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An elliptic analogue of a theorem of Hecke

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Dedicated to the memory of Marvin Knopp, with respect and admiration

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Abstract We revisit the classical theorem of Euler regarding special values of the Riemann zeta function as well as Hecke's generalization of this to Dirichlet's L-functions and derive an elliptic analogue. We also discuss transcendence questions that arise from this analogue.

Keywords Elliptic functions · Special values · Dirichlet's L-series · Hurwitz zeta function · Eisenstein series · Transcendence

Mathematics Subject Classification 11J89 · 11M35

1 Introduction

It is a pleasure to dedicate this paper to Professor Marvin Knopp. Marvin's 1970 book, Modular Functions in Analytic Number Theory, was an influential and inspirational book for many. He was also one of the founding fathers of the American school of modular forms.

The notion of modularity is central to number theory. Even for the study of special values of zeta and L-functions, the theme of modularity is a recurrent one. This may not be evident when one considers special cases or classical zeta functions. In this paper, we will revisit an old derivation of Euler regarding special values of the Riemann

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zeta function and Hecke's generalization of this two centuries later, and highlight the modular theme. This viewpoint allows us to generalize and derive an elliptic analogue of the work of Euler and Hecke. It also raises some interesting transcendence questions which we discuss at the end of the paper.

In 1734, Euler [2] used the infinite product for the sine function to compute the values of the Riemann zeta function $\zeta(s)$ at the first few even positive integers. In 1740, he [3] obtained a closed form for $\zeta(2k)$, k an integer, $k \geq 1$. A well written exposition of these results has been given by Ayoub in [1].

It is surprising that only as late as 1940, Euler's result was generalized to Dirichlet L-series by Hecke [4]. We formulate and prove an elliptic analogue of Hecke's result. We apply these results to derive some new results on special values of Eisenstein series.

2 The elliptic analogue

Let us first recall some classical functions. The Hurwitz zeta function is defined for positive integers s as

$$\zeta_{\pm}(s, x) = \sum_{\substack{n \in \mathbb{Z} \\ \operatorname{sgn} n = \pm}} \frac{1}{(n+x)^s},$$

where we adopt the convention that $n = 0$ is also included in the sum. (One can define this also for complex values of s provided we fix the branch of the logarithm, but this not needed for our discussion.) In the classical case, the lattice we are dealing with is $\mathbb{Z} = \{n.1 : n \in \mathbb{Z}\}$ and n runs over half of this lattice in the above expression. In order to generalize this to elliptic curves, we consider the lattice $L = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$. L will denote this standard lattice from now on. In analogy with the above, a natural way to define the elliptic Hurwitz zeta function is as follows. We define

$$\mathcal{Z}_L(k, x, \epsilon_1, \epsilon_2) = \sum_{\substack{m,n \in \mathbb{Z} \\ \operatorname{sgn} m = \epsilon_1, \operatorname{sgn} n = \epsilon_2}} \frac{1}{(m\omega_1 + n\omega_2 + x)^k},$$

where $\epsilon_1 = \pm$, $\epsilon_2 = \pm$, k is a positive integer, again noting our convention that $m = 0$ and $n = 0$ are included in the sum. In the classical case, for χ a Dirichlet character modulo q, where q is a positive integer, the L-function is given by (for positive integer values of s)

$$L_{\pm}(s, \chi) = \sum_{\substack{n \in \mathbb{Z} \\ \operatorname{sgn} n = \pm}}' \frac{\chi(n)}{n^s},$$

where the dash indicates that we sum over $n \neq 0$. If we emulate this for the lattice L, then we see that we have to introduce another character in the expression. Let χ and ψ be Dirichlet characters modulo q_1 and q_2, respectively. Then, we get the following L-functions:

$$\mathcal{L}(k, \chi, \psi, \epsilon_1, \epsilon_2) = \sum_{\substack{m\in\mathbb{Z} \\ \operatorname{sgn} m=\epsilon_1}}{}' \sum_{\substack{n\in\mathbb{Z} \\ \operatorname{sgn} n=\epsilon_2}}{}' \frac{\chi(m)\psi(n)}{(m\omega_1 + n\omega_2)^k},$$

where $\epsilon_1 = \pm, \epsilon_2 = \pm$ and k a positive integer as before. These L-functions are related to classical Eisenstein series, and we discuss this relationship in the next section. We define the elliptic L-function as

$$\mathcal{L}(k, \chi, \psi) := \mathcal{L}(k, \chi, \psi, +, +) + \mathcal{L}(k, \chi, \psi, +, -).$$

To understand how the elliptic versions of the L-function and the Hurwitz zeta function are related to each other, let us first inspect $\mathcal{L}(k, \chi, \psi, +, -)$. Note that

$$\begin{aligned}\mathcal{L}(k, \chi, \psi, +, -) &= \sum_{m=1}^{\infty} \sum_{n=-\infty}^{-1} \frac{\chi(m)\psi(n)}{(m\omega_1 + n\omega_2)^k} \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(m)\psi(n)\psi(-1)}{(m\omega_1 - n\omega_2)^k}.\end{aligned}$$

If we split this sum according to the residue classes of m and n, then the above expression can be written as

$$\begin{aligned}&\sum_{a(\text{mod } q_1)}{}' \sum_{b(\text{mod } q_2)}{}' \chi(a)\psi(b)\psi(-1) \sum_{\substack{m\equiv a \\ (\text{mod } q_1)}} \sum_{\substack{n\equiv b \\ (\text{mod } q_2)}} \frac{1}{(m\omega_1 - n\omega_2)^k} \\ &= \sum_{a=1}^{q_1}{}' \sum_{b=1}^{q_2}{}' \chi(a)\psi(b)\psi(-1) \sum_{t_1=0}^{\infty} \sum_{t_2=0}^{\infty} \frac{1}{((t_1q_1 + a)\omega_1 - (t_2q_2 + b)\omega_2)^k},\end{aligned}$$

where the dash over the sum indicates that a runs over residue classes co-prime to q_1 and b runs over residue classes co-prime to q_2. Thus,

$$\mathcal{L}(k, \chi, \psi, +, -) = \sum_{a=1}^{q_1}{}' \sum_{b=1}^{q_2}{}' \chi(a)\psi(b)\psi(-1) \left\{\mathcal{Z}_{L'}(k, a\omega_1 - b\omega_2, +, -)\right\}, \quad (1)$$

where L' is the lattice $\{m(q_1\omega_1) + n(q_2\omega_2) : m, n \in \mathbb{Z}\}$ generated by $q_1\omega_1$ and $q_2\omega_2$. Similarly, we get

$$\mathcal{L}(k, \chi, \psi, +, +) = \sum_{a=1}^{q_1}{}' \sum_{b=1}^{q_2}{}' \chi(a)\psi(b) \left\{\mathcal{Z}_{L'}(k, a\omega_1 + b\omega_2, +, +)\right\}. \quad (2)$$

In the classical case, to obtain an explicit expression for the L-function in terms of known quantities, the following well-known cotangent expansion is used:

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right), \quad z \in \mathbb{C}, \quad z \notin \mathbb{Z}.$$

In the elliptic case, our analogue for the cotangent function is the Weierstrass zeta function, defined as follows for the lattice L:

$$\zeta_L(z) = \frac{1}{z} + \sum_{\substack{\omega \in L \\ \omega \neq 0}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

Using the convention $\omega = m\omega_1 + n\omega_2$,

$$\zeta_L(z) = \frac{1}{z} + \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \left(\frac{1}{z-(m\omega_1+n\omega_2)} + \frac{1}{m\omega_1+n\omega_2} + \frac{z}{(m\omega_1+n\omega_2)^2} \right). \tag{3}$$

We conclude this section by defining the parity of a Dirichlet character. Recall that the parity of a character χ is said to be even if $\chi(-1) = 1$; it is odd if $\chi(-1) = -1$.

3 Results

Lemma 1 *Let r be a positive integer, $r \geq 2$. Let us denote by $\mathcal{Z}_L(r+1, z)$ the quantity,*

$$\mathcal{Z}_L(r+1, \omega_1+\omega_2+z, +, +) + (-1)^{r+1}\mathcal{Z}_L(r+1, \omega_1+\omega_2-z, +, +)$$
$$+\mathcal{Z}_L(r+1, \omega_1-\omega_2+z, +, -) + (-1)^{r+1}\mathcal{Z}_L(r+1, \omega_1-\omega_2-z, +, -).$$

Then, $\mathcal{Z}_L(r+1, z)$ can be computed explicitly in terms of the rth derivatives of the Weierstrass zeta function and the cotangent function. Namely, it is equal to

$$\frac{1}{z^{r+1}} + \frac{D^r(\zeta_L(z))}{r!(-1)^r} - \frac{\pi}{r!(-1)^r} \left(\frac{1}{\omega_2} D^r(\cot(\pi z/\omega_2)) + \frac{1}{\omega_1} D^r(\cot(\pi z/\omega_1)) \right).$$

Proof Consider Eq. (3) for the Weierstrass zeta function. Let us denote the second term on the right-hand side of (3) by S, that is, $\zeta_L(z) = \frac{1}{z} + S$. If we restrict the value of m to zero in S, then n ranges over the non-zero integers, giving us

$$S_n = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{z-n\omega_2} + \frac{1}{n\omega_2} + \frac{z}{(n\omega_2)^2} \right).$$

Repeating this for the other variable gives

$$S_m = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(\frac{1}{z - m\omega_1} + \frac{1}{m\omega_1} + \frac{z}{(m\omega_1)^2} \right).$$

If we denote by S' the sum

$$\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{z - (m\omega_1 + n\omega_2)} + \frac{1}{m\omega_1 + n\omega_2} + \frac{z}{(m\omega_1 + n\omega_2)^2} \right),$$

then it is clear that S equals $S_n + S_m + S'$.

As we are concerned with the rth derivative of $\zeta_L(z)$, let us examine individually the rth derivatives of S_n, S_m and S'. We have

$$\begin{aligned} S_n &= \sum_{n=1}^{\infty} \left(\frac{1}{z - n\omega_2} + \frac{1}{z + n\omega_2} + \frac{1}{n\omega_2} + \frac{1}{(-n\omega_2)} + \frac{2z}{(n\omega_2)^2} \right) \\ &= \frac{1}{\omega_2} \sum_{n=1}^{\infty} \left(\frac{1}{\frac{z}{\omega_2} - n} + \frac{1}{\frac{z}{\omega_2} + n} \right) + \sum_{n=1}^{\infty} \frac{2z}{(n\omega_2)^2} \\ &= \frac{1}{\omega_2} \left(\pi \cot(\pi z/\omega_2) - \frac{\omega_2}{z} \right) + \frac{2z}{\omega_2^2} \frac{\pi^2}{6}. \end{aligned}$$

Thus,for $r \geq 2$, differentiating r times with respect to z gives us

$$\frac{D^r(S_n)}{r!(-1)^r} = \frac{\pi}{(-1)^r r! \omega_2} D^r \left(\cot(\pi z/\omega_2)\right) - \frac{1}{z^{r+1}}. \tag{4}$$

It is clear that for S_m, we obtain in a similar way,

$$\frac{D^r(S_m)}{r!(-1)^r} = \frac{\pi}{(-1)^r r! \omega_1} D^r \left(\cot(\pi z/\omega_1)\right) - \frac{1}{z^{r+1}}. \tag{5}$$

Now, as S' involves m and n running over the non-zero integers, we note that this sum involves four kinds of lattice points $m\omega_1 + n\omega_2$, according to whether m and n are positive or negative. Thus, we can write

$$\begin{aligned} S' = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \ & \frac{1}{z - (m\omega_1 + n\omega_2)} + \frac{1}{z + m\omega_1 + n\omega_2} + \frac{1}{z + m\omega_1 - n\omega_2} \\ & + \frac{1}{z - m\omega_1 + n\omega_2} + \frac{2z}{(m\omega_1 + n\omega_2)^2} + \frac{2z}{(m\omega_1 - n\omega_2)^2}. \end{aligned}$$

Then, for $r \geq 2$,

$$\frac{D^r(S')}{r!(-1)^r} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(z - m\omega_1 - n\omega_2)^{r+1}} + \frac{1}{(z + m\omega_1 + n\omega_2)^{r+1}} + \frac{1}{(z + m\omega_1 - n\omega_2)^{r+1}} + \frac{1}{(z - m\omega_1 + n\omega_2)^{r+1}}.$$

Each of these terms is simply an elliptic Hurwitz zeta function. For example,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(z - m\omega_1 - n\omega_2)^{r+1}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{r+1}}{((m+1)\omega_1 + (n+1)\omega_2 - z)^{r+1}} = \mathcal{Z}_L(r+1, \omega_1 + \omega_2 - z, +, +).$$

Thus,

$$\begin{aligned}\frac{D^r(S')}{r!(-1)^r} &= \mathcal{Z}_L(r+1, \omega_1 + \omega_2 + z, +, +) + (-1)^{r+1}\mathcal{Z}_L(r+1, \omega_1 + \omega_2 - z, +, +) \\ &\quad + \mathcal{Z}_L(r+1, \omega_1 - \omega_2 + z, +, -) + (-1)^{r+1}\mathcal{Z}_L(r+1, \omega_1 - \omega_2 - z, +, -) \\ &= \mathcal{Z}_L(r+1, z). \end{aligned} \tag{6}$$

As $\zeta_L(z) = \frac{1}{z} + S_n + S_m + S'$, taking the rth derivative of $\zeta_L(z)$ and using the identities (4), (5) and (6) concludes the proof. □

Remark Recall that in the classical case, a suitable linear combination of the Hurwitz zeta functions could be expressed in terms of derivatives of the cotangent function [8]. The above lemma expresses a linear combination of the elliptic Hurwitz zeta functions in terms of derivatives of the classical cotangent function and its elliptic analogue—the Weierstrass zeta function. This completes the analogy.

Theorem 2 *Let k be a positive integer, $k \geq 3$. Then. $\mathcal{L}(k, \chi, \psi)$ can be computed explicitly in terms of derivatives of the Weierstrass zeta function and the cotangent function whenever k and $\chi\psi$ have the same parity, that is, whenever $(-1)^k = \chi(-1)\psi(-1)$.*

Proof Consider Eq. (2) for $\mathcal{L}(k, \chi, \psi, +, +)$. Noting that

$$\chi(q_1 - a) = \chi(-1)\chi(a)$$

and

$$\psi(q_2 - b) = \psi(-1)\psi(b),$$

we pair the residue class a with $q_1 - a$ and b with $q_2 - b$ in (2). This gives us an expression identical to (2), except for the fact that a and b now run over co-prime residue classes only up to $\lfloor q_1/2 \rfloor$ and $\lfloor q_2/2 \rfloor$, respectively, and the elliptic Hurwitz zeta function in (2) is replaced by

$$\begin{aligned} &\mathcal{Z}_{L'}(k, a\omega_1 + b\omega_2, +, +) + \chi(-1)\psi(-1)\mathcal{Z}_{L'}(k, q_1\omega_1 + q_2\omega_2 - a\omega_1 - b\omega_2, +, +) \\ &+ \chi(-1)\mathcal{Z}_{L'}(k, q_1\omega_1 - a\omega_1 + b\omega_2, +, +) + \psi(-1)\mathcal{Z}_{L'}(k, a\omega_1 + q_2\omega_2 - b\omega_2, +, +). \end{aligned}$$

The entire argument repeats for $\mathcal{L}(k, \chi, \psi, +, -)$, and the expression we derive is identical to (1), except that in this case, the elliptic Hurwitz zeta function in (1) is replaced by

$$\begin{aligned} &\mathcal{Z}_{L'}(k, a\omega_1 - b\omega_2, +, -) + \chi(-1)\psi(-1)\mathcal{Z}_{L'}(k, q_1\omega_1 - q_2\omega_2 - a\omega_1 + b\omega_2, +, -) \\ &+ \chi(-1)\mathcal{Z}_{L'}(k, q_1\omega_1 - a\omega_1 - b\omega_2, +, -) + \psi(-1)\mathcal{Z}_{L'}(k, a\omega_1 - q_2\omega_2 + b\omega_2, +, -), \end{aligned}$$

and a and b again run over half the co-prime residue classes as before. Now, we use the notation

$$z_1 := a\omega_1 + b\omega_2 - \frac{q_1\omega_1}{2} - \frac{q_2\omega_2}{2}$$
$$z_2 := a\omega_1 - b\omega_2 - \frac{q_1\omega_1}{2} + \frac{q_2\omega_2}{2}$$

and add the expressions obtained above for $\mathcal{L}(k, \chi, \psi, +, +)$ and $\mathcal{L}(k, \chi, \psi, +, -)$. If k and $\chi\psi$ have the same parity, then letting $r + 1 = k$ and replacing the lattice L by L' in Lemma 1, we can write

$$\mathcal{L}(k, \chi, \psi) = \sum_{\substack{a \,(\mathrm{mod}\, q_1) \\ (a, q_1)=1}}^{\lfloor q_1/2 \rfloor} \sum_{\substack{b \,(\mathrm{mod}\, q_2) \\ (b, q_2)=1}}^{\lfloor q_2/2 \rfloor} \chi(a)\psi(b) \left(\mathcal{Z}_{\frac{1}{2}L'}(k, z_1) + \psi(-1)\mathcal{Z}_{\frac{1}{2}L'}(k, z_2) \right). \tag{7}$$

Here, $\mathcal{Z}_{\frac{1}{2}L'}(k, z)$ is the quantity defined in Lemma 1, attached to the half lattice $\frac{1}{2}L' := \{m\frac{q_1\omega_1}{2} + n\frac{q_2\omega_2}{2} : m, n \in \mathbb{Z}\}$. Applying Lemma 1 to this quantity completes the proof. □

The explicit closed form for $\mathcal{L}(k, \chi, \psi)$ in terms of derivatives of the Weierstrass zeta function and the cotangent function can be obtained from (7) by applying Lemma 1.

4 Relation to Eisenstein series

In section 7.1 of [6], Miyake defines Eisenstein series of weight $k \geq 3$ as

$$E_k(z; \chi, \psi) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \chi(m)\psi(n)(mz + n)^{-k}, \qquad (z \in \mathfrak{H}),$$

where χ and ψ are Dirichlet characters modulo q_1 and q_2, respectively. As discussed in [6], $E_k(z; \chi, \psi)$ is a holomorphic function on $\mathfrak{H}$. We can relate this Eisenstein series to our elliptic L-function as

$$E_k(\omega_2/\omega_1; \chi, \psi) = \begin{cases} 2\omega_1^k \mathcal{L}(k, \chi, \psi) & if\ \chi(-1)\psi(-1) = (-1)^k, \\ 0 & if\ \chi(-1)\psi(-1) \neq (-1)^k. \end{cases}$$

We then have the following result on special values of Eisenstein series.

Theorem 3 *Let k be a positive integer, $k \geq 3$. Then, $E_k(\omega_1/\omega_2; \chi, \psi)$ can be computed explicitly in terms of derivatives of the Weierstrass zeta function and the cotangent function whenever $(-1)^k = \chi(-1)\psi(-1)$.*

Proof This follows directly from Theorem 2 and the above relation between the Eisenstein series and our elliptic L-function. □

5 Transcendence questions

These results lead to natural questions of transcendence. In particular, we are interested in the transcendence of special values of Eisenstein series. It is clear from (7) that the values of $\mathcal{Z}_{\frac{1}{2}L'}(k, z)$ at z_1 and z_2 determine the arithmetical nature of $\mathcal{L}(k, \chi, \psi)$ and hence of $E_k(\omega_1/\omega_2; \chi, \psi)$. By Lemma 1, we have

$$(k-1)!(-1)^{k-1}\mathcal{Z}_{\frac{1}{2}L'}(k, z) = \frac{1}{(k-1)!(-1)^{k-1}z^k} - \frac{2\pi}{q_2\omega_2} D^{k-1}\left(\cot\left(\frac{2\pi z}{q_2\omega_2}\right)\right) - \frac{2\pi}{q_1\omega_1} D^{k-1}\left(\cot\left(\frac{2\pi z}{q_1\omega_1}\right)\right) + D^{k-1}(\zeta_{\frac{1}{2}L'}(z)). \quad (8)$$

We will examine each term of the above expression for the CM case. Let us first note a few facts. It is known that for an elliptic curve with complex multiplication, the ratio

$$\tau = \frac{\omega_1}{\omega_2}$$

of the fundamental periods is an imaginary quadratic number. In this case, the field of endomorphisms of $\mathcal{E}$ is $\mathbf{k} = \mathbb{Q}(\tau) = \mathbb{Q}(\sqrt{D})$ for some negative integer D.

We also note for future reference the following result of Nesterenko [9]

Proposition 4 *Let $\wp(z)$ be the Weierstrass elliptic function with algebraic invariants g_2 and g_3 and complex multiplication by the field $\mathbf{k}$. If ω is any period of $\wp$ and $\tau' \in \mathbf{k}$, $\operatorname{Im}\tau' \neq 0$, then the set*

$$\{\pi, \omega, e^{2\pi i\tau'}\}$$

is algebraically independent over $\overline{\mathbb{Q}}$.

Proof See Corollary 1.6, chapter 3 of [9]. One may also refer to Proposition 15 and section 7 of [7], which not only gives the desired algebraic independence but also imparts additional insight by relating the period ω to a suitable product of Γ-functions. □

Recall that the Weierstrass $\wp$-function associated to L is defined by the series

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in L \\ \omega \neq 0}} \left\{ \frac{1}{(z-\omega^2)} - \frac{1}{\omega^2} \right\}.$$

It is known that $\wp(z)$ is analytic in $\mathbb{C}\backslash L$

We then have the following lemma about the arithmetical nature of certain linear combinations of elliptic Hurwitz zeta functions.

Lemma 5 *Let the elliptic curve $\mathcal{E}$ corresponding to the lattice $\frac{1}{2}L'$ be parametrized by the Weierstrass $\wp$-function of $\frac{1}{2}L'$ as $y^2 = 4x^3 - g_2 x - g_3$. We assume that g_2 and g_3 are algebraic and that $\mathcal{E}$ has complex multiplication by* **k**. *Then, for each pair of residue classes a and b modulo q_1 and q_2, respectively, the quantity*

$$\mathcal{Z}_{\frac{1}{2}L'}(k, z_1) + \psi(-1)\mathcal{Z}_{\frac{1}{2}L'}(k, z_2)$$

is of the form (up to algebraic factors)

$$\frac{1}{\omega_1^k} \frac{f(\pi, e^{2\pi i \sqrt{D}/q_{a,b}})}{g(e^{2\pi i \sqrt{D}/q_{a,b}})},$$

for some $q_{a,b} \in \mathbb{Q}$, where g and f are polynomials with algebraic coefficients, in one and two variables, respectively.

Proof We examine the value of each term on the right-hand side of (8) for $z = z_1$ and $z = z_2$. Plugging in $z = z_1$ for the first term and using $q_2(\omega_2/2) = \tau q_1(\omega_1/2)$ give us (up to an algebraic factor)

$$\frac{1}{a\omega_1 - q_1(\omega_1/2) + \frac{q_1}{q_2} b\tau\omega_1 - q_1\tau(\omega_1/2)}.$$

Recall from (7) that a and b are residue classes modulo q_1 and q_2, respectively. Thus, $a, b, q_1, q_2, \tau \in \overline{\mathbb{Q}}$, and we get an expression of the form

$$\frac{1}{\omega_1^k},$$

up to an algebraic factor. It is clear that this argument works for $z = z_2$ as well, to give us an expression of the same kind.

For the second and third term, we inspect first the general form of the rth derivative of the cotangent function. Let c be a constant. Then,

$$\begin{aligned} D^{k-1}(\cot(2\pi z/c)) &= D^{k-1}\left(i\,\frac{e^{4\pi i z/c}+1}{e^{4\pi i z/c}-1}\right) \\ &= i D^{k-1}\left(1+\frac{2}{e^{4\pi i z/c}+1}\right) \\ &= (k-1)!(-1)^{k-1}\frac{2^k i^k \pi^{k-1}}{c^{k-1}(e^{4\pi i z/c})^k}. \end{aligned}$$

To obtain the second term of (8) for z_1, we let $z = z_1$ and $c = q_2\omega_2$. Using the algebraic ratio of the periods and the fact that $z_1/q_2\omega_2 \in \mathbf{k}$, we get that

$$\frac{2\pi}{q_2\omega_2} D^{k-1}\left(\cot\left(\frac{2\pi z_1}{q_2\omega_2}\right)\right)$$

equals (up to an algebraic factor)

$$\frac{\pi^k}{\omega_1^k}\frac{1}{(c_1 e^{2\pi i\sqrt{D}d_1}-1)^k},$$

for some $c_1 \in \overline{\mathbb{Q}}$ and $d_1 \in \mathbb{Q}$. A similar argument works for the third term and then for z_2 instead of z_1, giving us like terms with different constants $c_i \in \overline{\mathbb{Q}}$ and $d_i \in \mathbb{Q}$

We now arrive at the final term on the right-hand side of (8). The first derivative of $\zeta_L(z)$ is $-\wp_L(z)$, the second is $-\wp_L'(z)$ and so on. It is clear that all the derivatives of $\zeta_{\frac{1}{2}L'}(z)$ are elliptic functions attached to the half lattice $\frac{1}{2}L'$ and are, hence, generated by $\wp_{\frac{1}{2}L'}(z)$ and $\wp_{\frac{1}{2}L'}(z)$ over $\mathbb{Q}$. Thus,

$$D^{k-1}(\zeta_{\frac{1}{2}L'}(z_1)) = \frac{g\left(\wp_{\frac{1}{2}L'}(z_1), \wp'_{\frac{1}{2}L'}(z_1)\right)}{h\left(\wp_{\frac{1}{2}L'}(z_1), \wp'_{\frac{1}{2}L'}(z_1)\right)},$$

where g and h are polynomials in two variables with coefficients in $\overline{\mathbb{Q}}$. Now, notice that z_1 is a q_1q_2 division point for our lattice, that is, $q_1q_2z_1 \in \frac{1}{2}L'$. Using this observation along with the condition of the theorem that the invariants g_2 and g_3 are algebraic, we can deduce that $\wp_{\frac{1}{2}L'}(z_1), \wp'_{\frac{1}{2}L'}(z_1) \in \overline{\mathbb{Q}}$. Thus, the entire term $D^{k-1}(\zeta_{\frac{1}{2}L'}(z_1))$ is algebraic, and can be neglected in our analysis of the arithmetical nature of (8). The same argument works for $z = z_2$ as it is also a division point.

Putting together our conclusion for each term and adding the terms, we find that for each pair of residue classes a and b in the sum of (7),

$$\mathcal{Z}_{\frac{1}{2}L'}(k, z_1) + \psi(-1)\mathcal{Z}_{\frac{1}{2}L'}(k, z_2)$$

has the form

$$\frac{1}{\omega_1^k}\frac{f(\pi, e^{2\pi i\sqrt{D}/q_{a,b}})}{g(e^{2\pi i\sqrt{D}/q_{a,b}})},$$

for some $q_{a,b} \in \mathbb{Q}$, g and f polynomials with coefficients in $\overline{\mathbb{Q}}$. □

Recall that z_1 and z_2 depend on the pair a, b. Note that the quantity $q_{a,b}$ in the lemma above depends upon a and b as well.

Theorem 6 *We assume the same conditions on the elliptic curve $\mathcal{E}$ attached to $\frac{1}{2}L'$ as stated in Lemma 5. Then, for $k \geq 3$, $E_k(\omega_2/\omega_1; \chi, \psi)$ and $\mathcal{L}(k, \chi, \psi)$ are both either zero or transcendental whenever $(-1)^k = \chi(-1)\psi(-1)$.*

Proof As a and b run over 'half' the co-prime residue classes modulo q_1 and q_2, respectively, finitely many expressions of the form given in Lemma 5 are being summed. Hence, we get for $\mathcal{L}(k, \chi, \psi)$ an expression of the form

$$\frac{1}{\omega_1^k}\frac{P(\pi, e^{2\pi i\sqrt{D}/\tilde{q}})}{Q(e^{2\pi i\sqrt{D}/\tilde{q}})},$$

for some $\tilde{q} \in \mathbb{Q}$ and P, Q polynomials with algebraic coefficients. If the above expression is non-zero and algebraic, then it gives a non-trivial algebraic dependence for the set $\{\pi, \omega_1, e^{2\pi i\sqrt{D}/\tilde{q}}\}$, thereby contradicting Proposition 4. Hence, $\mathcal{L}(k, \chi, \psi)$ is zero or transcendental. As

$$E_k(\omega_2/\omega_1; \chi, \psi) = 2\omega_1^k \mathcal{L}(k, \chi, \psi),$$

whenever the parity condition of the theorem is met, we see that

$$E_k(\omega_2/\omega_1; \chi, \psi) = \frac{2.P(\pi, e^{2\pi i\sqrt{D}/\tilde{q}})}{Q(e^{2\pi i\sqrt{D}/\tilde{q}})}.$$

Now, one does not have to use the full strength of Proposition 4 and the algebraic independence of π, and $e^{2\pi i\sqrt{D}/\tilde{q}}$ is enough to conclude that $E_k(\omega_2/\omega_1; \chi, \psi)$ must be zero or transcendental. □

6 Concluding remarks

In the preceding discussion, we have assumed $k \geq 3$. It would be interesting to study the case $k = 2$. Since the corresponding series do not converge, we are forced to study an appropriate limit of our elliptic L-function (using what is called 'Hecke's trick'). This naturally leads to higher level analogues of the Eisenstein series E_2, a topic discussed in some detail in chapters 7 and 8 of Schoenberg [10]. In this context, Marvin Knopp [5] gave a ten page review of Schoenberg's book in which he highlights the importance of the study of period polynomials arising initially from the study of the transformation formula for E_2. We thank the referee for pointing this out and relegate this study to a future paper.

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