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ON THE TRANSCENDENCE OF CERTAIN INFINITE SERIES

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We investigate the transcendental nature of the sum

$$\sum_{n\in\mathbb{Z}}{}' \frac{A(n)}{B(n)},$$

where $A(x), B(x)$ are polynomials with algebraic coefficients with $\deg A < \deg B$ and the sum is over integers n which are not zeros of $B(x)$. We relate this question to the celebrated conjectures of Gel'fond and Schneider. In certain cases, these conjectures are known, and this allows us to obtain some unconditional results of a general nature.

Keywords: Transcendental numbers; infinite series; Schneider's conjecture; algebraic independence.

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1. Introduction

Let $A(x)$ and $B(x)$ be polynomials in $\overline{\mathbb{Q}}[x]$ with $\deg A < \deg B$ so that $B(x)$ has no integral zeros. We will evaluate the infinite series

$$\sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)}, \tag{1}$$

interpreted as

$$\lim_{N\to\infty} \sum_{|n|\le N} \frac{A(n)}{B(n)}.$$

We seek to determine under what conditions the sum is a transcendental number. One could also allow $B(x)$ to have integral zeros and exclude these integral zeros from the sum (1). Our methods apply to this general setting also. We will relate these questions to a celebrated conjecture of Gel'fond and Schneider.

In 1934, Gel'fond ([4]) and Schneider ([8, 9]) independently solved Hilbert's seventh problem which predicts the following: if α is an algebraic number $\neq 0, 1$ and β is an irrational algebraic number, then α^β is transcendental. Throughout we define log as the principal value of the logarithm with argument in $(-\pi, \pi]$ and define α^β as $e^{\beta \log(\alpha)}$. This result has some interesting consequences. For example, by taking $\alpha = -1$ and $\beta = -i = -\sqrt{-1}$, we deduce the transcendence of e^π. Based on their investigations, Gel'fond and Schneider were led to formulate some general conjectures that provided a concrete goal for researchers in subsequent decades. For instance, Schneider conjectured that if $\alpha \neq 0, 1$ is algebraic and β is an algebraic irrational of degree $d \geq 2$, then

$$\alpha^\beta, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}},$$

are algebraically independent. In 1949, Gel'fond ([5]) proved that if $d \geq 3$, then the transcendence degree of

$$\mathbb{Q}(\alpha^\beta, \ldots, \alpha^{\beta^{d-1}})$$

is at least 2. Thus, in the case $d = 3$, this proves Schneider's conjecture. Building on earlier works of Chudnovsky ([2]) and Philippon ([7]), Diaz ([3]) showed that

$$\text{tr.deg.}\, \mathbb{Q}(\alpha^\beta, \ldots, \alpha^{\beta^{d-1}}) \geq \left[\frac{d+1}{2}\right].$$

Thus, we have crossed the "midway" point in our journey towards Schneider's conjecture.

Shortly after their solution to Hilbert's seventh problem, Gel'fond and Schneider were led to formulate a more general conjecture: if α is algebraic and unequal to $0, 1$, and β is algebraic of degree $d \geq 2$, then

$$\log \alpha, \alpha^\beta, \ldots, \alpha^{\beta^{d-1}}$$

are algebraically independent. We refer to this assertion as the Gel'fond–Schneider conjecture. We point out that, as will be seen, for our purposes α is a root of unity and we use these conjectures only in a special case.

In the study of the transcendence properties of the series (1), the case where the roots of $B(x)$ are rational and non-integral is easy. As will be evident from the discussion below, the sum in this case is equal to $\pi P(\pi)$, where $P(x) \in \overline{\mathbb{Q}}[x]$. Thus, if all the roots are rational and non-integral, the sum (1) is either zero or transcendental. We seek to establish a similar theorem in the general case when $B(x)$ has irrational roots.

Our main theorem is:

Theorem 1. *Let* $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ *with* $\deg A < \deg B$, $A(x)$ *coprime to* $B(x)$, *and* $A(x)$ *not identically zero. Suppose that the roots of* $B(x)$ *are*

$-r_1, \ldots, -r_l \in \mathbb{Q}\backslash\mathbb{Z}$ and $-\alpha_1, \ldots, -\alpha_k \notin \mathbb{Q}$ so that all roots are simple and $\alpha_i \pm \alpha_j \notin \mathbb{Q}$ for $i \neq j$. If $k = 0$ then the series

$$S = \sum_{n \in \mathbb{Z}} \frac{A(n)}{B(n)}$$

is an algebraic multiple of π. If $k \geq 1$, then Schneider's conjecture implies that S/π is transcendental and the Gel'fond–Schneider conjecture implies that S is algebraically independent from π.

The condition that $B(x)$ has only non-integral roots is not a serious constraint. In fact, it can easily be removed in some cases if we understand that we are considering sums

$$\sum_{n \in \mathbb{Z}}{}' \frac{A(n)}{B(n)} \tag{2}$$

where the dash on the sum means that we sum over only those integers n which are not roots of $B(x)$. Many of our theorems are also valid for sums of the form (2). We indicate below how this more general situation is easily handled.

Although the conditions from Theorem 1 are not necessary to conclude transcendence in general, we emphasize that some condition on the roots is necessary. For example, one can show that

$$\sum_{n \in \mathbb{Z}} \frac{2n - 1}{n^2 - n - 1} = 0,$$

even though all the roots of $x^2 - x - 1$ are irrational. Indeed, if ϕ and $1 - \phi$ are the roots of $x^2 - x - 1$, then

$$\sum_{n \in \mathbb{Z}} \frac{2n - 1}{n^2 - n - 1} = \sum_{n \in \mathbb{Z}} \left(\frac{1}{n - \phi} + \frac{1}{n - (1 - \phi)} \right) = 0.$$

Since the Gel'fond–Schneider conjecture is still far away from being established, and we are somewhat "nearer" to the Schneider conjecture, it is reasonable to ask what can be said about S assuming the "weaker" conjecture.

Theorem 2. *Fix nonconstant polynomials $A_1(x), A_2(x), B_1(x), B_2(x) \in \overline{\mathbb{Q}}[x]$ so that $A_i(x)$ has no common factors with $B_i(x)$, $\deg(A_i) < \deg(B_i)$ and the functions $A_1(x)/B_1(x), A_2(x)/B_2(x)$ are not scalar multiples. Write $B(x) = \operatorname{lcm}(B_1(x), B_2(x))$ and suppose that $B(x)$ has only simple irrational roots $-\alpha_1, \ldots, -\alpha_k$ such that $\alpha_i \pm \alpha_j \notin \mathbb{Q}$ for $i \neq j$. If Schneider's conjecture is true, then the quotient*

$$\left(\sum_{n \in \mathbb{Z}} \frac{A_1(n)}{B_1(n)} \right) \Big/ \left(\sum_{n \in \mathbb{Z}} \frac{A_2(n)}{B_2(n)} \right)$$

is transcendental.

A simple corollary of Theorem 2 is that by assuming Schneider's conjecture, along with our condition on the irrational roots of $B(x)$, one can establish the transcendence of S with "at most one exception."

In the above theorems, we restricted ourselves to the case of simple roots. We can also derive results in the case of multiple roots.

Theorem 3. *If the Gel'fond–Schneider conjecture is true, then for any* $A(x), B(x) \in \overline{\mathbb{Q}}[x]$, *with* $\deg(A) < \deg(B)$ *and* $B(n) \neq 0$ *for any* $n \in \mathbb{Z}$, *the series*

$$\sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)}$$

is either zero or transcendental.

Theorem 4. *Let* $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ *with* $\deg(A) < \deg(B)$, *and* $A(x)$ *coprime to* $B(x)$. *Suppose that the roots of* $B(x)$ *are* $-r_1, \ldots, -r_t \in \mathbb{Q}\backslash\mathbb{Z}$ *and* $-\alpha_1, \ldots, -\alpha_k \notin \mathbb{Q}$ *with* $k \geq 1$. *Let* N *be the maximum order of all the irrational roots and suppose that for distinct* α_i, α_j *of order* N, $\alpha_i \pm \alpha_j \notin \mathbb{Q}$. *If the Gel'fond–Schneider conjecture is true then the series*

$$\sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)}$$

and π *are algebraically independent.*

Both the Schneider conjecture and the Gel'fond–Schneider conjecture are special cases of the far-reaching Schanuel conjecture. This conjecture predicts that if the complex numbers $x_1, \ldots, x_n$ are linearly independent over $\mathbb{Q}$, then

$$\text{tr.deg.}\, \mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n.$$

An interesting consequence of this conjecture is that π and e are algebraically independent. If $x_1, \ldots, x_n$ are algebraic numbers, the assertion of the Schanuel conjecture is the celebrated Lindemann–Weierstrass theorem. Beyond this, the general conjecture seems unreachable at present. However, as mentioned in the introduction, progress has been made on the Schneider conjecture and it is this that allows us to make a portion of our results unconditional. To standardize the setting throughout, let $K_1 = \mathbb{Q}(\alpha_1, \ldots, \alpha_k)$, the field generated by the roots of $B(x)$, and let K_2 be K_1 adjoin the coefficients of $A(x)$ and $B(x)$. Restricting ourselves to the case where $B(x)$ has simple roots, the following are unconditional versions of Theorems 1 and 2, respectively.

Theorem 5. *In the same setting as Theorem 1, if* $[K_1 : \mathbb{Q}] = 2$ *or* 3, *then* S/π *is transcendental. If* K_1 *is an imaginary quadratic field, then* S *is algebraically independent from* π.

Theorem 6. *In the same setting as Theorem 2, if* $[K_1 : \mathbb{Q}] = 2$ *or* 3, *then the quotient*

$$\left(\sum_{n\in\mathbb{Z}} \frac{A_1(n)}{B_1(n)}\right) \Bigg/ \left(\sum_{n\in\mathbb{Z}} \frac{A_2(n)}{B_2(n)}\right)$$

is transcendental.

In the case of multiple roots our methods allow us to obtain the following theorem. It can be viewed as a natural generalization of Euler's famous theorem that $\zeta(2k) \in \pi^{2k}\mathbb{Q}$, where $\zeta(s)$ is the Riemann zeta function.

Theorem 7. *Let* $A(x), B(x)$ *be polynomials with algebraic coefficients,* $\deg A < \deg B$, *and* $A(x)$ *is coprime to* $B(x)$. *Let* K_1 *be either an imaginary quadratic field or* $\mathbb{Q}$. *If* $B(x)$ *has no integral roots, then*

$$\sum_{n\in\mathbb{Z}}{}' \frac{A(n)}{B(n)}$$

is either zero or transcendental. If $B(x)$ *has at least one integral root then the sum is either in* K_2 *or transcendental. If* $B(x)$ *has at least one irrational root and all irrational roots satisfy the conditions of Theorem 1 that* $\alpha_i \pm \alpha_j \notin \mathbb{Q}$ *for* $i \neq j$, *then the sum is transcendental.*

There are easily identifiable situations when one can say definitively that the sum is transcendental. For example, as in Theorem 7, if the irrational roots of $B(x)$ satisfy the conditions of Theorem 1 and generate an imaginary quadratic field, then the sum is transcendental. But there are other cases when the conditions of Theorem 1 may not be satisfied and still, one can check directly that the sum is transcendental. A superb example of this phenomenon is the assertion that the sum

$$\sum_{n\in\mathbb{Z}} \frac{1}{An^2 + Bn + C} = \frac{2\pi}{\sqrt{D}} \left(\frac{e^{2\pi\sqrt{D}/A} - 1}{e^{2\pi\sqrt{D}/A} - 2(\cos(\pi B/A))e^{\pi\sqrt{D}/A} + 1} \right)$$

is transcendental if $A, B, C \in \mathbb{Z}$ and $-D = B^2 - 4AC < 0$. We leave the details to the reader and simply point out a theorem of Nesterenko (see [6, Corollary 1.7]) that π and $e^{\pi\sqrt{D}}$ are algebraically independent.

Another illustration is given by a problem investigated by Bundschuh. In 1979, Bundschuh ([1]) studied the series

$$\sum_{|n|\geq 2} \frac{1}{n^k - 1} \tag{3}$$

and showed using Schanuel's conjecture that all of these sums are transcendental numbers for $k \geq 3$. An examination of his proof shows that the "weaker"

Gel'fond–Schneider conjecture is sufficient to deduce his result. As a consequence of our work, we can prove unconditionally:

Theorem 8. *The sum*

$$\sum_{|n|\geq 2} \frac{1}{n^k - 1}$$

is transcendental for $k = 3, 4, 6$.

In particular, at least one of

$$\sum_{n=2}^{\infty} \frac{1}{n^3+1} \quad \text{or} \quad \sum_{n=2}^{\infty} \frac{1}{n^3-1},$$

is transcendental.

Let us note that for $k = 2$, the sum (3) is a telescoping sum equal to $3/2$. For $k = 4$, the sum is equal to

$$\frac{7}{4} - \frac{\pi}{2}\coth \pi,$$

which is transcendental, since by the theorem of Nesterenko, π and e^{π} are algebraically independent. (See also [10, p. 274].) Our result is new in the case $k = 3$ and $k = 6$. In this case, we see that

$$\sum_{|n|\geq 2} \frac{1}{n^3-1} = \sum_{n=2}^{\infty} \frac{1}{n^3-1} - \sum_{n=2}^{\infty} \frac{1}{n^3+1} = 2\sum_{n=2}^{\infty} \frac{1}{n^6-1}.$$

By direct calculation (using the methods of the next section), we find that

$$\sum_{|n|\geq 2} \frac{1}{n^6-1} = \frac{11 + 11e^{\pi\sqrt{3}} + 2\pi\sqrt{3} - 2\pi\sqrt{3}e^{\pi\sqrt{3}}}{6(1+e^{\pi\sqrt{3}})}.$$

By Nesterenko's theorem we know that π and $e^{\pi\sqrt{3}}$ are algebraically independent. Thus the sum in question is transcendental. This proves Theorem 8.

2. Proof of the Main Theorem

Our method is based on two observations. The first is that

$$\pi \cot \pi x = \sum_{n\in\mathbb{Z}} \frac{1}{n+x},$$

which is valid for $x \notin \mathbb{Z}$. Now,

$$\cot \pi x = i\frac{e^{i\pi x} + e^{-i\pi x}}{e^{i\pi x} - e^{-i\pi x}} = i\frac{e^{2\pi i x}+1}{e^{2\pi i x}-1} = i + \frac{2i}{e^{2\pi i x}-1},$$

and this will be useful below. The second is that by the theory of partial fractions, we can write

$$\frac{A(x)}{B(x)} = \sum_{m=1}^{l} \frac{d_m}{x + r_m} + \sum_{j=1}^{k} \frac{c_j}{x + \alpha_j}.$$

Proof of Theorem 1. By direct calculation, our series is equal to

$$i\left(\sum_{j=1}^{k} c_j \frac{e^{2\pi i \alpha_j} + 1}{e^{2\pi i \alpha_j} - 1} + \sum_{m=1}^{l} d_m \frac{e^{2\pi i r_m} + 1}{e^{2\pi i r_m} - 1}\right),$$

where each c_j and d_m is in $\overline{\mathbb{Q}}\backslash\{0\}$. If all of the roots are rational, the first sum is empty and S/π is algebraic which proves the first assertion.

Assume that $B(x)$ has at least one irrational root and suppose the sum, $S/\pi \in \overline{\mathbb{Q}}$. We have

$$\begin{aligned}\frac{S}{\pi} - i\sum_{m=1}^{l} d_m \frac{e^{2\pi i r_m} + 1}{e^{2\pi i r_m} - 1} &= i\sum_{j=1}^{k} c_j \frac{e^{2\pi i \alpha_j} + 1}{e^{2\pi i \alpha_j} - 1} \\ &= i\sum_{j=1}^{k} c_j + 2i\sum_{j=1}^{k} \frac{c_j}{e^{2\pi i \alpha_j} - 1}\end{aligned}$$

so that

$$\sum_{j=1}^{k} \frac{c_j}{e^{2\pi i \alpha_j} - 1} = \frac{1}{2i}\left(\frac{S}{\pi} - i\sum_{m=1}^{l} d_m \frac{e^{2\pi i r_m} + 1}{e^{2\pi i r_m} - 1} - i\sum_{j=1}^{k} c_j\right) = \theta \in \overline{\mathbb{Q}}.$$

By assumption, $[\mathbb{Q}(\alpha_1, \ldots, \alpha_k) : \mathbb{Q}] = d > 1$, so by the theorem of the primitive element, there is a $\beta \in \overline{\mathbb{Q}}$ of degree d such that $\mathbb{Q}(\alpha_1, \ldots, \alpha_k) = \mathbb{Q}(\beta)$. Thus, we have the equations

$$\alpha_j = \sum_{a=0}^{d-1} r_{a,j} \beta^a$$

where each $r_{a,j} \in \mathbb{Q}$. Take any integer $M \in \mathbb{Z}$ such that

$$\alpha_j = \frac{1}{M} \sum_{a=0}^{d-1} n_{a,j} \beta^a$$

where each $n_{a,j} \in \mathbb{Z}$. Let $\alpha = e^{\pi i/M}$. If Schneider's conjecture is true then the numbers

$$\alpha^{\beta}, \ldots, \alpha^{\beta^{d-1}}$$

are algebraically independent, which implies that

$$\alpha^{2\beta}, \ldots, \alpha^{2\beta^{d-1}}$$

are also algebraically independent. Define $x_a := \alpha^{2\beta^a} = e^{2\pi i\beta^a/M}$ for $a = 1, \ldots, d-1$ so that

$$e^{2\pi i\alpha_j} = e^{\frac{2\pi i}{M}\sum_{a=0}^{d-1} n_{a,j}\beta^a} = \gamma_j x_1^{n_{1,j}} \cdots x_{d-1}^{n_{d-1,j}}$$

where $\gamma_j = e^{2\pi i n_{0,j}/M}$ is a root of unity.

Making this substitution we have

$$\theta = \sum_{j=1}^{k} \frac{c_j}{\gamma_j x_1^{n_{1,j}} \cdots x_{d-1}^{n_{d-1,j}} - 1}.$$

This implies that all of the x_i's cancel in some fashion leaving only an algebraic number. We will now show that this does not occur under the conditions of our theorem.

Let us examine the function

$$F(X_1, \ldots, X_{d-1}) = \sum_{j=1}^{k} \frac{c_j}{\gamma_j X_1^{n_{1,j}} \cdots X_{d-1}^{n_{d-1,j}} - 1}.$$

If we can show that F is not constant, then our sum actually contains some variables and we are done. We show that F is not constant by examining F at some special points. Let y be a new indeterminate. For some integral values $e_1, \ldots, e_{d-1}$ to be specified later, let $X_i = y^{e_i}$. We have that

$$F(y^{e_1}, \ldots, y^{e_{d-1}}) = \sum_{j=1}^{k} \frac{c_j}{\gamma_j y^{\overline{n_j}\cdot\overline{e}} - 1}$$

where $\overline{n_j} = (n_{1,j}, \ldots, n_{d-1,j})$ and $\overline{e} = (e_1, \ldots, e_{d-1})$. For any $\overline{n_j} \cdot \overline{e} < 0$, we have

$$\frac{1}{\gamma_j y^{\overline{n_j}\cdot\overline{e}} - 1} = -1 - \frac{1}{\gamma_j^{-1} y^{-\overline{n_j}\cdot\overline{e}} - 1}$$

so that

$$F(y^{e_1}, \ldots, y^{e_{d-1}}) = -\sum_{\overline{n_j}\cdot\overline{e}<0} c_j + \sum_{j=1}^{k} c_j \frac{\operatorname{sgn}(\overline{n_j}\cdot\overline{e})}{\gamma_j^{\operatorname{sgn}(\overline{n_j}\cdot\overline{e})} y^{|\overline{n_j}\cdot\overline{e}|} - 1} \tag{4}$$

where $\operatorname{sgn}(x) = 1$ if $x \geq 0$ and -1 otherwise. If every power of y that appears in the second sum is different and nonzero, then we can group each summand over a common denominator and notice that the degree of the numerator will be less than the degree of the denominator. It is easy to see that, if the function above in (4) (as a function of y) is constant then each $c_j = 0$, which is a contradiction. Hence, if we can guarantee the condition that each $|\overline{n_j} \cdot \overline{e}|$ is different and nonzero, then our function is not constant, and therefore the transcendental part of our original series does not vanish and we are done.

We now specify $\overline{e}$. We wish to choose integers e_i such that $\overline{n_j} \cdot \overline{e} \neq \pm\overline{n_{j'}} \cdot \overline{e}$ for $j \neq j'$. In some cases, we need each $\overline{n_j} \cdot \overline{e} \neq 0$ as well. Thus, we need $\overline{e}$ which simultaneously satisfies

$$(\overline{n_j} \pm \overline{n_{j'}}) \cdot \overline{e} \neq 0$$
$$\overline{n_j} \cdot \overline{e} \neq 0.$$

To find such an $\overline{e}$, we use a lattice point argument. For positive integer D, let $I_D = (0, D]$. Examine the box $B_D = I_D^{d-1}$ which contains a total of D^{d-1} lattice points. We wish to avoid points which satisfy the equations

$$(\overline{n_j} \pm \overline{n_{j'}}) \cdot \overline{e} = 0$$
$$\overline{n_j} \cdot \overline{e} = 0.$$

Our conditions on the irrational roots ensure that $\overline{n_j} \pm \overline{n_{j'}} \neq \overline{0}$ so that none of these equations is trivially satisfied. There are at most D^{d-2} lattice points in B_D which satisfy each equation. We have $2\binom{k}{2}$ equations of the first form, and k equations of the second type, thus, for D large enough, we have at least

$$D^{d-1} - \left(k + 2\binom{k}{2}\right) D^{d-2} > 1$$

lattice points to choose from for $\overline{e}$. Thus, there exists such an $\overline{e}$ which shows that our function F is not constant. This shows that θ, and therefore S/π, is transcendental and we have the second assertion of our theorem. To show the third assertion, we observe that the Gel'fond–Schneider conjecture predicts that the d numbers

$$\log(\alpha), \alpha^{\beta}, \ldots, \alpha^{\beta^{d-1}}$$

are algebraically independent. In our setting, this conjecture implies that π and $x_1, \ldots, x_{d-1}$ are algebraically independent which completes the argument. $\square$

We now indicate how the previous theorem is valid (partially) if we remove the restriction that $B(x)$ has no integral roots and we interpret the sum (2) as omitting the integral zeros of $B(x)$. Indeed, suppose that $-n_1, \ldots, -n_t$ are all the integral roots of $B(x)$. After expanding $A(x)/B(x)$ in partial fractions, we encounter three types of sums:

$$\sum_{n\in\mathbb{Z}}{}' \frac{1}{n+n_i}, \quad \sum_{n\in\mathbb{Z}}{}' \frac{1}{n+r_i} \quad \text{and} \quad \sum_{n\in\mathbb{Z}}{}' \frac{1}{n+\alpha_i}. \tag{5}$$

To see how the first sum of (5) affects our result, we observe that

$$\lim_{N\to\infty} \sum_{|n|\leq N}{}' \frac{1}{n+n_i} = \lim_{N\to\infty} \sum_{|n|\leq N, n\neq -n_i} \frac{1}{n+n_i} - \sum_{j=1, j\neq i}^{t} \frac{1}{n_i - n_j}.$$

The second sum on the right-hand side is rational. The limit of the first sum on the right-hand side is easily seen to be zero.

The second and third sums of (5) are

$$\pi \cot \pi r_i - \sum_{j=1}^{t} \frac{1}{n_j + r_i} \quad \text{and} \quad \pi \cot \pi \alpha_i - \sum_{j=1}^{t} \frac{1}{n_j + \alpha_i}.$$

Since the second sum for each is algebraic, it is clear that when $B(x)$ has at least one integral zero we will obtain a similar conclusion to the last part of Theorem 1 where there are no integral zeroes. More precisely, in the same setting of Theorem 1 with $k \geq 1$, allowing $B(x)$ to possibly have integral roots, the Gel'fond–Schneider conjecture implies that the sum (2) and π are algebraically independent. In order to extend the remaining results of Theorem 1 to include the case where $B(x)$ has integral roots, a careful treatment of the sums coming from rational (not integral) and irrational roots is needed. We suspect that there are many situations in which one can conclude transcendence results with extra conditions placed on the roots of $B(x)$, however we do not discuss that here.

3. Proof of Theorem 2

We first work with the case that $B_1(x)$ and $B_2(x)$ are scalar multiples. Without loss of generality, we can assume that $B_1(x) = B_2(x) = B(x)$. By partial fractions we write

$$\frac{A_1(x)}{B(x)} = \sum_{j=1}^{k} \frac{c_j}{x + \alpha_j}$$

and

$$\frac{A_2(x)}{B(x)} = \sum_{j=1}^{k} \frac{C_j}{x + \alpha_j}$$

for some $c_j, C_j \in \overline{\mathbb{Q}}$. As in the proof of Theorem 1, we have

$$\sum_{n \in \mathbb{Z}} \frac{A_1(n)}{B(n)} = \pi i(\beta_1 + 2\theta_1), \quad \sum_{n \in \mathbb{Z}} \frac{A_2(n)}{B(n)} = \pi i(\beta_2 + 2\theta_2)$$

where

$$\beta_1 = \sum_{j=1}^{k} c_j, \quad \beta_2 = \sum_{j=1}^{k} C_j, \quad \theta_1 = \sum_{j=1}^{k} \frac{c_j}{e^{2\pi i \alpha_j} - 1}, \quad \theta_2 = \sum_{j=1}^{k} \frac{C_j}{e^{2\pi i \alpha_j} - 1}.$$

Theorem 1 implies that θ_1 and θ_2 are transcendental. If the ratio of the two series is algebraic then

$$\sum_{n \in \mathbb{Z}} \frac{A_1(n)}{B(n)} - \lambda \sum_{n \in \mathbb{Z}} \frac{A_2(n)}{B(n)} = 0$$

for some algebraic $\lambda \neq 0$. Thus

$$2(\theta_1 - \lambda\theta_2) = \lambda\beta_2 - \beta_1.$$

We now focus on

$$\theta_1 - \lambda\theta_2 = \sum_{j=1}^{k} \frac{c_j - \lambda C_j}{e^{2\pi i \alpha_j} - 1}.$$

Similar to the proof of Theorem 1, we see that $\theta_1 - \lambda\theta_2$ is algebraic only if $c_j - \lambda C_j = 0$ for each j. This implies that $A_1(x) = \lambda A_2(x)$ which gives a contradiction.

Next we assume that $B_1(x) \neq \alpha B_2(x)$ for any algebraic number α. That is, without loss of generality, $B_2(x)$ has a root R such that $B_1(R) \neq 0$. Suppose that the quotient

$$\left(\sum_{n\in\mathbb{Z}} \frac{A_1(n)}{B_1(n)}\right) \Big/ \left(\sum_{n\in\mathbb{Z}} \frac{A_2(n)}{B_2(n)}\right)$$

is algebraic. Inserting the appropriate missing factors to each numerator, respectively, we have that the quotient

$$\left(\sum_{n\in\mathbb{Z}} \frac{\widetilde{A_1}(n)}{B(n)}\right) \Big/ \left(\sum_{n\in\mathbb{Z}} \frac{\widetilde{A_2}(n)}{B(n)}\right)$$

is algebraic. We see that we are in a situation close to the previous case. We remark that in the previous case, $A_i(x)$ need not be coprime with $B_i(x) = B(x)$. If there were common factors, some of the (say) c_j's would simply be zero and we would still obtain the same contradiction. With this in mind, if the quotient of series is algebraic then according to the previous case, there is a nonzero $\lambda \in \overline{\mathbb{Q}}$ such that

$$\frac{\widetilde{A_1}(x)}{B(x)} = \lambda \frac{\widetilde{A_2}(x)}{B(x)}$$

which simplifies to

$$\frac{A_1(x)}{B_1(x)} = \lambda \frac{A_2(x)}{B_2(x)}.$$

Since R is a pole of the right side but not the left, we have a contradiction and we are done. $\square$

4. Proofs of Theorems 5 and 6

Since Schneider's conjecture is true for $d = 2, 3$ (Gel'fond), we immediately have Theorem 6 and the first part of Theorem 5. To prove the second part of Theorem 5, we invoke a theorem of Nesterenko ([6]): if $K_1 = \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic field with $D > 0$, then π and $e^{\pi\sqrt{D}}$ are algebraically independent. Thus, S is algebraically independent from π.

5. The Case of Multiple Roots

We can relax the restriction that $B(x)$ has only simple roots and still obtain some conditional results.

First, we prove a lemma regarding derivatives of the cotangent function.

Lemma 9. *For $k \geq 2$,*

$$\frac{d^{k-1}}{dx^{k-1}}(\pi\cot(\pi x)) = (2\pi i)^k \left(\frac{A_{k,1}}{e^{2\pi ix}-1} + \cdots + \frac{A_{k,k}}{(e^{2\pi ix}-1)^k}\right)$$

where each $A_{i,j} \in \mathbb{Z}$ with $A_{k,1}, A_{k,k} \neq 0$.

Proof. We have that $\pi\cot(\pi x) = \pi i + 2\pi i/(e^{2\pi ix}-1)$. Differentiating this, we obtain the result for $k = 2$. Assuming that the equality is true for all $k < t$. then by induction we have $A_{t-1,1}, \ldots, A_{t-1,t-1} \in \mathbb{Z}$ with $A_{t-1,1}, A_{t-1,t-1} \neq 0$ such that

$$\frac{d}{dx}\left(\frac{d^{t-2}}{dx^{t-2}}(\pi\cot(\pi x))\right) = (2\pi i)^{t-1}\frac{d}{dx}\left(\frac{A_{t-1,1}}{e^{2\pi ix}-1} + \cdots + \frac{A_{t-1,t-1}}{(e^{2\pi ix}-1)^{t-1}}\right)$$

which equals

$$(2\pi i)^t \left(-A_{t-1,1}\frac{e^{2\pi ix}}{(e^{2\pi ix}-1)^2} - \cdots - (t-1)A_{t-1,t-1}\frac{e^{2\pi ix}}{(e^{2\pi ix}-1)^t}\right).$$

By subtracting and adding 1 from each numerator, we have

$$(2\pi i)^t \left(-A_{t-1,1}\frac{e^{2\pi ix}-1+1}{(e^{2\pi ix}-1)^2} - \cdots - (t-1)A_{t-1,t-1}\frac{e^{2\pi ix}-1+1}{(e^{2\pi ix}-1)^t}\right)$$

which equals

$$(2\pi i)^t \left(-\frac{A_{t-1,1}}{e^{2\pi ix}-1} - \frac{A_{t-1,1}}{(e^{2\pi ix}-1)^2} - \cdots - \frac{(t-1)A_{t-1,t-1}}{(e^{2\pi ix}-1)^{t-1}} - \frac{(t-1)A_{t-1,t-1}}{(e^{2\pi ix}-1)^t}\right)$$

which shows the result. □

Since

$$\sum_{n\in\mathbb{Z}} \frac{1}{n+x} = \pi\cot(\pi x) = \pi i + \frac{2\pi i}{e^{2\pi ix}-1},$$

a consequence of Lemma 9 is that for each $k \geq 2$,

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+x)^k} = \frac{(-1)^{k-1}(2\pi i)^k}{(k-1)!}\left(\frac{A_{k,1}}{e^{2\pi ix}-1} + \cdots + \frac{A_{k,k}}{(e^{2\pi ix}-1)^k}\right) \tag{6}$$

for $A_{k,j}$'s as above.

We can now prove Theorem 3. Before we start the proof, it is useful to remark that if $B(x)$ has only rational (and not integral) roots, then it is not hard to see from the previous lemma that the value of (1) is a polynomial in π with algebraic coefficients and zero constant term. Thus, again the sum is either zero or transcendental. So we can focus on the case of irrational roots. Indeed, if we also allow $n_1, \ldots, n_t$ to be integral roots and understand the sum over $\mathbb{Z}$ excludes these integral roots, we are led to study, as before, sums of three types:

$$\sum_{n\in\mathbb{Z}}{}' \frac{1}{(n+n_i)^k}, \quad \sum_{n\in\mathbb{Z}}{}' \frac{1}{(n+r_i)^k} \quad \text{and} \quad \sum_{n\in\mathbb{Z}}{}' \frac{1}{(n+\alpha_j)^k}. \tag{7}$$

The third sum is

$$\frac{(-1)^{k-1}}{(k-1)!} D^{k-1}(\pi\cot\pi x)\bigg|_{x=\alpha_i} - \sum_{j=1}^{t} \frac{1}{(n_j+\alpha_i)^k},$$

and the last sum is algebraic. A similar comment applies for the middle sum, which turns out to be an algebraic multiple of π^k plus a rational number. Finally, the first sum is easily seen to be a rational multiple of π^k plus a rational number. Thus, in the case that there are integral roots and we sum over those $n \in \mathbb{Z}$ which exclude those roots, we are able to assert the stronger theorem that the series is either given explicitly as an algebraic number, seen as the sum of the remainder terms $\sum_{j=1}^{t}$ above, or is transcendental under the assumption of the Gel'fond–Schneider conjecture.

Proof of Theorem 3. Let $-\alpha_1, \ldots, -\alpha_k \in \overline{\mathbb{Q}}\backslash\mathbb{Z}$ be the roots of $B(x)$ with multiplicities $m_1, \ldots, m_k$, respectively. By partial fractions, we write

$$\frac{A(x)}{B(x)} = \sum_{j=1}^{k}\sum_{l=1}^{m_j} \frac{c_{j,l}}{(x+\alpha_j)^l}.$$

By Lemma 9, we have that $\sum_{n\in\mathbb{Z}} A(n)/B(n)$ is equal to

$$\pi i \sum_{j=1}^{k} c_{j,1} \frac{e^{2\pi i\alpha_j}+1}{e^{2\pi i\alpha_j}-1}$$
$$+ \sum_{j=1}^{k}\sum_{l=2}^{m_j} \frac{c_{j,l}(-1)^{l-1}(2\pi i)^l}{(l-1)!} \left(\frac{A_{l,1}}{e^{2\pi i\alpha_j}-1} + \cdots + \frac{A_{l,l}}{(e^{2\pi i\alpha_j}-1)^l} \right). \tag{8}$$

Viewing this as a polynomial in π (with zero constant term), we analyze the coefficients. By the primitive element theorem, there is an algebraic β of degree d such that $\mathbb{Q}(\beta) = \mathbb{Q}(\alpha_1, \ldots, \alpha_k)$. Thus, as before, we can write each

$$\alpha_j = \frac{1}{M} \sum_{a=0}^{d-1} n_{a,j} \beta^a$$

for some integers $M, n_{a,j}$ so that

$$e^{2\pi i\alpha_j} = \prod_{a=0}^{d-1} e^{2\pi i n_{a,j}\beta^a/M}.$$

Let $\alpha = e^{\pi i/M}$ so that we have that each coefficient of a given power of π in Eq. (8) lies in the field $\overline{\mathbb{Q}}(\alpha^\beta, \ldots, \alpha^{\beta^{d-1}})$. The Gel'fond–Schneider conjecture implies that $\pi, \alpha^\beta, \ldots, \alpha^{\beta^{d-1}}$ are algebraically independent, so the sum is either zero or transcendental. $\square$

We can now prove Theorem 4.

Proof of Theorem 4. The case that $N = 1$ is dealt with in Theorem 1, so assume that $N > 1$. Let $v_1, \ldots, v_t$ and $m_1, \ldots, m_k$ be the orders of the roots, respectively. By partial fractions we have

$$\frac{A(x)}{B(x)} = \sum_{j=1}^{k}\sum_{l=1}^{m_j} \frac{c_{j,l}}{(x+\alpha_j)^l} + \sum_{s=1}^{t}\sum_{u=1}^{v_s} \frac{d_{s,u}}{(x+r_s)^u}$$

for some algebraic numbers $c_{j,l}, d_{s,u}$. By Lemma 9, the series $\sum_{n\in\mathbb{Z}} A(n)/B(n)$ equals

$$\begin{aligned}
&\pi i \sum_{j=1}^{k} c_{j,1}\frac{e^{2\pi i\alpha_j}+1}{e^{2\pi i\alpha_j}-1}\\
&\quad + \sum_{j=1}^{k}\sum_{l=2}^{m_j} \frac{c_{j,l}(-1)^{l-1}(2\pi i)^l}{(l-1)!}\left(\frac{A_{l,1}}{e^{2\pi i\alpha_j}-1} + \cdots + \frac{A_{l,l}}{(e^{2\pi i\alpha_j}-1)^l}\right)\\
&\quad + \pi i \sum_{s=1}^{t} d_{s,1}\frac{e^{2\pi i r_s}+1}{e^{2\pi i r_s}-1}\\
&\quad + \sum_{s=1}^{t}\sum_{u=2}^{v_s} \frac{d_{s,u}(-1)^{u-1}(2\pi i)^u}{(u-1)!}\left(\frac{A_{u,1}}{e^{2\pi i r_s}-1} + \cdots + \frac{A_{u,u}}{(e^{2\pi i r_s}-1)^u}\right).
\end{aligned}$$

We view this sum as a polynomial in π. We examine the coefficient of π^N. Note that the rational roots contribute algebraic numbers to this coefficient so we ignore them for now. We focus on the transcendental portion of this coefficient which comes from the irrational roots part of the above sum. That is, ignoring the common factor of $\frac{(-1)^{N-1}(2i)^N}{(N-1)!}$, we examine

$$\sum_{\mathrm{ord}(\alpha_j)=N} c_{j,N}\left(\frac{A_{N,1}}{e^{2\pi i\alpha_j}-1} + \cdots + \frac{A_{N,N}}{(e^{2\pi i\alpha_j}-1)^N}\right).$$

We proceed similar to the proof of Theorem 1 and let $M, \beta, d, n_{a,j}, \alpha, \gamma_j, x_a, X_a, y$ and $\overline{e}$ be as described there. By showing that there is an $\overline{e}$ so that the function

$$F(y) = \sum_{\mathrm{ord}(\alpha_j)=N} c_{j,N} \left(\frac{A_{N,1}}{\gamma_j y^{\overline{n_j}\cdot\overline{e}} - 1} + \cdots + \frac{A_{N,N}}{(\gamma_j y^{\overline{n_j}\cdot\overline{e}} - 1)^N} \right)$$

$$= \sum_{\mathrm{ord}(\alpha_j)=N} c_{j,N} \left(\frac{A_{N,1}(\gamma_j y^{\overline{n_j}\cdot\overline{e}} - 1)^{N-1} + \cdots + A_{N,N}}{(\gamma_j y^{\overline{n_j}\cdot\overline{e}} - 1)^N} \right)$$

is not constant, we show that the original coefficient of π^N is transcendental. By the remarks made above Eq. (4), we can assume that each $\overline{n_j} \cdot \overline{e}$ is positive (or else we could remove an algebraic number as we see in Eq. (4)). Note that we can choose $\overline{e}$ such that each $\overline{n_j} \cdot \overline{e}$ is distinct and nonzero. Thus, after placing everything over a common denominator, we have a function in y whose numerator has smaller degree than the denominator. If this function is constant (and therefore equal to zero), it is easy to see that this implies that each $c_{j,N}$ is zero which is a contradiction. Thus the coefficient of π^N is transcendental. Write

$$S = \sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)} = C_N\pi^N + \cdots + C_1\pi$$

where each $C_i \in \overline{\mathbb{Q}}(\alpha^\beta, \ldots, \alpha^{\beta^{d-1}})$ and $C_N \notin \overline{\mathbb{Q}}$. Similar to before, the Gel'fond–Schneider conjecture implies algebraic independence of π and the coefficients, C_j, thus S is transcendental. If there were a polynomial

$$P(x, y) = \sum_{i,j} D_{i,j} x^i y^j$$

with integer coefficients that was satisfied by $x = \pi$ and $y = S$, then we have

$$P(\pi, S) = \sum_{i,j} D_{i,j} \pi^i (C_N\pi^N + \cdots + C_1\pi)^j = 0.$$

Viewing this as a polynomial in π of degree q, we have that the coefficient of π^q is

$$\sum_{i+Nj=q} D_{i,j}(C_N)^j = 0.$$

The transcendence of C_N implies that each $D_{i,j}$ with $i + Nj = q$ is zero which in turn implies that $P(x, y)$ is identically 0 and we are done. □

6. Proof of Theorem 7

Suppose first that $K_1 = \mathbb{Q}$ and that $B(x)$ has no integral roots. Using (6), the sum of the series is $\pi P(\pi)$ for some polynomial $P(x) \in \overline{\mathbb{Q}}[x]$. If $P(x)$ is identically zero, the sum is zero. If $P(x)$ is not identically zero, then, the sum is a non-constant polynomial in π and hence transcendental. Suppose now that K_1 is an imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ with $D > 0$ and $B(x)$ has no integral roots. Again using (6) and the identity $\sum_{n\in\mathbb{Z}} \frac{1}{(n+x)} = \pi i \frac{e^{2\pi ix}+1}{e^{2\pi ix}-1}$, our sum is of the form

$$\pi R(\pi, e^{\pi\sqrt{D}/M})$$

where $R(x, y)$ is a rational function with algebraic coefficients which is polynomial in x and M is the same as was defined in the proof of Theorem 1. If $R(x, y)$ is identically zero, the sum is zero. If it is not identically zero, by Nesterenko's theorem, it is transcendental since π and $e^{\pi\sqrt{D}}$ are algebraically independent. This completes the first part of the proof. □

To treat the case that $B(x)$ may have integer roots, we argue as in the earlier sections. In this context, we inject the observation made earlier with the three sums (7) from which it was deduced that the sum in question is of the form

$$\pi P(\pi) + \pi R(\pi, e^{\pi\sqrt{D}/M}) + \text{algebraic number},$$

where the algebraic number lies in the field K_2 being essentially a finite sum of terms of the form

$$\frac{c_{j,l}}{(n_t + \alpha_j)^l}, \quad \frac{d_{s,u}}{(n_t + r_s)^u}, \quad \frac{e_{p,q}}{(n_t - n_p)^q}$$

where n_t is an integral root, α_j is an irrational root, r_s is a rational root, n_p is an integral root not equal to n_t, and $c_{j,l}, d_{s,u}, e_{p,q}$ are the coefficients arising from the partial fractions decomposition of $A(x)/B(x)$. It is clear that the algebraic number is an element of K_2. Thus, if $P(x) + R(x, y) = 0$, then the sum is in K_2, otherwise the sum is transcendental by the earlier argument using Nesterenko's Theorem.

Finally, if the irrational roots of $B(x)$ satisfy the conditions of Theorem 1, then $R(x, y)$ depends on the variable y in which case we can conclude the sum is transcendental.

7. Concluding Remarks

It is possible that in specific cases, where $A(x), B(x)$ are given, one can verify directly that the rational function $F(X_1, \ldots, X_{d-1})$ that occurs in the proof of our main theorem is not constant and therefore is not identically zero. In such cases, the transcendental nature of the series

$$\sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)}$$

can be established using the Gel'fond–Schneider conjecture. The methods developed in this paper give us a general method to sum such series enabling us to conclude something about the transcendental nature of these numbers. Thus, one can conclude that at least one of

$$\sum_{n=1}^{\infty} \frac{A(n)}{B(n)} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{A(-n)}{B(-n)}$$

is transcendental.

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