\beta/2+\varepsilon}).$$

REMARK. If in particular $\beta = 0$, then the Ramanujan conjecture follows. It was shown by Kuznetsov [6] that $G(x) \ll_{\varepsilon,m,n} x^{1/6+\varepsilon}$, but, unfortunately, the constant depends on m and n.

PROOF. Let us define the t-th Poincaré series

$$G_t(z) = \frac{1}{2} \sum_{(c,d)=1} (cz + d)^{-k} e^{2\pi i t (az+b)/(cz+d)},$$

for $1 \leqq t \leqq r$, where r is the dimension of the space of cusp forms of weight k for the full modular group. It is a result of Petersson that $G_1(z), \ldots, G_r(z)$ span this space of cusp forms. Expanding $G_t(z)$ in a Fourier series, we find that the n-th Fourier coefficient of $G_t(z)$ is given by

$$\left(\frac{n}{t}\right)^{(k-1)/2} \left\{\delta_{tn} + \pi \sum_{c=1}^{\infty} \frac{S(t, n, c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right)\right\},$$

where δ_{tn} is the Kronecker delta function and J_k is the Bessel function defined by

$$J_k(z) = \frac{(z/2)^k}{\sqrt{\pi}\,\Gamma\left(k + \frac{1}{2}\right)} \int_0^{\pi} \sin^{2k}\theta \cos(z \cos\theta) d\theta.$$

An easy partial summation reveals that

$$H(x) = \sum_{c \leq x} S(n, m, c) = O(x^{1+\beta+\varepsilon}).$$

In estimating the Fourier coefficient, it suffices to estimate

$$\sum_{c=1}^{\infty} \frac{S(t, n, c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right).$$

We easily find that

$$\sum_{c>\sqrt{n}} \frac{S(t, n, c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right) = \sum_{c>\sqrt{n}} G(c)\left(J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c+1}\right) - J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right)\right)$$
$$\ll \sqrt{n} \sum_{c>\sqrt{n}} \frac{G(c)}{c^2} J'_{k-1}(\xi_c).$$

for some $\xi_c \in (4\pi\sqrt{nt}/(c+1), 4\pi\sqrt{nt}/c)$. Since $J'_{k-1}(x) \ll x^{-1/2}$, we find that the above sum is

$$\ll n^{1/4} \sum_{c>\sqrt{n}} \frac{|G(c)|}{c^{3/2}} \ll n^{(\beta+\varepsilon)/2}.$$

It remains only to show that

$$\sum_{c \leq \sqrt{n}} \frac{S(t, n, c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right) = O(n^{(\beta+\varepsilon)/2}).$$

We show this by induction on k. For $k = 10$, there are no cusp forms of weight 10, therefore the estimate is valid in view of our earlier bound. By induction, we must therefore show

$$\sum_{c \leq \sqrt{n}} \frac{S(t, n, c)}{c} \left(J_{k+1}\left(\frac{4\pi\sqrt{nt}}{c}\right) + J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right)\right) = O(n^{(\beta+\varepsilon)/2}).$$

In view of the identity

$$\frac{2kJ_k(x)}{x} = J_{k+1}(x) + J_{k-1}(x),$$

it suffices to estimate

$$\frac{1}{\sqrt{n}} \sum_{c \leq \sqrt{n}} S(t, n, c) J_k\left(\frac{4\pi\sqrt{nt}}{c}\right).$$

By partial summation and the estimate $J'_k(x) \ll x^{-1/2}$, we find that the above sum is

$$\ll \frac{1}{n^{1/4}} \sum_{c \leq \sqrt{n}} \frac{|H(c)|}{c^{3/2}} \ll n^{(\beta+\varepsilon)/2},$$

as desired.

REMARK. It would be of interest to have an alternate proof of the above which avoids the induction step, for this would have application to other congruence subgroups of the modular group. In connection with λ_1, Selberg **[13]** has shown that the Dirichlet series $\sum_{c=1}^{\infty} S(m, n, c)/c^{2s}$ admits an analytic continuation to Re $s > \sigma_0 > 1/2$ if and only if $\lambda_1 \geqq \sigma_0(1 - \sigma_0)$. Weil's estimate therefore gives $\lambda_1 \geqq 3/16$.

5. Concluding remarks and conjectures. In this section, we explore the possibility of improving the error terms in Lemma 1. Certainly this would lead to better estimates for the eigenvalues. But, as we shall discover, the method has limitations.

Suppose $f(s)$ is an analytic function for Re $s > 0$, except for a pole of order k at $s = 1$, and that, for Re $s > 1$, it is given by the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n/n^s$. Define

$$\nu(\sigma) = \inf\left\{\nu : \int_T^{2T} |f(\sigma + it)|^2\, dt = O(T^{1+\nu})\right\}.$$

THEOREM 4. *Suppose that, for some* $\sigma > 0$, $\nu(\sigma) > 1$. *Let*

$$E(x) = \sum_{n \leqq x} a_n - xP_{k-1}(\log x)$$

be $o(x)$ *for a suitable polynomial of degree* $k - 1$. *Then*

$$E(x) = \Omega(x^{\sigma}/\log x).$$

PROOF. It is clear that

$$E(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(\sigma + it)}{\sigma + it} x^{\sigma + it}\, dt,$$

for any $\sigma > 0$. By Parseval's formula,

$$\int_0^{\infty} \left(\frac{E(x)}{x^{\sigma+1/2}}\right)^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\frac{f(\sigma + it)}{\sigma + it}\right|^2 dt.$$

If $E(x) = O(x^{\sigma}/\log x)$, then

$$\frac{1}{2\pi} \int_T^{2T} \left|\frac{f(\sigma + it)}{\sigma + it}\right|^2 dt < \infty,$$

for all T. Therefore, $\nu(\sigma) \leqq 1$, contrary to our assumption. This completes the proof.

Now, suppose that $f(s)$ satisfies a suitable functional equation, i.e., $c^s \Delta(s) f(s) = c^{1-s} \Delta(1 - s) f(1 - s)$, where $\Delta(s) = \prod_{i=1}^{N} \Gamma(\alpha_i s + \beta_i)$. By Stirling's formula and the functional equation, we deduce

$$\nu(\sigma) = p(1 - 2\sigma) + \nu(1 - \sigma),$$

where $p = \sum_{j=1}^{N} \alpha_j$. Thus, we find $p(1 - 2\sigma) > 1$ if $\sigma < (p - 1)/2p$, and so $\nu((p - 1)/2p) > 1$. It follows therefore from the theorem that

$$E(X) = \Omega(x^{(p-1)/2p}/\log x).$$

In analogy with the classical divisor and circle problems, we conjecture

$$E(x) = O(x^{(p-1)/2p+\varepsilon}).$$

This would lead to an estimate of $a_p = O(p^{1/9+\varepsilon})$, for the eigenvalues.

The above theorem, together with Lemma 1, easily yields the following corollary. In the holomorphic case, (1) was proved by Rankin **[12]** and (2) was proved by Walfisz **[16]**.

COROLLARY. (1) $\sum_{n \leq x} a_n = O(x^{1)/3+\varepsilon})$, *and*

(2) $\sum_{n \leq x} a_n = \Omega(x^{1/4-\varepsilon})$

for every $\varepsilon > 0$.

Indeed, it is easy to see that

$$\frac{E(e^{u+\delta}) - E(e^u)}{e^{u\sigma}} = \frac{1}{2\pi i} \int_{(c)} \frac{f(s)}{s} (e^{\delta s} - 1) e^{uit} \, ds.$$

Parseval's formula gives that

$$\int_0^\infty \left(\frac{E(e^{u-\delta}) - E(e^u)}{e^{u\sigma}} \right)^2 du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{f(\sigma + it)}{\sigma + it} \right|^2 |e^{\delta(\sigma+it)} - 1|^2 \, dt.$$

Utilising the fact that, for $\delta = \log(1 + 1/T)$, we have, for $|t| \leqq T$, the estimate $|e^{\delta(\sigma+it)} - 1| \leqq |t|/T$, we easily find, from the above formula,

$$\int_0^\infty \left(\frac{E(e^{u+\delta}) - E(e^u)}{e^{u\sigma}} \right)^2 du \ll T^{-1+\nu(\sigma)+\varepsilon} + \sum_{k=1}^{\infty} \int_{2^k T}^{2^{k+1} T} \left| \frac{f(\sigma + it)}{\sigma + it} \right|^2 dt$$
$$\ll T^{-1+\nu(\sigma)+\varepsilon}.$$

If we let $S(x) = \sum_{n \leq x} a_n$, we deduce that

$$\int_0^\infty \left(S\left(y + \frac{y}{T}\right) - S(y) - \frac{y}{T} \right)^2 y^{-2\sigma-1} \, dy \ll T^{-1+\nu(\sigma)+\varepsilon}.$$

Choosing $T = n$ and considering only the interval $[n, n + 1]$ in the integral immediately gives that $a_n \ll n^{\sigma+\nu(\sigma)/2+\varepsilon}$. In particular, if we assume the mean value conjecture that $\nu(\sigma) = 0$, for $\sigma \geqq 1/2$, then we deduce that $a_n = O(n^{1/2+\varepsilon})$. For the eigenvalues, we find, on the mean value conjecture, for the L-function $L(s, \pi, r_2 \otimes \tilde{r}_2)$, that $a_p = O(p^{1/8+\varepsilon})$. Under this general situation, one should be able to show that $\nu(\sigma) = 0$, for $\sigma \geqq p/(p + 1)$.

In the case of the eigenvalues, it is not difficult to see that for the corresponding L-function, we indeed have that $\nu(\sigma) = 0$, for $\sigma \geqq$

$(p-1)/(p+1)$. This fact equally implies the Serre bound of $p^{1/5}$. Consequently, any improvement in the range of vanishing of the ν-function, yields a correponding improvement in the eigenvalue bound.

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