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ON THE ASYMPTOTIC FORMULA FOR THE FOURIER COEFFICIENTS OF THE j-FUNCTION

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Abstract. We derive the asymptotic formula for the Fourier coefficients of the j-function using an arithmetic formula given by Kaneko based on Zagier's work on the traces of singular moduli. The key ingredient along with the Kaneko–Zagier formula is Laplace's method.

1. Introduction

Let $\mathbf{H}$ denote the upper half-plane $\{z \in \mathbf{C} : \Im(z) > 0\}$. The j-function defined by

$$j(\tau) = \frac{(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n)^3}{q \prod_{n=1}^{\infty} (1 - q^n)^{24}}, \quad q = e^{2\pi i \tau}, \quad \tau \in \mathbf{H}$$

is an $\mathrm{SL}_2(\mathbf{Z})$-invariant modular function on $\mathbf{H}$ whose q-expansion at $i\infty$ is

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots .$$

For definitions and basic facts about modular forms, see [**11**] or [**12**]. On the one hand, the values of the j-function at quadratic irrationalities, called *singular moduli*, generate Hilbert class fields while on the other, the coefficients $c(n)$ ($n \geqslant -1$, $n \neq 0$) in the q-expansion of $j(\tau)$ appear as dimensions of the head representations of the largest sporadic finite simple group, the Monster group. In what can be described as a marriage between these values and coefficients, Kaneko [**8**] discovered a closed formula for $c(n)$ based on Zagier's work [**14**] on traces of singular moduli.

Using the circle method introduced by Ramanujan and Hardy [**6**] to study the partition function $p(n)$, Petersson [**9**] and later Rademacher [**10**] independently derived the following asymptotic formula for $c(n)$:

$$c(n) = [q^n] j \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2} n^{3/4}} \quad \text{as } n \to \infty. \tag{1}$$

Here we use the convenient notation $[q^n] f$ for the coefficient of q^n in a q-series f.

In 2005, Brisebarre and Philibert [**1**] revisited this classical work to derive effective upper and lower bounds (as opposed to mere asymptotic formulas) more generally for powers of the j-function. In their paper, using Ingham's Tauberian theorem [**7**], they indicate a quick

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proof of the following asymptotic formula:

$$[q^n]j^m \sim \frac{m^{1/4}e^{4\pi\sqrt{nm}}}{\sqrt{2}n^{3/4}} \quad \text{as } n \to \infty. \tag{2}$$

In 2013, Dewar and Ram Murty [**4**] derived the asymptotic formula for $[q^n]j^m$ (and more generally, coefficients of weakly holomorphic modular forms) without using the circle method. They applied an algebraic formula due to Bruinier and Ono [**2**].

In this short note inspired by the earlier work [**4**], we derive the asymptotic formula for the coefficients of the j-function using the arithmetic formula of Kaneko [**8**] and Laplace's method.

2. Preliminaries

In this section, we shall set up the notation and review the key ingredients necessary for the proof.

The first fact we need is a well-known fact about binary quadratic forms (see [**5**, Chapter 6] for an introduction). We shall write $[a, b, c]$ for the form $aX^2 + bXY + cY^2$. The discriminant of the form $[a, b, c]$ is $b^2 - 4ac$, consequently, the discriminant of any form is 0 or 1 mod 4 according to whether b is even or odd.

Definition 2.1. (Principal form of discriminant D) The binary quadratic form

$$I_D = \begin{cases} \left[1, 0, -\frac{D}{4}\right], & D \equiv 0 \bmod 4 \\ \left[1, 1, \frac{1-D}{4}\right], & D \equiv 1 \bmod 4 \end{cases} \tag{3}$$

is a form with discriminant D and is called the principal form of discriminant D.

Recall that a form P is said to represent an integer m if there are $x, y \in \mathbf{Z}$ such that $P(x, y) = m$. The following lemma offers a key simplification to our proof.

LEMMA 2.2. *The following are equivalent for a form P of discriminant D:*

(i) *P represents* 1;

(ii) *P is* $\mathrm{SL}_2(\mathbf{Z})$*-equivalent to* $[1, B, C]$ *for some* $B, C \in \mathbf{Z}$; *and*

(iii) *P is* $\mathrm{SL}_2(\mathbf{Z})$*-equivalent to the principal form of discriminant D.*

Proof. (i) $\implies$ (ii). Suppose that x and y are integers such that $P(x, y) = 1$. Then, we have that $(x, y) = 1$, so there are integers r, s such that $xr - ys = 1$. Putting p for the matrix corresponding to P, we see that

$$\begin{pmatrix} x & s \\ y & r \end{pmatrix}^t p \begin{pmatrix} x & s \\ y & r \end{pmatrix} = \begin{pmatrix} P(x, y) & * \\ * & P(s, r) \end{pmatrix} = \begin{pmatrix} 1 & * \\ * & P(s, r) \end{pmatrix}$$

as required.

(ii) $\Longrightarrow$ (iii). The equations

$$\begin{pmatrix} 1 & 0 \\ -\frac{B}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \begin{pmatrix} 1 & -\frac{B}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{4C-B^2}{4} \end{pmatrix} \quad \text{when } B \text{ is even,}$$

$$\begin{pmatrix} 1 & 0 \\ -\frac{B-1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \begin{pmatrix} 1 & -\frac{B-1}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} + \frac{4C-B^2}{4} \end{pmatrix} \quad \text{when } B \text{ is odd,}$$

establish the claim.

(iii) $\Longrightarrow$ (i). Let i_D (respectively p) denote the matrix corresponding to the principal form (respectively P). Then, there exists $S \in \mathrm{SL}_2(\mathbf{Z})$ such that $S^t i_D S = p$. We note that

$$\begin{pmatrix} 1 & 0 \end{pmatrix} (S^{-1})^t p S^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} i_D \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

so that P represents 1 as claimed. □

We now review Kaneko's arithmetic formula for $c(n)$, one of the key ingredients in our proof.

2.1. *Kaneko–Zagier arithmetic formula for* $c(n)$

Let d be a positive integer with $d \equiv 0, 3 \bmod 4$. Let $\mathcal{Q}_d$ denote the set of all positive definite (not necessarily primitive) binary quadratic forms of discriminant $-d$. Associated to each quadratic form $Q \in \mathcal{Q}_d$ is an imaginary quadratic irrationality in $\mathfrak{H}$ given by the root of the quadratic equation $Q(t, 1) = 0$ with positive imaginary part. This correspondence satisfies the property that $\alpha_{S^t Q S} = S^{-1}\alpha_Q$ for $S \in \mathrm{SL}_2(\mathbf{Z})$ so that the modularity of j implies that the value of $j(\alpha_Q)$ depends only on the $\mathrm{SL}_2(\mathbf{Z})$-equivalence class of Q. It is classical [**3**] that $j(\alpha_Q)$ is an algebraic integer of degree $h(-d)$ over $\mathbf{Q}$, where $h(-d)$ is the number of $\mathrm{SL}_2(\mathbf{Z})$-equivalence classes of primitive positive definite binary quadratic forms of discriminant $-d$.

In [**14**], inspired by the question of determining the absolute trace of $j(\alpha_Q)$, Zagier introduced the modular trace function $\mathbf{t}$:

$$\mathbf{t}(d) := \sum_{Q \in \mathrm{SL}_2(\mathbf{Z})\backslash \mathcal{Q}_d} \frac{2}{|\mathrm{Aut}(Q)|} J(\alpha_Q), \quad d > 0,\ d \equiv 0, 3 \bmod 4, \tag{4}$$

for the normalized Hauptmodul $J(\tau) := j(\tau) - 744$ for $\mathrm{SL}_2(\mathbf{Z})$. Also, for convenience, define $\mathbf{t}(0) = 2$ and $\mathbf{t}(-1) = -1$ and set $\mathbf{t}(d) = 0$ if $d \equiv 1, 2 \bmod 4$ or if $d < -1$.

Recall the well-known fact about automorphs of binary quadratic forms [**5**, Theorem 6.1.9]:

$$|\mathrm{Aut}(Q)| = \begin{cases} 6, & Q \sim [a, a, a] \\ 4, & Q \sim [a, 0, a] \\ 2, & \text{otherwise.} \end{cases} \tag{5}$$

Based on Zagier's theorems, Kaneko [**8**] gave a closed form expression for the coefficients $c(n)$ of the j-function as follows.

THEOREM 2.3. (Kaneko) *For any $n \geqslant 1$, we have that*

$$c(n) = \frac{1}{n}\left\{\sum_{r\in\mathbf{Z}} \mathbf{t}(n-r^2) + \sum_{r\geqslant 1,\, r\text{ odd}} ((-1)^n \mathbf{t}(4n-r^2) - \mathbf{t}(16n-r^2))\right\}. \tag{6}$$

We conclude this section with the statement of Laplace's theorem in a formulation that is required in our proof.

2.2. *Laplace's method*

Laplace's method is one of the most fundamental methods useful in deriving asymptotic expansions of integrals of the form (7). We need the following key theorem.

THEOREM 2.4. (Laplace's method [**13**, p. 57]) *Suppose that h is a real-valued C^2-function defined on the interval (a, b) (with $a, b \in \mathbf{R}$). If we further suppose that h has a unique maximum at $\xi = c$ with $a < c < b$ so that $h'(c) = 0$ and $h''(c) < 0$, then, we have*

$$\int_a^b e^{\lambda h(t)}\, dt \sim e^{\lambda h(c)} \left(\frac{-2\pi}{\lambda h''(c)}\right)^{1/2} \tag{7}$$

as $\lambda \to \infty$.

3. Asymptotic formula for $j(\tau)$

In this section, we shall prove (1) using Kaneko's arithmetic formula (6). We begin by noting that the contribution to $\mathbf{t}(d)$ comes only from I_{-d}.

LEMMA 3.1. *With the same notation as before, we have the following asymptotic formula:*

$$c(n) \sim \frac{1}{n} \sum_{\substack{1\leqslant r\leqslant\sqrt{16n-1}\\ r\text{ odd}}} e^{\pi\sqrt{16n-r^2}} \tag{8}$$

as $n \to \infty$.

Proof. In view of Kaneko's formula (6), let us analyse the modular trace $\mathbf{t}(d)$, $d > 0$, to get started: recall from (4) that

$$\mathbf{t}(d) = \sum_{Q\in \mathrm{SL}_2(\mathbf{Z})\backslash \mathcal{Q}_d} \frac{2}{|\mathrm{Aut}(Q)|} J(\alpha_Q).$$

We claim that the only class that contributes to this sum is the class $[I_{-d}]$. Indeed, if $Q = [a, b, c]$ is a form of discriminant $-d$, then, we have

$$e^{2\pi i\alpha_Q} = \exp\left(2\pi i\left(\frac{-b+i\sqrt{d}}{2a}\right)\right) = \exp\left(-\frac{\pi i b}{a}\right)\exp\left(\frac{-\pi\sqrt{d}}{a}\right)$$

and consequently

$$\begin{aligned} J(\alpha_Q) &= j(\alpha_Q) - 744 \\ &= q^{-1} + O(q) \\ &= \exp\left(\frac{\pi i b}{a}\right)\exp\left(\frac{\pi\sqrt{d}}{a}\right) + O\left(\exp\left(\frac{-\pi\sqrt{d}}{a}\right)\right). \end{aligned}$$

In view of this, the contribution to $\mathbf{t}(d)$ comes only from classes that have forms with $a = 1$. By Lemma 2.2, any such form is equivalent to the principal form I_{-d} so that we have

$$\mathbf{t}(d) = O(\exp(-\pi\sqrt{d})) + \begin{cases} \exp(\pi\sqrt{d}), & d \equiv 0 \bmod 4,\ d \neq 4 \\ -\exp(\pi\sqrt{d}), & d \equiv 3 \bmod 4,\ d \neq 3. \end{cases} \tag{9}$$

Since $e^{\pi\sqrt{n}} = o(e^{4\pi\sqrt{n}})$ and $e^{2\pi\sqrt{n}} = o(e^{4\pi\sqrt{n}})$, the contribution to $c(n)$ comes only from the last summand of the formula. Since $16n - r^2 \equiv 3 \bmod 4$ when r is odd, the claim follows from (6) and (9) on a moment's reflection. □

In view of this lemma, we consider the sum

$$S_n := \frac{1}{\sqrt{n}} \sum_{\substack{1 \leqslant r \leqslant \sqrt{16n-1} \\ r \text{ odd}}} e^{4\pi\sqrt{n}\sqrt{1-r^2/16n}} = \frac{1}{2\sqrt{n}} \sum_{k=1}^{\lfloor \frac{1}{2}(1+\sqrt{16n-1})\rfloor} 2e^{4\pi\sqrt{n}\sqrt{1-(2k-1)^2/16n}} \tag{10}$$

and view this sum as a Riemann sum for the function $t \mapsto 2e^{4\pi\sqrt{n}\sqrt{1-t^2}} : [0, 1] \to \mathbf{R}$ corresponding to the following partition of [0, 1]:

$$0 =: x_0 < x_1 < \cdots < x_{b_n} < x_{b_n+1} := 1,$$

where $b_n := \lfloor \frac{1}{2}(1 + \sqrt{16n-1})\rfloor$ is the upper limit in the sum S_n and

$$x_k = \frac{2k-1}{4\sqrt{n}}, \quad 1 \leqslant k \leqslant b_n. \tag{11}$$

Our strategy is as follows. We shall show that S_n is asymptotic to the corresponding Riemann integral J_n where

$$J_n := 2\int_0^1 e^{4\pi\sqrt{n}\sqrt{1-t^2}}\, dt = \int_{-1}^1 e^{4\pi\sqrt{n}\sqrt{1-t^2}}\, dt;$$

then, determine the asymptotic behaviour of J_n and finally deduce that the asymptotic formula for $c(n)$ is given by (1) using the fact that $c(n) \sim (1/\sqrt{n})S_n$ as $n \to \infty$.

The asymptotic behaviour of J_n is easily determined using Laplace's method.

LEMMA 3.2. *We have the asymptotic formula*

$$J_n \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{1/4}} \tag{12}$$

as $n \to \infty$.

Proof. We apply Theorem 2.4 in this case to our integral with the obvious candidate function: put $h(t) = 4\pi\sqrt{1-t^2}$ on $(-1, 1)$. Then, the function h has a unique maximum at $\xi = 0$. Its first two derivatives are given by

$$h'(t) = -4\pi\frac{t}{\sqrt{1-t^2}} \quad \text{and} \quad h''(t) = \frac{-4\pi}{(1-t^2)^{3/2}}$$

and (7) completes the proof. □

COROLLARY 3.3. *We have the asymptotic formula*

$$\frac{1}{\sqrt{n}} J_n \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}} \tag{13}$$

as $n \to \infty$.

This corollary leaves us needing to show that $S_n \sim J_n$ as $n \to \infty$. This is done in the following lemma.

LEMMA 3.4. *With the same notation as before, we have that*

$$\lim_{n\to\infty} \left| \frac{S_n}{J_n} - 1 \right| = 0. \tag{14}$$

In other words, the sum S_n *is asymptotic to the integral* J_n.

Proof. Let f be the function defined on [0, 1] by $f(x) = 2e^{4\pi\sqrt{n}\sqrt{1-x^2}}$. Then f is a decreasing function. Let $b := b_n = \lfloor (1 + \sqrt{16n - 1})/2 \rfloor$ denote the upper limit in S_n. We estimate the difference $S_n - J_n$ (with x_k as in (11)) as

$$\begin{aligned} S_n - J_n &= \frac{1}{2\sqrt{n}} \sum_{k=1}^{b} f(x_k) - \int_0^1 f(t)\, dt \\ &= \left(\frac{1}{2\sqrt{n}} f(x_1) - \int_0^{x_1} f(t)\, dt \right) + \sum_{k=2}^{b} \int_{x_{k-1}}^{x_k} (f(x_k) - f(t))\, dt - \int_{x_b}^{1} f(t)\, dt \\ &= \Sigma_1(n) + \Sigma_2(n) + \Sigma_3(n) \quad \text{(say).} \end{aligned}$$

We shall show that $|\Sigma_1(n) + \Sigma_2(n) + \Sigma_3(n)|$ is bounded by a function in n of order lower than that of J_n; then, as $n \to \infty$, the claim will follow.

We begin with $\Sigma_1(n)$:

$$\begin{aligned} \Sigma_1(n) &= \frac{1}{2\sqrt{n}} f(x_1) - \int_0^{x_1} f(t)\, dt \\ &= \frac{1}{4\sqrt{n}} f(x_1) + \frac{1}{4\sqrt{n}} f(x_1) - \int_0^{x_1} f(t)\, dt. \end{aligned}$$

Since $x_1 = 1/4\sqrt{n}$, we may introduce the integral sign

$$\begin{aligned} \Sigma_1(n) &= \frac{1}{4\sqrt{n}} f(x_1) + \int_0^{x_1} f(x_1)\, dt - \int_0^{x_1} f(t)\, dt \\ |\Sigma_1(n)| &\leqslant \frac{1}{4\sqrt{n}} f(x_1) + \int_0^{x_1} |f(t) - f(x_1)|\, dt. \end{aligned} \tag{T1}$$

The first term in (T1) is evaluated and seen to have an order lower than that of J_n:

$$\frac{1}{4\sqrt{n}} f(x_1) = \frac{1}{4\sqrt{n}} e^{\pi\sqrt{16n-1}} = O\left(\frac{\exp(4\pi\sqrt{n})}{\sqrt{n}} \right).$$

To estimate the integral in (T1), we use the following bound from the mean value theorem:

$$|e^x - e^y| \leqslant e|x - y| \quad \text{for } x,\, y \in [0, 1].$$

We now look at the integrand (bearing in mind that $t \in [0, x_1] \subseteq [0, 1]$)

$$|f(t) - f(x_1)| \leqslant 8\pi e\sqrt{n}\left|\sqrt{1-t^2} - \sqrt{1 - \frac{1}{16n}}\right|.$$

Since $\sqrt{1-t^2}$ is a decreasing function on [0, 1], we note that

$$|f(t) - f(x_1)| \leqslant 8\pi e\sqrt{n}\left(1 - \sqrt{1 - \frac{1}{16n}}\right).$$

Thus, the integral in (T1) is $O(1/n)$ and it follows that

$$\Sigma_1(n) = O\left(\frac{\exp(4\pi\sqrt{n})}{\sqrt{n}}\right).$$

This completes the analysis of the sum $\Sigma_1(n)$. We study the sum $\Sigma_2(n)$:

$$|\Sigma_2(n)| \leqslant \sum_{k=2}^{b} \int_{x_{k-1}}^{x_k} |f(x_k) - f(t)|\, dt. \tag{T2}$$

As before, we study the integrand

$$\begin{aligned} |f(t) - f(x_k)| &\leqslant 8\pi e\sqrt{n}|\sqrt{1-t^2} - \sqrt{1-x_k^2}| \\ &\leqslant 8\pi e\sqrt{n}(\sqrt{1-x_{k-1}^2} - \sqrt{1-x_k^2}). \end{aligned}$$

Using the inequality $1 - x/2 - x^2/2 \leqslant \sqrt{1-x} \leqslant 1 - x/2$, we obtain

$$|f(t) - f(x_k)| \leqslant 4\pi e\sqrt{n}(x_k^4 + x_k^2 - x_{k-1}^2).$$

In view of equation (T2), we have

$$\begin{aligned} |\Sigma_2(n)| &\leqslant 2\pi e\left(\sum_{k=2}^{b} x_k^4 + \sum_{k=2}^{b}(x_k^2 - x_{k-1}^2)\right) \\ &= 2\pi e\left(\frac{1}{256n^2}\sum_{k=2}^{b}(2k-1)^4 + x_b^2 - x_1^2\right) \\ &\leqslant 2\pi e\left(\frac{1}{256n^2}\sum_{k=2}^{b}(2k-1)^4 + \frac{8n-1}{8n}\right) \end{aligned}$$

which is indeed bounded by a rational function in n. Let us now consider $\Sigma_3(n)$:

$$\Sigma_3(n) = -\int_{x_b}^{1} f(t)\, dt$$

$$|\Sigma_3(n)| \leqslant 2\int_{x_b}^{1} e^{4\pi\sqrt{n}\sqrt{1-t^2}}\, dt. \tag{T3}$$

For $x \in \mathbf{R}$, writing $\{x\}$ for the fractional part $x - \lfloor x \rfloor$ of x, we note that

$$\begin{aligned} \left\{\frac{1+\sqrt{16n-1}}{2}\right\} &\leqslant \max\left(\frac{1}{2} + \frac{\{\sqrt{16n-1}\}}{2}, \frac{\{\sqrt{16n-1}\}}{2}\right) \\ &= \frac{1}{2} + \frac{\{\sqrt{16n-1}\}}{2} \end{aligned}$$

which gives us that

$$x_b \geqslant \frac{1}{4\sqrt{n}}(\lfloor\sqrt{16n-1}\rfloor - 1) > \frac{1}{4\sqrt{n}}(\sqrt{16n-1} - 2).$$

Now, we have

$$|\Sigma_3(n)| \leqslant 2\int_{(1/4\sqrt{n})(\sqrt{16n-1}-2)}^{1} e^{4\pi\sqrt{n}\sqrt{1-t^2}}\, dt.$$

Using Taylor's theorem and the fact that the integrand is decreasing, we have

$$|\Sigma_3(n)| = O\left(\frac{\exp(\sqrt{2\sqrt{16n-1}-3})}{\sqrt{n}}\right).$$

This shows that $|\Sigma_1(n) + \Sigma_2(n) + \Sigma_3(n)|$ is bounded by a function in n of order lower than that of J_n and the proof is complete. □

Putting Lemmas 3.1, 3.2, 3.4 and Corollary 3.3 together, we have the following.

THEOREM 3.5. *The Fourier coefficients $c(n)$ of the j-function have the asymptotic formula*

$$c(n) \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}} \tag{15}$$

as $n \to \infty$.

Remark 3.6. The asymptotic formula for j can now be used to deduce an asymptotic formula for the sequence $[q^n]j_m$ as $n \to \infty$, where j_m is the unique modular function holomorphic on the upper half-plane whose q-expansion at $i\infty$ is $q^{-m} + O(q)$. This is done by computing the q-expansion of j_m (see [**11**]) by noting that $j_m = j_1 \mid_0 T_m$, where T_m is the normalized mth Hecke operator and $j_1 = j - 744$.

Concluding remarks

The method presented in this paper can perhaps be applied in other contexts. Whenever there is an algebraic formula for a quantity that is derived via arithmetic methods, one can discuss its asymptotic behaviour using this method.

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