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On the Uniform Distribution of Certain Sequences

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Abstract. We investigate the uniform distribution of the sequence n^α as n ranges over the natural numbers and α is a fixed positive real number which is not an integer. We then apply this in conjunction with the Linnik-Vaughan method to study the uniform distribution of the sequence p^α as p ranges over the prime numbers.

Key words: exponential sums, uniform distribution, the Linnik-Vaughan method

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1. Introduction

In this paper, we will investigate the uniform distribution of the sequence $\{n^\alpha\}$, where n ranges over natural numbers and $\{p^\alpha\}$, where p ranges over prime numbers. We will focus our attention on $0 < \alpha < 1$, though we will make remarks for $\alpha > 1$ as well. The sequence $\{n^\alpha\}$ has been investigated in the literature [4], though no explicit error terms have been written down. We will do so below in Theorem 1. Then we will apply Linnik-Vaughan method to obtain estimates for

$$\sum_{p \leq x} e^{2\pi i p^\alpha \theta}.$$

Such sums for $\theta = a/q$ rational and $\alpha = 1/2$ have arisen recently in the work [3], where an interesting connection is made between sharp estimates for such sums and the absence of zeros of $L(s, f)$ where f is a Hecke eigenform, on a certain segment of the real line close to the edge of the critical strip.

We now elucidate the precise nature of the results we prove.

For a real number x, let $[x]$ denote the *integral* part of x; let $\{x\} = x - [x]$ be the *fractional* part of x or the residue of x modulo 1.

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Let $\omega = (x_n)$, $n = 1, 2, \ldots$ be a given sequence of real numbers.

For a positive integer N and a subset E of $I (= [0, 1))$, let the counting function $A(E; N; \omega)$ be defined as the number of terms x_n, $1 \leq n \leq N$, for which $\{x_n\} \in E$.

The sequence (x_n), $n = 1, 2 \ldots$ is said to be *uniformly distributed modulo 1* (in short u.d. mod 1) if for every sub-interval E of I, we have

$$\lim_{N \to \infty} \frac{A(E; N; \omega)}{N} = |E|.$$

In other words, (x_n) is u.d. mod 1 if every half open sub-interval of I eventually gets its "proper share" of fractional parts.

There is a deep connection between the theory of u.d. mod 1 and the estimation of *exponential sums* as envisaged by Weyl, which we mention below.

Weyl's Criterion (see page 7 of [4]) says that the sequence (x_n), $n = 1, 2, \ldots$ is u.d. mod 1 if and only if

$$\sum_{n=1}^{N} e^{2\pi i h x_n} = o(N) \text{ for all integers } h \neq 0.$$

Remark. Using Weyl's criterion it is easy to show that the sequence $(n\theta)$, $n = 1, 2, \ldots$ is u.d. mod 1 whenever θ is irrational and is not u.d. mod 1 if θ is rational number.

In this paper we shall investigate the distribution of the fractional parts of the sequence (n^α) for $\alpha > 0$ not an integer.

More precisely we prove the following

Theorem 1. *Let* $S(N) = \sum_{n=1}^{N} e^{2\pi i n^\alpha h}$. *Then for all integers* $h \neq 0$, *we have*

(a)

$$S(N) = O\left(|h|^{\frac{1}{4}} N^{\frac{\alpha+2}{4}} (\log(N + 10|h|)^2\right),$$

(*whenever* $0 < \alpha < 2, \alpha \neq 1$)

(b)

$$S(N) = O\left(|h|^{\frac{1}{12}} N^{\frac{\alpha+9}{12}} (\log(N + 10)|h|)^2\right)$$

(*whenever* $2 < \alpha < 3$)

$$S(N) = O\left(\max(|h|^{\frac{1}{2^{k-2}}} N^{1-\frac{k-\alpha}{2^k-2}}, N^{1-\left(\frac{1}{2^{k-2}} - \frac{k-\alpha}{2^k-2}\right)}\right)(\log(N + 10)|h|)^2)$$

(*whenever* $k - 1 < \alpha < k, k \geq 4$ *is an integer*)

As a corollary we obtain the following well-known

Theorem 2. *The sequence* (n^α), $n = 1, 2, \ldots$ *is uniformly distributed modulo 1 for* $\alpha (> 0)$ *not an integer.*

Remark. Theorem 2 can be obtained by using Fejér's theorem (see page 29 of [4]) or by using Van der Corput's lemma (see page 17 of [4]) which gives the exponent of N in the O-term for theorem 1 (a) as 3/4. However, our treatment yields the exponent 5/8.

Remark. It would be nice to ask if Theorem 1 throws any light on the bounds for the exponential sum

$$\sum_{p \leq X} e^{2\pi i p^{\alpha} h}.$$

Following Eratosthenes sieve, one can proceed in the following way:

Let

$$z = \sqrt{X},\ P(z) = \prod_{p \leq z} p$$

then the above exponential sum is

$$O(\sqrt{X}) + \sum_{n \leq X} e^{2\pi i n^{\alpha} h} \left(\sum_{d | (P(z), n)} \mu(d) \right).$$

We use a sophisticated version of the above idea as exemplified by Vaughan (see page 138 of [1]) and elaborated by the first author and Sankaranarayana in [5] and obtain the following

Theorem 3. *We have*

$$\sum_{1 \leq n \leq N} \Lambda(n) e^{2\pi i n^{\alpha} h} = O\left(|h|^{1/8} N^{\frac{14+2\alpha}{16}} (\log(N+10)|h|)^3\right)$$

uniformly in α *for* $0 < \alpha < 1$.

Remark. As stated above this, exponential sum seems to come up in the recent work of Iwaniec, Luo and Sarnak [3] concerning the Siegel zeros of Hecke L-functions attached to certain eigenforms. For details see [3].

2. Some lemmas

We will estimate the sums in question using the *Poisson summation formula*, but with an effective version of it. We use this occasion to point out that the Poisson summation formula can be derived from the simplest case of Euler-Maclaurin sum formula

$$\sum_{j=1}^{N} f(j) = \int_1^N f(t)\, dt + \int_1^N f'(t) \left(\{t\} - \frac{1}{2} \right) dt \tag{1}$$

by writing down the Fourier series for $\{t\} - \frac{1}{2}$ and inserting this in the integral.

In fact, it is not hard to show that

$$\{x\} - \frac{1}{2} = \sum_{0<|m|\le M} \frac{e(mx)}{2\pi i m} + O\left(\min\left(1, \frac{1}{M||x||}\right)\right)$$

so that one can write the right hand side of (1) as

$$\int_1^N f'(t)\left(\sum_{0<|m|\le M} \frac{e(mt)}{2\pi i m} + O\left(\min\left(1, \frac{1}{M||t||}\right)\right)\right) dt$$
$$= \sum_{0<|m|\le M} \int_1^N f(t)e(mt)\,dt + O\left(\frac{NK\log M}{M}\right)$$

where $|f'(t)| \le K$ for all $t \in [1, N]$.

This proves

Lemma 1. (*Effective Poisson Summation Formula*).
Let $f(t)$ be differentiable on $[1, N]$ satisfying $|f'(t)| \le K$. Then

$$\sum_{j=1}^{N} f(j) = \sum_{0\le|m|\le M} \int_1^N f(t)e(mt)\,dt + O\left(\frac{NK\log M}{M}\right)$$

where $e(x) = e^{2\pi i x}$.

We will also need the following well-known result

Lemma 2. *Let $F(x)$ be real, twice differentiable function in $[a, b]$ such that $F''(x) \ge m > 0$ or $F''(x) \le -m < 0$. Then*

$$\left|\int_a^b e^{iF(x)}dx\right| \le \frac{8}{\sqrt{m}}.$$

Proof: See for example page 56 of [2]. □

3. Proof of the theorems

Proof of Theorem 1: First we split the interval $[1, N]$ into dyadic intervals of the type $[W, 2W]$. Clearly there are $O(\log N)$ such intervals. Therefore, it is enough to estimate the sum

$$S(W, 2W) = \sum_{W\le n\le 2W} e^{2\pi i n^{\alpha} h}.$$

By taking $f(t) = e^{2\pi i n^\alpha h}$ in lemma 1 we obtain

$$S(W, 2W) = \sum_{0 \le |m| \le M} \int_W^{2W} e^{2\pi i (t^\alpha h + mt)} dt + O\left(\frac{|h| W^\alpha \log M}{M}\right), \tag{2}$$

where M is a large positive constant to be chosen later.

Now we take $F(t) = \frac{t^\alpha h + mt}{2\pi}$ in lemma 2.
Observe that

$$F''(t) = \frac{\alpha(\alpha - 1) t^{\alpha - 2} h}{2\pi}.$$

Therefore

$$F''(t) \ge C_1 h W^{\alpha - 2} > 0 \quad \text{or} \quad F''(t) \le -C_2 h W^{\alpha - 2} < 0$$

depending on whether $h < 0$ or $h > 0$ respectively, provided $0 < \alpha < 1$. Here C_1, C_2 are positive constants which may depend on α.

Hence from lemma 2, we obtain

$$\left| \int_W^{2W} e^{2\pi i (t^\alpha h + mt)} dt \right| = O_\alpha (W^{1 - \alpha/2} |h|^{-1/2}) \tag{3}$$

By (2) and (3), we obtain

$$S(W, 2W) = O\big(M W^{1 - \alpha/2} |h|^{-1/2}\big) + O\left(\frac{|h| W^\alpha \log M}{M}\right).$$

Choosing $M = C_3 [|h|^{3/4} W^{\frac{3\alpha - 2}{4}}]$, where C_3 is a large positive constant, we obtain

$$S(W, 2W) = O\Big(|h|^{1/4} W^{\frac{\alpha + 2}{4}} \log W|h|\Big).$$

This proves

$$\sum_{1 \le n \le N} e^{2\pi i n^\alpha h} = O_\alpha\big(|h|^{1/4} N^{\frac{\alpha + 2}{4}} (\log N)(\log N|h|)\big)$$

provided $0 < \alpha < 1$. □

Remark. The estimates for $S(N)$ for $\alpha > 1, \alpha$ not an integer are obtained by Van der Corput's lemma (see Theorem 5.13 of [6]) and then using the exponent pair method (see page 72 of [2]).

Proof of Theorem 3: To prove the theorem we invoke Vaughan's method as illustrated in [1] and recently elaborated in [5]. □

Let $\Lambda(n)$ denotes the usual von Mangoldt function defined as $\Lambda(n) = \log p$ if $n = p^m$ for some prime p and some integer $m \geq 0$, 0 otherwise.

With $f(n) = e^{2\pi i n^\alpha h}$, $0 < \alpha < 1$, we form the sum

$$\begin{aligned}\sum_{n\leq N} \Lambda(n) f(n) &= \sum_{n\leq N} (a_1(n) + a_2(n) + a_3(n) + a_4(n)) f(n)\\ &= S_1(N) + S_2(N) + S_3(N) + S_4(N) \quad \text{(say)}\end{aligned}$$

Here $a_i(n)$'s are as given in page 139 of [1].

We now begin our estimations of the sums $S_i(N)$ for $i = 1, 2, 3$, and 4.

Lemma 3. *We have*

$$S_1(N) \leq U \log U$$

Proof: This is clear. □

Lemma 4. *We have*

$$S_2(N) = O\big(|h|^{\frac{1}{4}} U^{\frac{1}{2}} V^{\frac{1}{2}} N^{\frac{\alpha+2}{4}} (\log(N)|h|)^3\big)$$

uniformly in α *for* $0 < \alpha < 1$.

Proof: We have

$$\begin{aligned}S_2(N) &= -\sum_{n\leq N}\left(\sum_{\substack{mdr=n\\ m\leq U\\ d\leq V}} \Lambda(m)\mu(d)\right) f(n)\\ &= \sum_{\substack{mdr=n\\ m\leq U\\ d\leq V}} \Lambda(m)\mu(d) \sum_{r\leq \frac{N}{md}} f(mdr)\\ &= O\left(\sum_{\substack{m\leq U\\ d\leq V}} \Lambda(m)\left(\frac{N}{md}\right)^{\frac{\alpha+2}{4}} \big(|h|(md)^\alpha\big)^{\frac{1}{4}} (\log(N)|h|)^2\right)\\ &= O\left(\left(\sum_{\substack{m\leq U\\ d\leq V}} \frac{1}{(md)^{1/2}}\right) |h|^{\frac{1}{4}} N^{\frac{\alpha+2}{4}} (\log(N)|h|)^3\right)\\ &= O\big(|h|^{\frac{1}{4}} U^{\frac{1}{2}} V^{\frac{1}{2}} N^{\frac{\alpha+2}{4}} (\log(N)|h|)^3\big)\end{aligned}$$

□

Lemma 5. *We have*

$$S_3(N) = O\big(|h|^{\frac{1}{4}} V^{\frac{1}{2}} N^{\frac{\alpha+2}{4}} (\log(N)|h|)^3\big)$$

Proof:

$$\begin{aligned} S_3(N) &= \sum_{n \le N} \left(\sum_{\substack{ld=n \\ d \le V}} \mu(d) \log l \right) e^{2\pi i h n^\alpha} \\ &= \sum_{d \le V} \mu(d) \sum_{l \le N/d} e^{2\pi i h (ld)^\alpha} \int_1^l \frac{dt}{t} \\ &= \int_1^N \sum_{d \le V} \mu(d) \left\{ \sum_{t \le l \le N/d} e^{2\pi i h (ld)^\alpha} \right\} \frac{dt}{t} \end{aligned}$$

From Theorem 1, it follows that the right hand side is

$$\begin{aligned} &O\left(\sum_{d \le V} N^{\frac{\alpha+2}{4}} \frac{1}{d^{1/2}} |h|^{1/4} (\log(N)|h|)^3 \right) \\ &= O\big(|h|^{\frac{1}{4}} N^{\frac{\alpha+2}{4}} V^{\frac{1}{2}} (\log(N)|h|^3\big) \end{aligned}$$

□

Lemma 6. *We have*

$$S_4(N) = O\Big(|h|^{\frac{1}{8}} N^{1+\alpha/8} U^{-1/4} (\log(N)|h|)^4 + N V^{-1/2} (\log(N)|h|)^3\Big)$$

Proof: Following [1], we have

$$S_4(N) = O\big(N^{1/2} \log(N)^3 \max_{U \le M \le N/V} \Delta\big)$$

where

$$\Delta = O\left(\max_{V \le j \le N/M} \sum_{V < k \le N/M} \left| \sum_{\substack{M < m \le 2M \\ m \le N/j \\ m \le N/k}} f(mj)\overline{f(mk)} \right| \right)^{\frac{1}{2}}$$

Now the innermost sum on the right hand side of the above expression is evaluated as follows

$$\left| \sum_m e^{2\pi i m^\alpha (j^\alpha - k^\alpha) h} \right| = O\big(|h|^{\frac{1}{4}} (j^\alpha - k^\alpha)^{\frac{1}{4}} M^{\frac{\alpha+2}{4}} (\log(M)|h|)^2\big) + O(\min N/j, M)$$

Therefore,

$$\sum_{V<k\le N/M}\left|\sum_{M<m\le 2M} e^{2\pi i m^{\alpha}(j^{\alpha}-k^{\alpha})h}\right| = O\left(|h|^{\frac{1}{4}}\frac{N}{M}\left(\frac{N}{M}\right)^{\alpha/4} M^{\frac{\alpha+2}{4}}(\log(M)|h|)^2\right) + \min\left(\frac{N}{j}, M\right)$$

Hence

$$\max_{V<j\le N/M}\sum_{V<k\le N/M}\left|\sum_{M<m\le 2M} e^{2\pi i m^{\alpha}(j^{\alpha}-k^{\alpha})h}\right| = O\left(|h|^{\frac{1}{4}}\frac{N}{M}\left(\frac{N}{M}\right)^{\alpha/4} M^{\frac{\alpha+2}{4}}(\log(M)|h|)^2\right) + O\left(\frac{N}{V}\right)$$

Thus,

$$\Delta = O\big(|h|^{\frac{1}{8}}N^{1/2+\alpha/8}M^{-1/4}(\log(M)|h|)\big) + O\big(N^{1/2}V^{-1/2}\big)$$

This proves the lemma. □

Proof of Theorem 3: We choose the parameter U and V as follows:

$$U = C_4 N^{1/2} \quad \text{and} \quad V = C_5 N^{\frac{1-\alpha}{4}}$$

for $0 < \alpha < 1$. Here C_4, C_5 denotes positive constants. □

With these choices, the theorem follows from Lemma 3–6.

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