A Generalization of Euler's Theorem for $\zeta(2k)$

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Abstract. We extend Euler's celebrated theorem evaluating $\zeta(2k)$. We replace the terms n^{-2k} in the infinite sum for $\zeta(2k)$, with $(n^2 + Bn + C)^{-k}$ where B, C are complex and k is a positive integer. We explicitly evaluate these sums and also briefly discuss their transcendence.

1. INTRODUCTION. An important example from the theory of special functions is that of the Riemann zeta function defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re(s) > 1$. The story of how Euler proved that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and more generally that

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k-1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!},$$

where B_k denotes the kth Bernoulli number, is fascinating and can be found in many places (for example, [**1**], [**2**], [**15**], etc.). The transcendental nature of these special values of the Riemann zeta function, $\zeta(s)$, is determined by the transcendence of π, first proved by Lindemann [**5**]. The allure of these results is eternal and is evidenced by the numerous papers written by many mathematicians giving simpler and elegant derivations. For example, in this MONTHLY alone, we have the papers by Apostol [**1**] and Williams [**15**].

In this note, we focus on evaluating the convergent series

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^2 + Bn + C} \tag{1}$$

and more generally

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^2 + Bn + C)^k} \tag{2}$$

for parameters $B, C \in \mathbb{C}$ and positive integer k. We take summation over $\mathbb{Z}$ to be defined as in [**14**] as

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$$\sum_{n\in\mathbb{Z}} a_n = \lim_{N\to\infty} \sum_{|n|\le N} a_n.$$

To examine sums of the forms (1) and (2) we will introduce some methods from Fourier analysis. The specific results of this paper are contained in the more general works [**7**] and [**13**]. The methods there are elaborate and we present here a shorter and more elegant derivation.

2. FOURIER ANALYSIS. We assume the reader is familiar with the properties of the Fourier transform and Fourier series. If not, we refer the reader to [**9**] for an introduction to the subject. To keep this article self-contained, we review some basic facts. For a function $f(x) \in L^1[0, 1]$, the collection of 1-periodic, Lebesgue integrable functions with

$$\int_0^1 |f(x)|dx < \infty,$$

we define the *n*th Fourier coefficient by

$$f_n = \int_0^1 f(x)e^{-2\pi inx}dx.$$

With this definition we have the Fourier series associated with $f(x)$,

$$\sum_{n\in\mathbb{Z}} f_n e^{2\pi inx}.$$

It is well known that if f is differentiable, then the Fourier series for f converges to f. Also, if f is piecewise smooth, then the Fourier series of f at x converges to

$$\frac{f(x_+) + f(x_-)}{2}.$$

As described in [**12**], such a function is said to satisfy *Dirichlet's conditions*. We take piecewise smooth to mean that the function f can be broken into distinct pieces for which both f and f' are continuous and where jump discontinuities are the only allowable discontinuities. Determining exactly when the Fourier series for a given function f converges to f is not such an easy task. We refer the interested reader to [**6**] (Appendix D) for a discussion of convergence for functions with *bounded variation*.

Related to Fourier series is the notion of a Fourier transform. For convergence purposes we examine piecewise smooth functions $f \in L^1(\mathbb{R})$ which are the Lebesgue integrable functions satisfying

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty.$$

For such functions we define the Fourier transform as

$$\widehat{f}(u) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixu}dx.$$

For $f \in L^1(\mathbb{R})$, such that $\widehat{f} \in L^1(\mathbb{R})$, we have an inversion formula in which we can relate f to the Fourier transform of $\widehat{f}$. The proof of the following inversion formula is not obvious and we refer the reader to [**9**] for a full treatment.

Lemma 1. *For $f, \widehat{f} \in L^1(\mathbb{R})$, we have the inversion formula*

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(u) e^{2\pi i x u} du,$$

which implies that $\widehat{\widehat{f}}(x) = f(-x)$.

Using what we have developed thus far, we now state and prove a version of Poisson's summation formula. There are stronger versions of the theorem, though the version presented here is particularly useful for our purposes, and has a proof that is concise and elementary. See [**3**] (Theorem 3.1.17) for a stronger version of the following theorem where the assumptions taken on the function f are simply that $f, \widehat{f} \in L^1(\mathbb{R})$ and satisfy

$$|f(x)| + |\widehat{f}(x)| \leq C(1 + |x|)^{-1-\delta}$$

for some $C, \delta > 0$.

Theorem 1 (Poisson Summation). *If $f \in L^1(\mathbb{R})$ is continuous and piecewise smooth where the sum*

$$\sum_{n \in \mathbb{Z}} f(n + v)$$

converges absolutely and uniformly in v, and if

$$\sum_{n \in \mathbb{Z}} |\widehat{f}(n)| < \infty,$$

then

$$\sum_{n \in \mathbb{Z}} f(n + v) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n v}.$$

Proof. Set $F(v) = \sum_{n \in \mathbb{Z}} f(n + v)$. By the assumption of absolute and uniform convergence, F is a continuous and piecewise smooth periodic function in v with period 1. Thus we can view F as a function on $[0, 1]$ and compute the Fourier coefficients directly:

$$\begin{aligned} F_n &= \int_0^1 F(v) e^{-2\pi i n v} dv \\ &= \sum_{m \in \mathbb{Z}} \int_0^1 f(m + v) e^{-2\pi i n v} dv. \end{aligned}$$

Replacing $x = m + v$ we obtain

$$\begin{aligned} F_n &= \sum_{m \in \mathbb{Z}} \int_m^{m+1} f(x) e^{-2\pi i n x} dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx = \widehat{f}(n). \end{aligned}$$

Since $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty$, the Fourier series for F converges and we have that

$$\sum_{n\in\mathbb{Z}} f(n+v) = F(v) = \sum_{n\in\mathbb{Z}} F_n e^{2\pi inv} = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi inv}. \qquad \blacksquare$$

The components of Fourier analysis introduced here will allow us to find explicit finite closed forms for the sums (1) and (2).

3. EVALUATING THE SUM $\sum_{N\in\mathbb{Z}} 1/(N^2 + BN + C)^K$.

We begin with an example calculation which will prove very useful.

Lemma 2. *If* $\Re(c) > 0$*, and we let* $g(x) = e^{-c|x|}$*, then*

$$\widehat{g}(u) = \frac{2c}{c^2 + 4\pi^2 u^2}.$$

Proof. It is clear that $g \in L^1(\mathbb{R})$, therefore we compute the Fourier transform. We have

$$\begin{aligned}
\widehat{g}(u) &= \int_{-\infty}^{\infty} e^{-c|x|}e^{-2\pi iux}dx \\
&= \int_{0}^{\infty} e^{x(-2\pi iu-c)}dx + \int_{-\infty}^{0} e^{x(-2\pi iu+c)}dx \\
&= \frac{1}{2\pi iu + c} + \frac{1}{-2\pi iu + c} \\
&= \frac{2c}{c^2 + 4\pi^2u^2}. \qquad \blacksquare
\end{aligned}$$

By making use of Lemma 2, as well as Poisson summation, we now evaluate the sum $\sum_{n\in\mathbb{Z}} 1/(n^2 + Bn + C)$.

Theorem 2. *For* $B, C \in \mathbb{C}$ *with* $-D = B^2 - 4C$ *such that* $n^2 + Bn + C \neq 0$ *for any integer* n*, we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^2 + Bn + C} = \frac{2\pi}{\sqrt{D}} \left(\frac{e^{2\pi\sqrt{D}} - 1}{e^{2\pi\sqrt{D}} - 2\cos(B\pi)e^{\pi\sqrt{D}} + 1} \right).$$

Proof. Suppose that $D = 4C - B^2$ such that $\Re(D) > 0$. Let g be as in Lemma 2 with $c = \pi\sqrt{D}$ so that we have

$$\widehat{g}(u) = \frac{\sqrt{D}}{2\pi} \left(\frac{1}{u^2 + D/4} \right).$$

The sum in question can be written

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{1}{n^2 + Bn + C} &= \sum_{n\in\mathbb{Z}} \frac{1}{(n + B/2)^2 + D/4} \\
&= \frac{2\pi}{\sqrt{D}} \sum_{n\in\mathbb{Z}} \widehat{g}(n + B/2).
\end{aligned}$$

By the Weierstrass M-test, this sum converges uniformly as a function of D and will yield an analytic function on $\Re(D) > 0$ as its uniform limit. By Poisson summation, with $v = B/2$, this last sum is equal to

$$\frac{2\pi}{\sqrt{D}} \sum_{n\in\mathbb{Z}} \widehat{\widehat{g}}(n) e^{2\pi i n(B/2)}.$$

By Lemma 1 we have that $\widehat{\widehat{g}}(n) = g(-n)$, which is equal to $g(n)$ since g is even. Our sum now becomes

$$\frac{2\pi}{\sqrt{D}} \sum_{n\in\mathbb{Z}} e^{-\pi\sqrt{D}|n|} e^{B\pi i n},$$

which is a combination of two geometric series

$$\frac{2\pi}{\sqrt{D}} \left(\sum_{n=0}^{\infty} \left(e^{B\pi i - \pi\sqrt{D}}\right)^n + \sum_{n<0} \left(e^{B\pi i + \pi\sqrt{D}}\right)^n \right).$$

Evaluating these geometric sums we obtain

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^2 + Bn + C} = \frac{2\pi}{\sqrt{D}} \left(\frac{1}{e^{B\pi i + \pi\sqrt{D}} - 1} - \frac{1}{e^{B\pi i - \pi\sqrt{D}} - 1} \right), \tag{3}$$

which simplifies to the desired result. By the theory of analytic continuation, this closed form holds for all $D \in \mathbb{C}$ such that $n^2 + Bn + C \neq 0$ for any $n \in \mathbb{Z}$. ∎

Remark. The methods demonstrated here give us a general approach for attempting to find closed forms of infinite sums. We need only recognize the terms of a series as the Fourier transform of a particular function and then invoke Poisson summation. Here Poisson summation leads to summation of a geometric series, however, in general something more complicated may arise.

From the closed form given in (3), we can derive an evaluation of the more general sum (2). We write

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^2 + Bn + C)^k} = \sum_{n\in\mathbb{Z}} \frac{1}{((n + B/2)^2 + D/4)^k},$$

which can be computed by treating the sum in Theorem 2 as a function of D and differentiating $k - 1$ times. That is, $\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+Bn+C)^k}$ is equal to

$$\frac{2\pi(-4)^{k-1}}{(k-1)!} \left[\frac{1}{\sqrt{D}} \left(\frac{1}{e^{B\pi i + \pi\sqrt{D}} - 1} - \frac{1}{e^{B\pi i - \pi\sqrt{D}} - 1} \right) \right]^{(k-1)}. \tag{4}$$

Writing

$$f(D) = \frac{1}{\sqrt{D}}, \quad g(D) = \frac{1}{e^D - 1}, \quad h_{\pm}(D) = B\pi i \pm \pi\sqrt{D}, \tag{5}$$

we see that we must evaluate higher derivatives of $f \cdot g(h_+) - f \cdot g(h_-)$. We will require the derivatives of f, g, h_+, and h_- explicitly and it is easy to see for $m \geq 0$ and $n \geq 1$ that

$$f^{(m)} = \frac{(-1)^m (2m)! D^{-(2m+1)/2}}{4^m m!}, \quad h_{\pm}^{(n)} = \pm \frac{2\pi(-1)^{n-1}(2n-2)! D^{-(2n-1)/2}}{4^n (n-1)!}.$$

The following lemma characterizes the form of all derivatives of g. These derivatives involve the Stirling numbers of the second kind, $S(n, j)$, which are defined to be the number of ways to partition a set of n objects into j nonempty subsets. These Stirling numbers of the second kind satisfy the recurrence relation

$$S(n+1, j) = jS(n, j) + S(n, j-1)$$

and with the convention that $S(0, 0) = 1$ and $S(n, 0) = 0$ for $n > 0$. They are explicitly given by

$$S(n, j) = \frac{1}{j!} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} k^n \in \mathbb{Z}$$

for $0 \leq j \leq n$, while $S(n, j) = 0$ for $j > n$. Note that the previous sum surprisingly gives the correct value for $S(n, j)$ for all nonnegative values n and j, even for $j > n$, where $S(n, j)$ is 0.

Lemma 3. *For $m \geq 0$,*

$$\left(\frac{1}{e^x - 1} \right)^{(m)} = (-1)^m \sum_{j=1}^{m+1} \frac{(j-1)! S(m+1, j)}{(e^x - 1)^j},$$

where $S(m+1, j) \in \mathbb{Z}$ is a Stirling number of the second kind.

Proof. We induct on m with the case $m = 0$ being clear since $S(1, 1) = 1$. Assume the equality is true for all $m < t$. By induction we have

$$\left(\left(\frac{1}{e^x - 1} \right)^{(t-1)} \right)' = (-1)^{t-1} \left(\sum_{j=1}^{t} \frac{(j-1)! S(t, j)}{(e^x - 1)^j} \right)',$$

which equals

$$(-1)^t \sum_{j=1}^{t} j! S(t, j) \frac{e^x}{(e^x - 1)^{j+1}} = (-1)^t \sum_{j=1}^{t} j! S(t, j) \frac{e^x - 1 + 1}{(e^x - 1)^{j+1}}.$$

Writing $(e^x - 1 + 1)/(e^x - 1)^{j+1} = 1/(e^x - 1)^j + 1/(e^x - 1)^{j+1}$, we can write our sum as

$$(-1)^t \left(\frac{1}{e^x - 1} + \sum_{j=2}^{t} \frac{(j-1)!(jS(t, j) + S(t, j-1))}{(e^x - 1)^j} + \frac{t!}{(e^x - 1)^{t+1}} \right).$$

Making use of the recurrence for the Stirling numbers, $S(t+1, j) = jS(t, j) + S(t, j-1)$, which implies that $S(t+1, 1) = 1$ and $S(t+1, t+1) = 1$, we have the desired result. ■

The following lemmas will be helpful for evaluating derivatives of the expression $f \cdot g(h_+) - f \cdot g(h_-)$.

Lemma 4. *If f and g are functions with a sufficient number of derivatives, then*

$$(fg)^{(m)} = \sum_{i=0}^{m} \binom{m}{i} f^{(m-i)} g^{(i)}.$$

Proof. We induct on m with the case $m = 0$ being clear. For $m > 0$ write $(fg)^{(m)} = ((fg)')^{(m-1)}$, which is equal to

$$\left(f'g\right)^{(m-1)} + \left(fg'\right)^{(m-1)} = \sum_{i=0}^{m-1} \binom{m-1}{i} \left(f^{(m-i)} g^{(i)} + f^{(m-1-i)} g^{(i+1)}\right)$$

by induction. This last sum can be written

$$f^{(m)} g + \sum_{i=1}^{m-1} \left(\binom{m-1}{i} + \binom{m-1}{i-1} \right) f^{(m-i)} g^{(i)} + f g^{(m)},$$

which simplifies to the result by Pascal's identity for binomial coefficients. ■

The following result commonly credited to Faà di Bruno gives a method of explicitly evaluating higher derivatives of a composition of functions. We state the result without proof, and refer the reader to [**4**] of this MONTHLY for proofs and history of the result.

Lemma 5 (Faà di Bruno's Formula). *If g and h are functions with a sufficient number of derivatives, then*

$$(g(h))^{(m)} = \sum_{b_1, \ldots, b_m} \frac{m! g^{(b_1 + \cdots + b_m)}(h)}{b_1! \cdots b_m!} \prod_{j=1}^{m} \left(\frac{h^{(j)}}{j!} \right)^{b_j},$$

where the sum is over all nonnegative integers $b_1, \ldots, b_m$ such that $b_1 + 2b_2 + \cdots + mb_m = m$.

We now have all components in place to explicitly compute the value of the sum (2). The explicit evaluation involves the Catalan numbers C_m, which count the number of ways of dividing a regular $m+2$ sided polygon into m triangles. The Catalan numbers, with initial convention $C_0 = 1, C_1 = 1$, are explicitly given by $\binom{2m}{m}/(m+1)$. The reader may be interested in seeing six different interpretations of C_n in Corollary 6.2.3 of [**10**], an additional 66 interpretations in exercise 6.19 of [**10**], and many more interpretations on Stanley's website [**11**].

Theorem 3. *For $B, C \in \mathbb{C}$ with $-D = B^2 - 4C$ such that $n^2 + Bn + C \neq 0$ for any integer n, and k a positive integer, we have that $\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+Bn+C)^k}$ is equal to*

$$\frac{2\pi\sqrt{D}}{D^k}\sum_{r=0}^{k-1}\binom{2k-2-2r}{k-1-r}\sum_{b_1,\ldots,b_r}\frac{(2\pi)^R D^{\frac{R}{2}} C_0^{b_1}\cdots C_{r-1}^{b_r}}{b_1!\cdots b_r!}$$
$$\times\sum_{s=1}^{R+1}\left(\frac{(s-1)!S(R+1,s)}{(e^{B\pi i+\pi\sqrt{D}}-1)^s}-\frac{(-1)^R(s-1)!S(R+1,s)}{(e^{B\pi i-\pi\sqrt{D}}-1)^s}\right),$$

where the summation in b_i is over all nonnegative solutions to $b_1 + 2b_2 + \cdots + rb_r = r$ and where $R = b_1 + \cdots + b_r$, C_m is the mth Catalan number and $S(R+1, s)$ is a Stirling number of the second kind. The sum lies in $\pi\mathbb{Q}(\sqrt{D}, e^{\pi\sqrt{D}}, e^{B\pi i})[\pi]$, with π^k being the largest power of π present.

Proof. We proceed with f, g, h_+, and h_- defined in (5). From (4) our sum is equal to $\frac{2\pi(-4)^{k-1}}{(k-1)!}$ times

$$[f\cdot g(h_+) - f\cdot g(h_-)]^{(k-1)}.$$

By Lemma 4 we have that this derivative expression is equal to

$$\sum_{r=0}^{k-1}\binom{k-1}{r} f^{(k-1-r)}\left((g(h_+))^{(r)} - (g(h_-))^{(r)}\right).$$

By Faà di Bruno's formula, for each $0 \le r \le k-1$, we take all nonnegative integers $b_1, \ldots, b_r$ such that $b_1 + 2b_2 + \cdots rb_r = r$ and we have

$$\sum_{r=0}^{k-1}\binom{k-1}{r} f^{(k-1-r)}\sum_{b_1,\ldots,b_r}\frac{r!}{b_1!\cdots b_r!}$$
$$\times\left(g^{(R)}(h_+)\prod_{t=1}^{r}\left(\frac{h_+^{(t)}}{t!}\right)^{b_t} - g^{(R)}(h_-)\prod_{t=1}^{r}\left(\frac{h_-^{(t)}}{t!}\right)^{b_t}\right),$$

where we write R in place of $b_1 + \cdots + b_r$. By Lemma 3, inserting the explicit derivatives for f, g, h_+, and h_- and simplifying slightly, we have

$$(-4)^{1-k}D^{(1-2k)/2}(k-1)!\sum_{r=0}^{k-1}\frac{(-4D)^r(2k-2-2r)!}{((k-1-r)!)^2}$$
$$\times\sum_{b_1,\ldots,b_r}\frac{1}{b_1!\cdots b_r!}\sum_{s=1}^{R+1}\left(\frac{(-1)^R(s-1)!S(R+1,s)}{(e^{B\pi i+\pi\sqrt{D}}-1)^s}-\frac{(s-1)!S(R+1,s)}{(e^{B\pi i-\pi\sqrt{D}}-1)^s}\right)$$
$$\times\prod_{t=1}^{r}\left(\frac{2\pi(-1)^{t-1}(2t-2)!D^{-(2t-1)/2}}{t!4^t(t-1)!}\right)^{b_t}.$$

Note that the product with index t can be written

$$2^{R-2r}\pi^R(-1)^{R-r}D^{R/2-r}\prod_{t=1}^{r}\left(\frac{(2t-2)!}{t!(t-1)!}\right)^{b_t},$$

where each factor $(2t-2)!/t!(t-1)!$ can be recognized as the $(t-1)$th Catalan number. Inserting this, our final evaluation of the sum (2) is equal to

$$\frac{2\pi\sqrt{D}}{D^k}\sum_{r=0}^{k-1}\binom{2k-2-2r}{k-1-r}\sum_{b_1,\ldots,b_r}\frac{(2\pi)^R D^{\frac{R}{2}}C_0^{b_1}\cdots C_{r-1}^{b_r}}{b_1!\cdots b_r!}$$
$$\times\sum_{s=1}^{R+1}\left(\frac{(s-1)!S(R+1,s)}{(e^{B\pi i+\pi\sqrt{D}}-1)^s}-\frac{(-1)^R(s-1)!S(R+1,s)}{(e^{B\pi i-\pi\sqrt{D}}-1)^s}\right)$$

as desired. This value is clearly in $\pi\mathbb{Q}(\sqrt{D}, e^{\pi\sqrt{D}}, e^{B\pi i})[\pi]$, with π^k being the largest power of π present. ■

Theorems 2 and 3 avoid cases when $p(x) = x^2 + Bx + C$ has an integral root, but these cases can be handled as well. For instance, it is easy to see that if $p(x) = (x-m)^2$ for some integer m, then

$$\sum_{n\in\mathbb{Z},n\neq m}\frac{1}{(n-m)^{2k}} = 2\zeta(2k).$$

If p has two distinct integer roots, m_1, m_2, then by partial fractions there are rational numbers a_i, b_i so that

$$\sum_{n\in\mathbb{Z},n\neq m_1,m_2}\frac{1}{(n-m_1)^k(n-m_2)^k}$$
$$=\sum_{i=1}^{k}a_i\sum_{n\neq m_1}\frac{1}{(n-m_1)^i}+\sum_{i=1}^{k}b_i\sum_{n\neq m_2}\frac{1}{(n-m_2)^i}-\sum_{i=1}^{k}\frac{a_i+(-1)^ib_i}{(m_2-m_1)^i}.$$

The sums $\sum_{n\neq m_j}1/(n-m_j)^i$ are zero for odd i and $2\zeta(i)$ for even i, while the final sum is rational. If $p(x) = (x-m)(x-\alpha)$ for integer m and $\alpha \notin \mathbb{Z}$, then by partial fractions there are $a_i, b_i \in \mathbb{Q}(\alpha)$ such that

$$\sum_{n\in\mathbb{Z},n\neq m}\frac{1}{(n-m)^k(n-\alpha)^k}$$
$$=\sum_{i=1}^{k}a_i\sum_{n\neq m}\frac{1}{(n-m)^i}+\sum_{i=1}^{k}b_i\sum_{n\in\mathbb{Z}}\frac{1}{(n-\alpha)^i}-\sum_{i=1}^{k}\frac{b_i}{(m-\alpha)^i}.$$

The last sum lies in $\mathbb{Q}(\alpha)$ and for each $1 \leq i \leq k$, the first sums are zero for odd i, and $2a_i\zeta(i) \in \pi^i\mathbb{Q}(\alpha)$ for even i. We note that the sum

$$\sum_{n\in\mathbb{Z}} \frac{1}{n-\alpha} = -\pi\cot(\pi\alpha) \tag{6}$$

and so each sum $\sum_{n\in\mathbb{Z}} 1/(n-\alpha)^i$ can be written in terms of derivatives of the cotangent function. We refer the reader to exercise 4 on page 316 of [**9**] and section 2 of [**13**] for a detailed derivation of equation (6). For each $1 \le i \le k$, these sums lie in $\pi^i\mathbb{Q}(e^{\pi i\alpha})$. Sums of this form, related to derivatives of the cotangent function, have been discussed extensively in [**7**] and [**13**] and we refer the reader to those works for more details on computing explicit values in these cases.

4. TRANSCENDENCE OF THE SUMS. In some cases we can characterize the sums (1) and (2) discussed in Theorems 2 and 3 as being transcendental. Namely, we specify the case when B, C and therefore $D = 4C - B^2$ are all rational with $D > 0$. Writing $D = p/q$ so that $\sqrt{D} = \sqrt{pq}/q$, we have that the sum (1) lies in $\pi\mathbb{Q}(\sqrt{pq}, \cos(B\pi), e^{\pi\sqrt{pq}/q})$. By a theorem of Nesterenko [**8**], it is known that π and $e^{\pi\sqrt{d}}$ are algebraically independent for positive integer d. That is, π and $e^{\pi\sqrt{d}}$ are each transcendental over $\mathbb{Q}$ (or $\overline{\mathbb{Q}}$) and there is no polynomial relationship between them, with coefficients from $\mathbb{Q}$ (or $\overline{\mathbb{Q}}$). Thus, the sums (1) in Theorem 2 are all transcendental in this case. Similarly, in this special case, the sums (2) discussed in Theorem 3 are all polynomials in π with coefficients from $\mathbb{Q}(\sqrt{pq}, e^{B\pi i}, e^{\pi\sqrt{pq}/q})$ with zero being the coefficient for π^0. The final form given in Theorem 3 could be grouped over one common denominator involving integers and powers of $e^{B\pi i}$ and $e^{\pi\sqrt{pq}/q}$, the former being algebraic. The numerator would be a polynomial in π and $e^{\pi\sqrt{pq}/q}$ with coefficients in $\mathbb{Q}(\sqrt{pq}, e^{B\pi i}, e^{\pi\sqrt{pq}/q}) \subseteq \overline{\mathbb{Q}}$. Again, by the algebraic independence given by the theorem of Nesterenko, these expressions are transcendental when they do not vanish. In particular, we immediately obtain a transcendental number for the sums (2) for any even value of k. We now show for any positive integer k that these sums never vanish and therefore are all transcendental. The following lemma will prove useful.

Lemma 6. *For Stirling numbers of the second kind, $S(k, s)$, and $k \ge 2$ we have*

$$\sum_{s=1}^{k} (-1)^s (s-1)! S(k, s) = 0.$$

Proof. Inserting the recurrence for the Stirling numbers, we have that the sum is equal to

$$\sum_{s=1}^{k} (-1)^s (s-1)! s S(k-1, s) + \sum_{s=1}^{k} (-1)^s (s-1)! S(k-1, s-1).$$

Since $S(k-1, k) = 0 = S(k-1, 0)$, we have

$$\sum_{s=1}^{k-1} (-1)^s s! S(k-1, s) + \sum_{s=2}^{k} (-1)^s (s-1)! S(k-1, s-1),$$

which is zero after shifting the second sum to match the index of the first sum. ■

We note that Lemma 6 can also be shown by one of the generating functions for these Stirling numbers, which relates powers to sums of falling factorials. Namely,

$$x^k = \sum_{s=0}^{k} S(k,s)(x)_s,$$

where $(x)_s = x(x-1)\cdots(x-s+1)$. We leave the details to the reader, and now focus on our main transcendence result.

Theorem 4. *For any positive integer k and rational $B, C, D = 4C - B^2 > 0$, the sum*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+Bn+C)^k}$$

is transcendental.

Proof. By Theorems 2 and 3, these sums are polynomials in π with coefficients in $\mathbb{Q}(\sqrt{D}, e^{B\pi i}, e^{\pi\sqrt{D}})$ and by the above remarks either vanish or are transcendental since π and $e^{\pi\sqrt{D}}$ are algebraically independent. The sum does not vanish for $k=1$ or any even k, and is therefore transcendental in these cases. Thus, we assume that $k \geq 3$ is odd and we need only show that at least one coefficient for some power of π is nonzero. In particular, we isolate the coefficient of π^k. From the closed form given in Theorem 3, it is easy to see that we obtain π^k only when $R = k-1$, which implies that $r = k-1$, which in turn implies that the terms of interest from the sum over $b_1, \ldots, b_r$ will be those in which $b_1 = k-1$ and $b_i = 0$ for $i = 2, \ldots, r$. Thus, the coefficient of π^k is

$$\frac{2\sqrt{D}}{D^k}\binom{0}{0}\frac{2^{k-1}D^{(k-1)/2}C_0^{k-1}}{(k-1)!}\left(\sum_{s=1}^{k}(s-1)!S(k,s)\left(\frac{1}{(\alpha y-1)^s} - \frac{(-1)^{k-1}}{(\alpha y^{-1}-1)^s}\right)\right),$$

where we write α in place of the algebraic $e^{B\pi i}$ and y in place of the transcendental $e^{\pi\sqrt{D}}$. We need only show that the sum with index s is nonzero. This sum over s can be written

$$\sum_{s=1}^{k}(s-1)!S(k,s)\left(\frac{1}{(\alpha y-1)^s} + \frac{(-1)^k y^s}{(\alpha - y)^s}\right)$$

and after finding a common denominator for this expression we have

$$\sum_{s=1}^{k}(s-1)!S(k,s)\left(\frac{(\alpha y-1)^{k-s}(\alpha-y)^k + (-1)^k y^s(\alpha y-1)^k(\alpha-y)^{k-s}}{(\alpha y-1)^k(\alpha-y)^k}\right).$$

The coefficient of y^{2k} in the denominator is nonzero while the coefficient of y^{2k} from the numerator is

$$\alpha^k\sum_{s=1}^{k}(-1)^s(s-1)!S(k,s),$$

which is zero by Lemma 6. Thus, the degree of the numerator is less than the degree of the denominator and this rational expression in y is in fact transcendental and algebraically independent from π if it does not vanish. To show nonvanishing, we isolate the coefficient of y^1 which is

$$\alpha^{k-1} + (-1)^{k-1}(\alpha^2+1)\alpha^{k-1}\sum_{s=1}^{k-1}(-1)^s(s-1)!S(k,s) - (k-1)!\alpha^{k-1}.$$

Making use of Lemma 6 we have that

$$\sum_{s=1}^{k-1}(-1)^s(s-1)!S(k,s) = (-1)^{k+1}(k-1)!$$

so that the coefficient of y simplifies to

$$\alpha^{k-1}(1+\alpha^2(k-1)!).$$

This coefficient is not zero for $k \geq 3$ since $|\alpha| = 1$, therefore the coefficient of π^k is not zero, and we have established transcendence. ■

Remark. In the case that D is algebraic, but not positive rational, $e^{\pi\sqrt{D}} = (e^{\pi i})^{-i\sqrt{D}}$ is transcendental by Gel'fond's theorem. In this setting it is not known whether π and $e^{\pi\sqrt{D}}$ are algebraically independent, so we cannot conclude transcendence of the sums (2). In a more general setting, there are conjectures of Gel'fond and Schneider that π and the numbers $\alpha^{\beta}, \dots, \alpha^{\beta^{d-1}}$ are algebraically independent where α, β are algebraic with β having degree d at least 2. If these conjectures are assumed to be true, then we can conclude transcendence of the sum (2) in a more general setting. Details about these conjectures and their implication on these as well as more general infinite series can be found in [**7**] and [**13**].

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100 Years Ago This Month in The American Mathematical Monthly
Edited by Vadim Ponomarenko

The American Mathematical Society held its twenty-second annual meeting in New York, December 27–28, 1915. There were seventy-two members present at the four sessions and thirty-seven papers were presented. The total membership of the Society is now 736. The number of members attending at least one meeting of the Society or its sections during the year 1915 was 253. The total number of registrations at all meetings during the year was 418.

There is a bill before Congress to make the use of the Centigrade thermometer scale obligatory in all government publications, in the hope of bringing about its adoption for all purposes in place of the Fahrenheit scale. This is a move in a good direction, but it raises again the larger question of the metric system as a whole, and we wonder whether the United States will be the last of the civilized nations to adopt that system. An article in *The Scientific Monthly* for December, 1915, by Dr. Joseph V. Collins, of Stevens Point, Wis., discusses the question under the title: "A metrical tragedy," showing that at least two thirds of a year for every child in the land is wasted in the study of our cumbersome system of weights and measures, and that this waste entails an economic loss of possibly three hundred millions of dollars annually.

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