Forum Math. **14** (2002), 65–83

Forum Mathematicum

Explicit formulas for the pair correlation of zeros of functions in the Selberg class

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(Communicated by Peter Sarnak)

Abstract. For any two functions F and G in the Selberg class we prove explicit formulas which relate sums over pairs of zeros, of the form:

$$\sum_{-T \leq \gamma_F, \gamma_G \leq T} f(\rho_F + \rho_G)$$

to sums over prime powers, of the form:

$$\frac{T}{\pi} \sum_{n \geq 2} \Lambda_F(n) \Lambda_G(n) g(n)$$

where f and g are test functions such that f is the Mellin transform of g. As a consequence we find that the Weak Pair Correlation Conjecture for functions in the Selberg class is essentially equivalent to the Selberg Orthonormality Conjectures.

1991 Mathematics Subject Classification: 11M41.

1 Introduction

In 1989, Selberg [11] defined a general class $\mathcal{S}$ of Dirichlet series that admit analytic continuation, functional equation and an Euler product. Presumably, this class includes all the automorphic L-functions, but this has not been established since we do not yet know the Ramanujan conjecture for GL_n for $n \geq 2$. Maybe $\mathcal{S}$ even coincides with the class of automorphic L-functions in GL_n.

The class $\mathcal{S}$ consists of Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_F(n)/n^s$ satisfying the following axioms:

(i) there exists an integer $m \geq 0$ such that $(s-1)^m F(s)$ is an entire function of finite order;

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* Research supported by a Killam Research Fellowship and the Bankers Trust Company Foundation by a grant to the Institute for Advanced Study.

(ii) F satisfies a functional equation of the type:

$$\Phi(s) = w\bar{\Phi}(1 - s),$$

where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

with $Q > 0$, $\lambda_j > 0$, $\mathrm{Re}(\mu_j) \geq 0$ and $|w| = 1$. (Here, $\bar{f}(s) = \overline{f(\bar{s})}$);

(iii) we have

$$\log F(s) = \sum_{n=1}^{\infty} b_F(n)/n^s,$$

where $b_F(n) = 0$ unless $n = p^m$ with $m \geq 1$ and $b_F(n) \ll n^\theta$ for some $\theta = \theta_F < 1/2$;

(iv) for every $\varepsilon > 0$, $a_F(n) = O(n^\varepsilon)$.

We usually refer to (iv) as the Ramanujan hypothesis. We also use the following notation:

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \Lambda_F(n)/n^s, \quad \Lambda_F(n) = b_F(n) \log n.$$

As pointed out in [4], the functional equation is not uniquely determined in view of the duplication formula of the Γ-function. However, the sum $2\sum_{j=1}^{r} \lambda_j$ is well-defined and we denote it by d_F and call it the degree of F.

A function $F \in \mathscr{S}$ is called primitive if it cannot be written as a non-trivial product of two elements in $\mathscr{S}$. Selberg [11] proved that every element can be written as a product of primitive functions and conjectured that this factorization is unique. This latter conjecture would follow from his two conjectures (a) and (b):

(a) for any primitive function F, we have

$$\sum_{p \leq x} \frac{|a_F(p)|^2}{p} = \log\log x + O(1);$$

(b) for any two distinct primitive functions F and G, we have

$$\sum_{p \leq x} \frac{a_F(p)\overline{a_G(p)}}{p} = O(1),$$

where the summations are over primes $p \leq x$.

Selberg [11] also conjectures that the analogue of the Riemann Hypothesis (GRH, for short) holds for every element of $\mathcal{S}$. In [4], Murty described several surprising consequences of conjectures (a) and (b), most notable being the Artin conjecture about holomorphy of non-abelian L-series attached to Galois representations and the solvable case of the Langlands reciprocity conjecture over the rational number field. One could derive these consequences from slightly weaker formulations of conjectures (a) and (b).

In [8], Murty and Perelli investigate to what extent GRH for elements of $\mathcal{S}$ implies conjectures (a) and (b). They found that if in addition, a pair-correlation conjecture is formulated for elements of $\mathcal{S}$, then conjectures (a) and (b) can be derived from it. In pursuing this idea, they followed Montgomery who was the first to formulate a pair-correlation conjecture for the Riemann zeta function $\zeta(s)$. To do this, Montgomery [3] assumed the Riemann Hypothesis for $\zeta(s)$. The authors in [8] followed this line of thought.

The approach of this paper diverges from theirs. First, it is not necessary to assume any form of the Riemann Hypothesis to formulate a pair-correlation conjecture for elements of $\mathcal{S}$. This is consistent with the viewpoint of Rudnick and Sarnak [10] and Katz and Sarnak [2] who consider higher correlation functions in a very general context. This approach liberates us from any unproved hypothesis. It also allows us to formulate weaker versions of the pair-correlation conjecture that will suffice to deduce some form of the unique factorization conjecture needed to deduce the Artin and Langlands' conjectures. In fact, our weak PC conjecture (see below) is essentially equivalent to the Selberg conjectures.

If F and G in $\mathcal{S}$ are given, it seems natural to conjecture that there is an element $F \otimes G$ in $\mathcal{S}$ with the property that $a_{F\otimes G}(p) = a_F(p)a_G(p)$ for all but finitely many p. All known examples of elements of $\mathcal{S}$ are automorphic L-functions and such a construction is given by the Rankin-Selberg L-function of two automorphic representations in these cases. After having formulated a 'GRH-free' pair-correlation conjecture, we derive below new explicit formulas. A striking consequence of our work is that for any $F, G \in \mathcal{S}$ with $F \otimes G \in \mathcal{S}$ and any α in a certain range the quantity $\mathcal{F}_{F,G}(\alpha)$ which appears in our weak PC conjecture (see below) is unconditionally related to the corresponding quantity $\mathcal{F}_{\zeta, F\otimes G}(\alpha)$ obtained by replacing the pair (F, G) with the pair $(\zeta, F \otimes G)$.

2 Statement of results

By an 'explicit formula' for an element $F \in \mathcal{S}$, one means a relationship of sums over primes of the form

$$\sum_n \Lambda_F(n) f(n)$$

to sums over the zeros of F, of the form

$$\sum_\rho g(\rho),$$

where ρ runs over zeros of F. Here f and g are related by the Fourier transform. Similarly, for any two functions $F, G \in \mathscr{S}$, there are explicit formulas for the pair correlation of zeros of F and G. It turns out that these formulas relate sums over pairs of zeros, of the form:

$$(2.1) \qquad \sum_{-T \le \gamma_F, \gamma_G \le T} f(\rho_F + \rho_G)$$

to sums of the form:

$$(2.2) \qquad \sum_n \Lambda_F(n)\Lambda_G(n)g(n)$$

for certain pairs of weights f and g. For example, assuming GRH in $\mathscr{S}$, such a formula is provided in Proposition 2 of [8]. In this paper we present a GRH-free treatment of the pair correlation of zeros of L functions in the Selberg class $\mathscr{S}$. First, let us extend Montgomery's weight to a function of a complex variable: $w(z) = \dfrac{4}{4 - z^2}$ and set:

$$(2.3) \qquad \mathscr{F}_{F,G}(\alpha) = \frac{\pi}{d_F T \log T} \sum_{-T \le \gamma_F, \gamma_G \le T} T^{\alpha d_F(\rho_F + \rho_G - 1)} w(\rho_F + \rho_G - 1),$$

where $\rho_F = \beta_F + i\gamma_F$ runs over 'non-trivial' zeros of F. Note that if RH holds true for F and G then $\rho_F = \frac{1}{2} + i\gamma_F$, $\rho_G = \frac{1}{2} + i\gamma_G$ and $\mathscr{F}_{F,G}(\alpha)$ coincides with $\mathscr{F}_{F,\bar{G}}(\alpha)$ defined in [8]. We make two pair correlation conjectures. For primitive $F, G \in \mathscr{S}$ define $\delta_{F,G} = 1$ if $F = \overline{G}$ and $\delta_{F,G} = 0$ if $F \neq \overline{G}$.

The strong PC (the GRH "full" PC). *For any primitive $F, G \in \mathscr{S}$, as $T \to \infty$ one has*

$$(2.4) \qquad \mathscr{F}_{F,G}(\alpha) = \begin{cases} \delta_{F,G}|\alpha| + d_G T^{-2|\alpha|d_F} \log T(1 + o(1)) + o(1) & \text{if } |\alpha| \le 1 \\ \delta_{F,G} + o(1) & \text{if } |\alpha| \ge 1 \end{cases}$$

uniformly for α in any bounded interval.

This is the PC from [8] if we assume RH for F and G. This is GRH "full" in the sense that any zero $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2}$ produces in $\mathscr{F}_{F,\bar{F}}(\alpha)$ a term of the form: $\dfrac{T^{\alpha d_F(2\beta - 1)}}{T \log T}$ which goes to infinity as $T \to \infty$ for any large enough α.

The weak PC (the GRH "free" PC). *For any primitive $F, G \in \mathscr{S}$ there exists a constant $c_{F,G} > 0$ such that for any $0 < \alpha < c_{F,G}$ we have $\mathscr{F}_{F,G}(\alpha) = \delta_{F,G}\alpha + o(1)$ as $T \to \infty$.*

From the work of Rudnick and Sarnak [10] it follows that the weak PC holds under the additional assumption that F and G are automorphic. In that case one exploits properties of the Rankin-Selberg convolutions, established by Jacquet, Piatetski-Shapiro and Shalika [1], which are not known for general elements in the Selberg class. Note that if F and G are primitive elements in $\mathscr{S}$ and are automorphic, they have to be attached to irreducible automorphic representations. Conversely, every irreducible automorphic representation should give a primitive function in the Selberg class. This is not known in general. So far it has been proved for GL_1 and GL_2 (see [4] and [6] and the references therein). Although it is much weaker than the strong PC, the weak PC still has important consequences.

Theorem 1. *Assume that the weak PC holds for any primitive* $F, G \in \mathscr{S}$. *Then:*
(i) *one has unique factorization in the Selberg class* $\mathscr{S}$*;*
(ii) *Artin conjecture on the holomorphy of non-abelian L-functions holds true;*
(iii) *Langlands reciprocity conjecture for solvable extensions of* $\mathbb{Q}$ *holds true.*

Theorem 1 is obtained via the following unconditional form of Proposition 2 of [8]. First define

$$\Lambda_F(n,x) = \begin{cases} \Lambda_F(n)\left(\dfrac{n}{x}\right)^{1/2} & n \le x \\ \Lambda_F(n)\left(\dfrac{n}{x}\right)^{3/2} & n > x. \end{cases}$$

Theorem 2. *For any* $F, G \in \mathscr{S}$, *any* $T \ge 2$, *any* $\delta > 0$ *and any* $\delta \le \alpha d_F \le 1 - \delta$ *we have:*

$$\mathscr{F}_{F,G}(\alpha) = \frac{1}{d_F x \log T} \sum_{n=1}^{\infty} \Lambda_F(n,x)\Lambda_G(n,x) + O_{\delta,F,G}(T^{-\delta_1})$$

where $x = T^{\alpha d_F}$ *and* $\delta_1 = \delta \min\{\frac{1}{2}, 1 - \theta_F - \theta_G\}$.

The original orthonormality conjecture of Selberg involves the sum

$$\Psi_{F,G}(x) = \sum_{n \le x} \Lambda_F(n)\Lambda_G(n)$$

instead of the above sum $\sum_{n=1}^{\infty} \Lambda_F(n,x)\Lambda_G(n,x)$. For $\Psi_{F,G}(x)$ too we have an expression in terms of the zeros of F and G. To be precise one has the following:

Theorem 3. *For any* $F, G \in \mathscr{S}$, *any* $T, x \ge 2$ *and any* $\varepsilon > 0$ *we have:*

$$\Psi_{F,G}(x) = \frac{\pi}{T} \sum_{-T \le \gamma_F, \gamma_G \le T} \frac{x^{\rho_F + \rho_G}}{\rho_F + \rho_G} + O_{\varepsilon,F,G}(x^{\theta_F + \theta_G} + x^{2+2\varepsilon} T^{-1} \log^4 T).$$

It is expected that $\theta_F = 0$ for any F in the Selberg class. For such L-functions we obtain the following:

Corollary 1. *For any $F, G \in \mathcal{S}$ with $\theta_F = \theta_G = 0$, any constant $c > 2$, any $x \geq 2$ and any $T \geq x^c$ we have:*

$$\Psi_{F,G}(x) = \frac{\pi}{T} \sum_{-T \leq \gamma_F, \gamma_G \leq T} \frac{x^{\rho_F + \rho_G}}{\rho_F + \rho_G} + O_{c,F,G}(1).$$

Theorems 2 and 3 above give examples of explicit formulas of the type described at the beginning of this section. Here one has some freedom in choosing the weights f and g, obtaining thus more general explicit formulas for the correlation of zeros of a pair of functions in the Selberg class. Restricting to functions $g \in C_c^1(2, \infty)$ we prove the following:

Theorem 4. *For any $F, G \in \mathcal{S}$, any $T \geq 2$, any $\varepsilon > 0$ and any $g \in C_c^1(\mathbb{R})$ with $\operatorname{supp} g \subset (2, \infty)$ one has:*

$$\begin{aligned} &\sum_{-T \leq \gamma_F, \gamma_G \leq T} f(\rho_F + \rho_G) \\ &\quad = \frac{T}{\pi} \sum_{n \geq 2} \Lambda_F(n) \Lambda_G(n) g(n) \\ &\qquad + O_{\varepsilon,F,G}(T \| y^{\theta_F + \theta_G} g'(y) \|_1 + \log^4 T \| y^{2+\varepsilon} g'(y) \|_1) \end{aligned}$$

where f is the Mellin transform of g.

Corollary 2. *For any $F, G \in \mathcal{S}$, any $T, N > 2$, any $\varepsilon > 0$ and any $g \in C_c^1(\mathbb{R})$ with $\operatorname{supp} g \subset (2, N)$ one has:*

$$\begin{aligned} &\sum_{-T \leq \gamma_F, \gamma_G \leq T} f(\rho_F + \rho_G) \\ &\quad = \frac{T}{\pi} \sum_{n \geq 2} \Lambda_F(n) \Lambda_G(n) g(n) + O_{\varepsilon,F,G}((TN^{\theta_F + \theta_G} + N^{2+\varepsilon} \log^4 T) \| g' \|_1) \end{aligned}$$

where f is the Mellin transform of g.

For any $F, G \in \mathcal{S}$, Narayanan [9] defines the tensor product $F \otimes G$ by:

$$F \otimes G(s) = \prod_p H_p(s)$$

where

$$H_p(s) = \exp\left(\sum_{k=1}^{\infty} \frac{k b_F(p^k) b_G(p^k)}{p^{ks}} \right). \tag{2.5}$$

One actually has to be a little bit more flexible here in order to have a chance to obtain an element $F \otimes G \in \mathcal{S}$. We only require here that $H_p(s)$ is given by the above formula for all but finitely many primes p. Then we know from "strong multiplicity one" on Selberg class (see [7]) that there is at most one element in $\mathscr{S}$ with the above property. If such an element exists, we denote it by $F \otimes G$ and call it the tensor product of F and G. By taking logs on both sides in the above equalities and expressing $b_F(p^k), b_G(p^k)$ and $b_{F\otimes G}(p^k)$ in terms of $\Lambda_F(p^k), \Lambda_G(p^k)$ and $\Lambda_{F\otimes G}(p^k)$ we find that

$$\Lambda_F(n)\Lambda_G(n) = \Lambda(n)\Lambda_{F\otimes G}(n) \tag{2.6}$$

for all positive integers n with the possible exception of prime powers $n = p^k$ with p in a finite set of primes which depends on F and G. Then from Corollary 2 we obtain the following:

Corollary 3. *For any $F, G \in \mathscr{S}$ such that $F \otimes G \in \mathscr{S}$, any $T, N > 2$, any $\varepsilon > 0$ and any $g \in C_c^1(\mathbb{R})$ with $\operatorname{supp} g \subset (2, N)$ one has:*

$$\sum_{-T \le \gamma_F, \gamma_G \le T} f(\rho_F + \rho_G)$$

$$= \sum_{-T \le \gamma_\zeta, \gamma_{F\otimes G} \le T} f(\rho_\zeta + \rho_{F\otimes G}) + O_{\varepsilon, F, G}((TN^{\theta_F+\theta_G} + N^{2+\varepsilon}\log^4 T)\|g'\|_1)$$

where f is the Mellin transform of g.

The thrust of Corollary 3 is that for any $F, G \in \mathscr{S}$ with $F \otimes G \in \mathscr{S}$ and any α in a certain range the quantity $\mathscr{F}_{F,G}(\alpha)$ which appears in our weak PC conjecture is unconditionally related to $\mathscr{F}_{\zeta, F\otimes G}(\alpha)$. More precisely, from Theorem 2 we obtain:

Corollary 4. *For any $F, G \in \mathscr{S}$ such that $F \otimes G \in \mathscr{S}$, any $T \ge 2$, any $\delta > 0$ and any $\delta \le \alpha d_F \le 1 - \delta$ we have:*

$$\mathscr{F}_{F,G}(\alpha) = \mathscr{F}_{\zeta, F\otimes G}(\alpha) + O_{\delta, F, G}(T^{-\delta_1})$$

where $x = T^{\alpha d_F}$ and $\delta_1 = \delta \min\{\frac{1}{2}, 1 - \theta_F - \theta_G\}$.

The above results enable us to prove that the weak PC holds under certain assumptions on the tensor product.

Theorem 5. *Let $F, G \in \mathscr{S}$ be primitive and such that $F \otimes G \in \mathscr{S}$ and $F \otimes G$ is entire. Assume that there exists $c = c(F, G) > 0$ such that $F \otimes G(\sigma + it) \ne 0$ for any $\sigma, t \in \mathbb{R}$ with*

$$\sigma > 1 - \frac{c \log\log(|t| + 3)}{\log(|t| + 3)}.$$

Then the weak PC holds true for (F, G).

Acknowledgments. We thank Alberto Perelli and Peter Sarnak for their comments on an earlier version of this manuscript.

3 Generalizing the Landau–Gonek formula

In this section we prove a more general version of Proposition 1 of [8]. For convenience we simply write θ instead of θ_F when it is clear to which function F we refer. We have the following:

Proposition 1. *Let $F \in \mathcal{S}, T, x \geq 2, \varepsilon > 0$ and let n be an integer, closest to x. Then:*

$$(3.1) \qquad \sum_{-T \leq \gamma \leq T} x^{\rho} = -\frac{\Lambda_F(n)}{\pi} \frac{\sin\left(T \log \frac{x}{n}\right)}{\log \frac{x}{n}} + O_{\varepsilon, F}(n^{1+\varepsilon} \log^2 T)$$

$$+ O\left(n^{1+\theta} \sum_{\substack{|n-p^k|<n^{\theta} \\ p<p(\varepsilon,F)}} \frac{1}{|n-p^k|} \right)$$

where $\rho = \beta + i\gamma$ runs over the set of nontrivial zeros of F and $p(\varepsilon, F)$ depends on F and ε only.

Proof. Let R be the rectangle, oriented counterclockwise, with vertices $1 + \frac{\varepsilon}{2} - iT$, $1 + \frac{\varepsilon}{2} + iT, -\frac{\varepsilon}{2} + iT, -\frac{\varepsilon}{2} - iT$. Clearly:

$$(3.2) \qquad \frac{1}{2\pi i} \int_R \frac{F'}{F}(s) x^s \, ds = \sum_{\rho} x^{\rho} - m_F x$$

where ρ runs over the zeros of F inside R. We denote by I_1, I_2, I_3, I_4 the four parts of the integral in (3.2) relative to the sides of R, starting with the right vertical one and proceeding counterclockwise. Here T is chosen in a way such that one has:

$$(3.3) \qquad \left| \frac{F'}{F}(\sigma \pm iT) \right| \ll \log^2 T$$

uniformly for $\sigma \in [-1, 2]$, say. Then we have:

$$(3.4) \qquad |I_2|, |I_4| \ll x^{1+\varepsilon/2} \log^2 T.$$

In order to estimate I_1 we use the Dirichlet series for $\frac{F'}{F}(s)$ to derive:

$$I_1 = -\frac{1}{2\pi}\int_{-T}^{T}\sum_{m=1}^{\infty}\Lambda_F(m)\left(\frac{x}{m}\right)^{1+\varepsilon/2+it}dt$$

$$= \frac{1}{2\pi}\int_{-T}^{T}\Lambda_F(n)\left(\frac{x}{n}\right)^{1+\varepsilon/2+it}dt + O\left(\sum_{m\neq n}|\Lambda_F(m)|\left(\frac{x}{m}\right)^{1+\varepsilon/2}\frac{1}{\left|\log\frac{x}{m}\right|}\right).$$

Here we used the fact that:

$$(3.5)\qquad \sum_{m=1}^{\infty}\frac{|\Lambda_F(m)|}{m^{1+\varepsilon/2}}$$

is convergent. To see this, recall that:

$$(3.6)\qquad |\Lambda_F(m)| \ll m^{\theta},$$

thus for any fixed prime number p the series:

$$\sum_{k=1}^{\infty}\frac{|\Lambda_F(p^k)|}{p^{k(1+\varepsilon/2)}}$$

is bounded by

$$\sum_{k}\frac{1}{p^{k(1-\theta+\varepsilon/2)}}$$

and hence it converges. On the other hand we have (see [5]):

$$(3.7)\qquad |\Lambda_F(m)| \ll_{\varepsilon,F} m^{\varepsilon/4}$$

for any m, $m = p^k$, for which p is larger than a constant $p(\varepsilon, F)$ depending on F and ε only. This proves (3.5). It also proves that the sum:

$$\sum_{1\leq m\leq n/2}|\Lambda_F(m)|\left(\frac{x}{m}\right)^{1+\varepsilon/2}\frac{1}{\left|\log\frac{x}{m}\right|} + \sum_{m\geq 2n}|\Lambda_F(m)|\left(\frac{x}{m}\right)^{1+\varepsilon/2}\frac{1}{\left|\log\frac{x}{m}\right|}$$

being bounded by the sum:

$$\sum_{m=1}^{\infty}|\Lambda_F(m)|\left(\frac{x}{m}\right)^{1+\varepsilon/2}$$

will also be bounded by $n^{1+\varepsilon}$. Therefore, in order to show that the remainder term in the computation of I_1 is

$$O_{\varepsilon,F}(n^{1+\varepsilon}) + O\left(n^{1+\theta} \sum_{\substack{|n-p^k|<n^\theta \\ p<p(\varepsilon,F)}} \frac{1}{|n-p^k|}\right)$$

it remains to prove that:

$$\sum_{\substack{n/2\le m\le 2n \\ m\neq n}} |\Lambda_F(m)| \left(\frac{x}{m}\right)^{1+\varepsilon/2} \frac{1}{\left|\log\frac{x}{m}\right|} \ll_{\varepsilon,F} n^{1+\varepsilon} + n^{1+\theta} \sum_{\substack{|n-p^k|<n^\theta \\ p<p(\varepsilon,F)}} \frac{1}{|n-p^k|}. \tag{3.8}$$

Here the factor $\left(\frac{x}{m}\right)^{1+\varepsilon/2}$ is $O(1)$ while $\frac{1}{\left|\log\frac{x}{m}\right|} \ll \frac{1}{\left|\log\frac{n}{m}\right|}$ is bounded by $\frac{n}{|n-m|}$, thus the LHS of (3.8) is

$$\ll n \sum_{\substack{n/2\le m\le 2n \\ m\neq n}} \frac{|\Lambda_F(m)|}{|n-m|}.$$

Again for $m = p^k, p \ge p(\varepsilon,F)$ we use (3.7) while for $m = p^k, p < p(\varepsilon,F)$ we use (3.6). In this last case we also note that the contribution in the LHS of (3.8) of those terms $m = p^k, p < p(\varepsilon,F)$ for which $|n-p^k| \ge p^\theta$ is bounded by $n\#\{p < p(\varepsilon,F)\}$ which is swallowed in the first term in the RHS of (3.8). This takes care of the remainder term in I_1. The other term in I_1 equals:

$$\frac{1}{2\pi}\Lambda_F(n)\left(\frac{x}{n}\right)^{1+\varepsilon/2} \frac{e^{iT\log(x/n)} - e^{-iT\log(x/n)}}{i\log\frac{x}{n}} = \frac{\Lambda_F(n)}{\pi}\left(\frac{x}{n}\right)^{1+\varepsilon/2} \frac{\sin T\log\frac{x}{n}}{\log\frac{x}{n}}.$$

Here the factor $\left(\frac{x}{n}\right)^{1+\varepsilon/2}$ can be removed since

$$\left|\frac{\Lambda_F(n)}{\pi}\left(\frac{x}{n}\right)^{1+\varepsilon/2} \frac{\sin T\log\frac{x}{n}}{\log\frac{x}{n}} - \frac{\Lambda_F(n)}{\pi}\frac{\sin T\log\frac{x}{n}}{\log\frac{x}{n}}\right|$$

$$= \left|\frac{\Lambda_F(n)}{\pi}\frac{\sin T\log\frac{x}{n}}{\log\frac{x}{n}}\left(\left(\frac{x}{n}\right)^{1+\varepsilon/2} - 1\right)\right| \le \frac{|\Lambda_F(n)|}{\pi}\left|\frac{\left(\frac{x}{n}\right)^{1+\varepsilon/2} - 1}{\log\frac{x}{n}}\right|$$

which is $\ll |\Lambda_F(n)| \ll n^\theta$ because both of $\left|\left(\frac{x}{n}\right)^{1+\varepsilon/2} - 1\right|$ and $\left|\log\frac{x}{n}\right|$ are bounded above and below by $\left|\frac{x-n}{n}\right|$. Finally, in order to estimate I_3 we use the functional equation of $F(s)$ to relate $\frac{F'}{F}\left(-\frac{\varepsilon}{2}+it\right)$ to $\frac{F'}{F}\left(1+\frac{\varepsilon}{2}-it\right)$. The Γ-factors give rise to a term of the form $c_1 \log c_2(|t|+2) + O\left(\frac{1}{|t|+2}\right)$. The contribution of $\frac{F'}{F}\left(1+\frac{\varepsilon}{2}-it\right)$ can be estimated in the same way we estimated I_1. This completes the proof of Proposition 1.

4 Explicit formulas

Before we start to discuss explicit formulas let us remark that the second error term in Proposition 1 will be swallowed in the first error term in any computation which involves averaging the formula (3.1) over any interval of the form $[x, 2x]$. In fact it is enough to average over a shorter interval, of the type $[x, x+x^\theta]$. Indeed, the contribution of any prime $p < p(\varepsilon, F)$ in a sum of the form:

$$(4.1)\qquad \sum_{x \le n \le x+x^\theta} \left(n^{1+\theta} \sum_{\substack{|n-p^k|<n^\theta \\ p<p(\varepsilon,F)}} \frac{1}{|n-p^k|} \right) \ll x^{1+\theta} \sum_{\substack{|x-p^k|<2x^\theta \\ p<p(\varepsilon,F)}} \sum_{\substack{x\le n\le x+x^\theta \\ n\ne p^k}} \frac{1}{|n-p^k|}$$

is bounded by:

$$x^{1+\theta} \sum_{0\ne|m|\le x^\theta} \frac{1}{|m|} \ll x^{1+\theta}\log x$$

thus the LHS of (4.1) is $\ll_\varepsilon x^{1+\theta+\varepsilon}$ which is equal to the corresponding average of the first error term in (3.1). If we now write (3.1) as:

$$|S_F(x) - M_F(x)| \ll E_1(x) + E_2(x)$$

where S_F stands for the sum over the zeros of F, M_F stands for the "main term" in (3.1) while E_1 and E_2 represent the two error terms, then we can take two functions $F, G \in S$ and estimate the product $S_F(x)S_G(x)$ by:

$$\begin{aligned}(4.2)\qquad &|S_F(x)S_G(x) - M_F(x)M_G(x)| \\ &\le (|M_F(x)| + |M_G(x)|)(E_1(x)+E_2(x)) + (E_1(x)+E_2(x))^2 \\ &\le (M_F(x)+M_G(x))(E_1(x)+E_2(x)) + 2(E_1^2(x)+E_2^2(x)).\end{aligned}$$

To get "explicit formulas" we take a test function $h(x)$ which satisfies certain conditions, which allows us to bound successfully the error terms in (4.2). One such condition for example would be that its support is contained in the interval $[2, T^{1-\varepsilon}]$, or at least we should require that the function $h(x)$ is very small outside this interval, otherwise we will have to use Proposition 1 for values of x which are larger than T and then clearly the main term in (3.1) will be smaller than the bound we obtained for the first error term. This being said, we now take any such function h, then we multiply (4.2) by $h(x)$ and finally we integrate it with respect to x, to obtain:

$$(4.3)\qquad \left|\int_2^{\infty} h(x)S_F(x)S_G(x)\,dx - \int_2^{\infty} h(x)M_F(x)M_G(x)\,dx\right|$$
$$\ll \int_2^{\infty}((|M_F(x)|+|M_G(x)|)(E_1(x)+E_2(x))+E_1^2(x)+E_2^2(x))|h(x)|\,dx.$$

Let us choose now

$$h(x) = \begin{cases} \dfrac{1}{x} & \text{if } 2 \le x \le y \\ 0 & \text{else} \end{cases}$$

where y is some number satisfying $T^{\varepsilon} \le y \le T^{1-\varepsilon}$. Then we have on one hand:

$$(4.4)\qquad \int_2^{\infty} h(x)S_F(x)S_G(x)\,dx = \sum_{-T\le\gamma_F,\gamma_G\le T} \frac{y^{\rho_F+\rho_G} - 2^{\rho_F+\rho_G}}{\rho_F+\rho_G}.$$

We now show that on the other hand one has:

$$(4.5)\qquad \int_2^{\infty} h(x)M_F(x)M_G(x)\,dx$$
$$= \frac{T}{\pi}\sum_{m\le y} \Lambda_F(m)\Lambda_G(m) + O_{\varepsilon,F,G}(y^{2+\varepsilon}) + O_{F,G}(y^{\theta_F+\theta_G}T)$$

Indeed, the LHS of (4.5) equals:

$$\int_2^{y} \frac{\Lambda_F(n)\Lambda_G(n)}{\pi^2}\left(\frac{\sin T\log\frac{x}{n}}{\log\frac{x}{n}}\right)^2 \frac{dx}{x}$$

where $n = n(x) = [x+\frac{1}{2}]$. We break this integral in pieces of the form $\int_{m-1/2}^{m+1/2}$, so that on any such interval n is constant, equal to m, and we obtain:

$$\int_{m-1/2}^{m+1/2} = \frac{\Lambda_F(m)\Lambda_G(m)}{\pi^2}\int_{m-1/2}^{m+1/2}\left(\frac{\sin T\log\frac{x}{m}}{\log\frac{x}{m}}\right)^2\frac{dx}{x}.$$

Here we make a change of variable $t = T\log\frac{x}{m}$ and get

$$\int_{m-1/2}^{m+1/2} = \frac{\Lambda_F(m)\Lambda_G(m)}{\pi^2}\,T\int_{T\log(1-1/2m)}^{T\log(1+1/2m)}\frac{\sin^2 t}{t^2}\,dt.$$

Our y is assumed to be smaller than $T^{1-\varepsilon}$ therefore $\left|T\log\left(1+\frac{1}{2m}\right)\right|$ and $\left|T\log\left(1-\frac{1}{2m}\right)\right|$ are large and the last integral will be a good approximation of $\int_{-\infty}^{\infty}\frac{\sin^2 t}{t^2}\,dt = \pi$, more precisely one has:

$$\int_{T\log(1-1/2m)}^{T\log(1+1/2m)}\frac{\sin^2 t}{t^2}\,dt = \pi + O\left(\frac{m}{T}\right).$$

It follows that:

$$\int_{m-1/2}^{m+1/2} = \frac{\Lambda_F(m)\Lambda_G(m)T}{\pi} + O(m|\Lambda_F(m)|\,|\Lambda_G(m)|).$$

We sum this equality for $m = 2, 3, \ldots [y]$ and note that the error which comes from the fact that y is not necessarily a half integer is bounded by

$$\int_{n(y)-1/2}^{n(y)+1/2}\frac{|\Lambda_F(n(y))\Lambda_G(n(y))|}{\pi^2}\left(\frac{\sin T\log\frac{x}{n(y)}}{\log\frac{x}{n(y)}}\right)^2\frac{dx}{x}$$

$$\leq \frac{|\Lambda_F(n(y))\Lambda_G(n(y))|}{\pi}\,T$$

We obtain:

$$\int_2^{\infty} h(x)M_F(x)M_G(x)\,dx$$

$$= \frac{T}{\pi}\sum_{m\leq y}\Lambda_F(m)\Lambda_G(m) + O\left(\sum_{m\leq y} m|\Lambda_F(m)\Lambda_G(m)|\right)$$

$$+ O(|\Lambda_F(n(y))\Lambda_G(n(y))|T).$$

Here the first error term is bounded by

$$y \sum_{m \le y} |\Lambda_F(m)\Lambda_G(m)| \le y \left(\sum_{m \le y} |\Lambda_F(m)|^2 \right)^{1/2} \left(\sum_{m \le y} |\Lambda_G(m)|^2 \right)^{1/2} \ll_{\varepsilon, F, G} y^{2+\varepsilon}$$

by (3.6) and (3.7). The second error term is bounded by $y^{\theta_F + \theta_G} T$ by (3.6), and this proves (4.5). Note that the RHS of (4.3) breaks into $O(\log y)$ integrals of the form $\int_N^{2N}$ and on each such interval $h(x)$ is bounded by $\frac{1}{N}$, hence:

$$\int_N^{2N} \ll \frac{1}{N} \int_N^{2N} \{(|M_F(x)| + |M_G(x)|)(E_1(x) + E_2(x)) + E_1^2(x) + E_2^2(x)\}\, dx$$

The contribution of $E_1^2(x)$ is

$$\text{(4.6)} \qquad \frac{1}{N} \int_N^{2N} x^{2+2\varepsilon} \log^4 T\, dx \ll N^{2+2\varepsilon} \log^4 T$$

while the contribution of $E_2^2(x)$ is bounded as follows:

$$\text{(4.7)} \qquad \frac{1}{N} \int_N^{2N} E_2^2(x)\, dx \ll \frac{1}{N} \int_N^{2N} N^{2+2\theta} \sum_{\substack{p_1,\, p_2 < p(\varepsilon, F, G) \\ 1 \le |x - p_1^{k_1}|,\, |x - p_2^{k_2}| < x^\theta}} \frac{1}{|x - p_1^{k_1}|} \frac{1}{|x - p_2^{k_2}|}\, dx$$

where we set $\theta = \max\{\theta_F, \theta_G\}$ and $p(\varepsilon, F, G) = \max\{p(\varepsilon, F), p(\varepsilon, G)\}$. The RHS of (4.7) is bounded by

$$N^{1+2\theta} p(\varepsilon, F, G)^2 \max_{p_1 p_2 < p(\varepsilon, F, G)} \int_N^{2N} \frac{1}{\max\{1, |x - p_1^k|\}} \frac{1}{\max\{1, |x - p_2^k|\}}\, dx$$

which is $\ll_{\varepsilon, F, G} N^{1+2\theta}$. This is smaller than the RHS of (4.6) since $\theta < \frac{1}{2}$. Finally, we have:

$$\text{(4.8)} \qquad \frac{1}{N} \int_N^{2N} (|M_F(x)| + |M_G(x)|)(E_1(x) + E_2(x))\, dx$$

$$\ll \frac{1}{N} \left(\int_N^{2N} |M_F^2(x)| + |M_G^2(x)|\, dx \right)^{1/2} \left(\int_N^{2N} |E_1^2(x)| + |E_2^2(x)|\, dx \right)^{1/2}$$

The last factor in the RHS of (4.8) is $\ll_{\varepsilon, F, G} N^{3/2+\varepsilon} \log^2 T$ by (4.6). On the other hand

$$\int_N^{2N} |M_F^2(x)|\,dx = \int_N^{2N} \frac{|\Lambda_F(n(x))|^2}{\pi^2} \left(\frac{\sin T \log \dfrac{x}{n(x)}}{\log \dfrac{x}{n(x)}} \right)^2 \frac{dx}{x}$$

$$\leq \sum_{N \leq n \leq 2N} \frac{|\Lambda_F(n)|^2 T}{\pi} \ll_{\varepsilon, F} N^{1+\varepsilon} T$$

by (3.6) and (3.7), and similarly for $\int_N^{2N} |M_G^2(x)|\,dx$. In conclusion, the RHS of (4.8) is $\ll_{\varepsilon,F,G} N^{1+\varepsilon} T^{1/2} \log^2 T$ and so the RHS of (4.3) (for this particular function h) is $\ll_{\varepsilon,F,G} (y^{1+\varepsilon} T^{1/2} + y^{2+2\varepsilon} \log^4 T)$. This is clearly bounded by $y^{\theta_F+\theta_G} T + y^{2+2\varepsilon} \log^4 T$ which is essentially what we have on the RHS of (4.5). On combining this with (4.4) and (4.5) we obtain:

$$\text{(4.9)} \qquad \sum_{-T \leq \gamma_F, \gamma_G \leq T} \frac{y^{\rho_F+\rho_G}}{\rho_F+\rho_G}$$
$$= \frac{T}{\pi} \sum_{m \leq y} \Lambda_F(m) \Lambda_G(m) + O_{\varepsilon,F,G}(y^{\theta_F+\theta_G} T + y^{2+2\varepsilon} \log^4 T).$$

This is precisely the statement of Theorem 3, hence this theorem is now proved.

5 Proof of Theorem 4

We first note that the error term in (4.9) is smaller than yT (which is the order of magnitude of the main term, at least in case $F = \bar{G}$), provided we take $y < T^{1-3\varepsilon}$. We now take any test function $h : [2, \infty) \to \mathbb{C}$, multiply (4.9) by $h(y)$ and integrate with respect to y, to obtain “explicit formulas” for the correlation of zeros of F and G. One has:

$$\text{(5.1)} \qquad \sum_{-T \leq \gamma_F, \gamma_G \leq T} \int_2^\infty \frac{h(y) y^{\rho_F+\rho_G}}{\rho_F+\rho_G}\,dy = \frac{T}{\pi} \int_2^\infty \psi_{F,G}(y) h(y)\,dy$$
$$+ O_{\varepsilon,F,G}\left(T \int_2^\infty y^{\theta_F+\theta_G} |h(y)|\,dy \right)$$
$$+ O_{\varepsilon,F,G}\left(\log^4 T \int_2^\infty y^{2+2\varepsilon} |h(y)|\,dy \right).$$

We now apply (5.1) in the following context. We start with a function $g \in C_c^1(\mathbb{R})$ with $\operatorname{supp} g \subset (2, \infty)$ and set $h(y) = -g'(y)$ for any $y \in \mathbb{R}$. Then $g(n) = \int_n^\infty h(y)\,dy$ for all n, from which we derive

$$\sum_{n \geq 2} \Lambda_F(n)\Lambda_G(n)g(n) = \sum_{n \geq 2} \Lambda_F(n)\Lambda_G(n) \int_n^\infty h(y)\,dy$$

$$= \int_2^\infty h(y) \sum_{2 \leq n \leq y} \Lambda_F(n)\Lambda_G(n)\,dy = \int_2^\infty \psi_{F,G}(y)h(y)\,dy.$$

On the other hand, if we define f by

$$(5.2) \qquad f(s) = \frac{1}{s}\int_2^\infty h(y)y^s\,dy$$

then we will have

$$\sum_{-T \leq \gamma_F, \gamma_G \leq T} f(\rho_F + \rho_G) = \sum_{-T \leq \gamma_F, \gamma_G \leq T} \int_2^\infty \frac{h(y)\,y^{\rho_F + \rho_G}}{\rho_F + \rho_G}\,dy.$$

Using the above relations we eliminate h from (5.1), which we write in terms of f and g:

$$\sum_{-T \leq \gamma_F, \gamma_G \leq T} f(\rho_F + \rho_G) = \frac{T}{\pi} \sum_{n \geq 2} \Lambda_F(n)\Lambda_G(n)g(n)$$

$$+ O_{\varepsilon, F, G}(T\|y^{\theta_F + \theta_G} g'(y)\|_1 + \log^4 T\|y^{2+\varepsilon}g'(y)\|_1).$$

This is precisely the relation from the statement of Theorem 4. Finally, we obtain f in terms of g via their expressions in terms of h. Integrating by parts on the RHS of (5.2) we obtain:

$$f(s) = \frac{1}{s}\int_2^\infty h(y)y^s\,dy = -\frac{1}{s}\int_2^\infty g'(y)y^s\,dy = \int_2^\infty g(y)y^{s-1}\,dy.$$

Thus f is the Mellin transform of g, which completes the proof of Theorem 4.

6 Proof of Theorem 2

We take a number $x \in [T^\delta, T^{1-\delta}]$ and consider the test function h given by:

$$h(y) = \begin{cases} -\dfrac{1}{x} & \text{if } y \leq x \\[2ex] \dfrac{3x^3}{y^4} & \text{if } y > x. \end{cases}$$

Then by the relation (31) of [8], the main term in the RHS of (5.1) equals

$$\frac{T}{\pi}\sum_{n=1}^{\infty}\Lambda_F(n,x)\Lambda_G(n,x)$$

where

$$\Lambda_F(n,x)=\begin{cases}\Lambda_F(n)\left(\dfrac{n}{x}\right)^{1/2} & \text{if } n\le x\\[2ex] \Lambda_F(n)\left(\dfrac{x}{n}\right)^{3/2} & \text{if } n>x\end{cases}$$

On the other hand, using (1) of [8] the LHS of (5.1) equals:

$$\sum_{-T\le\gamma_F,\gamma_G\le T}\left(-\frac{1}{x}\int_1^x\frac{y^{\rho_F+\rho_G}}{\rho_F+\rho_G}\,dy+3x^3\int_x^\infty\frac{y^{\rho_F+\rho_G-4}}{\rho_F+\rho_G}\,dy\right)$$

$$=-\sum_{-T\le\gamma_F,\gamma_G\le T}\frac{x^{\rho_F+\rho_G}}{\rho_F+\rho_G}\left(\frac{1}{\rho_F+\rho_G+1}+\frac{3}{\rho_F+\rho_G-3}\right)$$

$$+\sum_{-T\le\gamma_F,\gamma_G\le T}\frac{1}{(\rho_F+\rho_G)(\rho_F+\rho_G+1)}$$

$$=-4\sum_{-T\le\gamma_F,\gamma_G\le T}\frac{x^{\rho_F+\rho_G}}{(\rho_F+\rho_G+1)(\rho_F+\rho_G-3)}+O_{F,G}(T\log^2 T)$$

$$=\sum_{-T\le\gamma_F,\gamma_G\le T}x^{\rho_F+\rho_G}w(\rho_F+\rho_G-1)+O_{F,G}(T\log^2 T).$$

Now the remainder terms in the RHS of (5.1) are bounded as follows:

$$T\int_2^\infty y^{\theta_F+\theta_G}|h(y)|\,dy$$

$$=\frac{T}{x}\int_2^\infty y^{\theta_F+\theta_G}\,dy+3Tx^3\int_x^\infty y^{\theta_F+\theta_G-4}\,dy$$

$$=T\frac{x^{\theta_F+\theta_G}}{\theta_F+\theta_G+1}-\frac{T}{x}\frac{2^{\theta_F+\theta_G}}{\theta_F+\theta_G+1}+\frac{3Tx^{\theta_F+\theta_G}}{3-\theta_F-\theta_G}\ll Tx^{\theta_F+\theta_G},$$

$$(\log^4 T)\int_2^\infty y^{2+2\varepsilon}|h(y)|\,dy$$

$$=\frac{\log^4 T}{x}\int_2^x y^{2+2\varepsilon}\,dy+x^3(\log^4 T)\int_x^\infty y^{2\varepsilon-2}\,dy\ll(\log^4 T)x^{2+2\varepsilon}$$

and

$$T^{1/2}\int_2^\infty y^{1+\varepsilon}|h(y)|\,dy = \frac{T^{1/2}}{x}\int_2^x y^{1+\varepsilon}\,dy + 3T^{1/2}x^3\int_x^\infty y^{\varepsilon-3}\,dy \ll T^{1/2}x^{1+\varepsilon}.$$

For a fixed $\delta > 0$ and $\varepsilon > 0$ small enough in terms of δ all the above error terms will be bounded by $xT^{1-\delta_1}$ where $\delta_1 = \delta\min\{\frac{1}{2}, 1-\theta_F-\theta_G\}$. This completes the proof of Theorem 2.

7 Concluding remarks

1. Theorem 1 can be derived from Theorem 2 by using the method employed by Murty in [4].

2. In order to prove Theorem 5 one uses standard arguments from analytic number theory familiar from the prime number theorem to control the sum:

$$\sum_{n=1}^\infty \Lambda(n,x)\Lambda_{F\otimes G}(n,x). \tag{7.1}$$

By (2.6) this sum equals

$$\sum_{n=1}^\infty \Lambda_F(n,x)\Lambda_G(n,x) \tag{7.2}$$

and the conclusion of Theorem 5 follows from Theorem 2.

3. In Corollary 3 one doesn't need to assume that $F\otimes G$ satisfies all the axioms from the definition of the Selberg class, to be precise the assumption $\theta_{F\otimes G} < \frac{1}{2}$ is not needed here.

4. Much of this work can be generalized and similar results can be obtained for the correlation of zeros of a k-tuple $(F_1,\dots,F_k)$ of functions in the Selberg class. Thus one may define $\mathscr{F}_{F_1,\dots,F_k}$ in the same way as in the case $k=2$, and then prove a result similar to Theorem 2. We point out however that in the error terms of such results an exponent $\theta_{F_1}+\cdots+\theta_{F_k}$ will appear. For k-tuples for which $\theta_{F_1}+\cdots+\theta_{F_k}\geq 1$, if there are any such k-tuples, this prevents us from obtaining asymptotic results. Clearly this phenomenon does not appear in case $k=2$. Finally, the same remark applies to the tensor product $F_1\otimes\cdots\otimes F_k$. In practice we have $\theta_{F_i}=0$. This should not prevent one from deriving interesting results.

5. Regardless of whether the weak PC holds true or not for a given pair of primitive functions F, G in the Selberg class, Theorem 2 shows that the pair correlation of F and G is unchanged under some vertical shifts of the interval from which the zeros are selected. More precisely one has the following result, valid for any $F, G\in\mathscr{S}$ not necessarily primitive.

Theorem 6. *For any $F, G\in\mathscr{S}$, any $T\geq 2$, any $\delta>0$, any $x\in[T^\delta, T^{1-\delta}]$ and any $T_1, T_2\geq 0$ with $T_1+T_2=T$ one has:*

$$\sum_{-T_1 \le \gamma_F, \gamma_G \le T_1} x^{\rho_F+\rho_G-1} w(\rho_F+\rho_G-1) + \sum_{-T_2 \le \gamma_F, \gamma_G \le T_2} x^{\rho_F+\rho_G-1} w(\rho_F+\rho_G-1)$$

$$= \sum_{-T \le \gamma_F, \gamma_G \le T} x^{\rho_F+\rho_G-1} w(\rho_F+\rho_G-1) + O_{\delta,F,G}(T^{1-\delta'})$$

for some $\delta' > 0$ *which depends only on* F, G *and* δ.

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Received in revised form April 16, 2000

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