MODULAR FORMS AND THE CHEBOTAREV DENSITY THEOREM

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Introduction. Let $N \geq 1$, $k \geq 2$ be integers and let f be a cusp form of weight k for $\Gamma_0(N)$, which is a normalized eigenform for all the Hecke operators T_p, $p \nmid N$. Let us write

$$f(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$$

for its Fourier expansion at $i\infty$, and for simplicity of discussion, let us suppose that the a_n are rational integers. For each integral value of a, set

$$\pi_{f,a}(x) = \#\{p \leq x : a_p = a\}.$$

If $a = 0$ and f is of CM-type (in the sense of Ribet [11]), we know that $\pi_{f,a}(x) \sim \pi(x)/2$. In the remaining cases, (i.e. f is not of CM type or $a \neq 0$), Lang and Trotter [5] conjecture that

$$\pi_{f,a}(x) \sim c_{f,a} \begin{cases} x^{1/2}/\log x & \text{if} \quad k = 2 \\ \operatorname{loglog} x & \text{if} \quad k = 3 \\ 1 & \text{if} \quad k \geq 4 \end{cases}$$

where $c_{f,a}$ is a constant which is generally (though not always) nonzero ($c_{f,a}$ can be zero for example if $a = 1$, $k = 2$ and f corresponds to an elliptic curve having a nontrivial $\mathbf{Q}$-rational torsion point). Moreover, Atkin and Serre [15] conjecture that for $k \geq 4$,

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$$|a_p| \gg_\epsilon p^{((k-3)/2)-\epsilon}$$

for every $\epsilon > 0$. This is known to hold in the CM-case (cf. Serre [15]).

From now on, we shall assume that f is not of CM-type. Assuming the Riemann Hypothesis for all Artin L-functions (GRH), Serre [16] has recently shown that

$$\pi_{f,a}(x) \ll \begin{cases} x^{7/8}(\log x)^{-1/2} & \text{if} \quad a \neq 0 \\ x^{3/4} & \text{if} \quad a = 0. \end{cases}$$

(Note that there is a misprint in the statement of [16, p. 174, Corollary 1]). His main tools were the Chebotarev density theorem in the effective form given by Lagarias and Odlyzko [3] and the ℓ-adic representations attached to f by Deligne [2].

One of our main results here is the following (Section 4).

THEOREM. *Suppose that GRH holds. Then*

$$\pi_{f,a}(x) \ll \begin{cases} x^{4/5}(\log x)^{-1/5} & \text{if} \quad a \neq 0 \\ x^{3/4} & \text{if} \quad a = 0. \end{cases}$$

We then apply this in Section 5 to the Atkin-Serre conjecture to show the following.

THEOREM. *Suppose that GRH holds. Then for any* $\epsilon > 0$,

$$|a_p| \geq p^{(1/4)-\epsilon}$$

holds for a set of primes p *of density* 1. *Unconditionally, there is a constant* $c > 0$ *such that*

$$|a_p| \geq (\log p)^c$$

holds for a set of primes p *of density* 1.

We use this, together with results obtained by methods of transcendental number theory [10] and a sieve-theoretic argument to deduce the following.

THEOREM. *Suppose the GRH holds. There is a constant* $c > 0$ *such that the set*

$$\{n : a_n = 0 \text{ or } |a_n| > n^c\}$$

has density 1. *Unconditionally, the set*

$$\{n : a_n = 0 \text{ or } |a_n| > (\log n)^c\}$$

has density 1.

This result should be compared with Serre's observations on "non-lacunarity" of L-series attached to modular forms (cf. [16, Section 6]).

To prove these results, we start as in Serre [16] with the ℓ-adic representations

$$\rho_{\ell,f} : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathbf{Z}_\ell)$$

of Deligne [2]. We know that if $p \nmid \ell N$, then $\rho_{\ell,f}$ is unramified at p, and $\rho_{\ell,f}(\sigma_p)$ has characteristic polynomial $x^2 - a_p x + p^{k-1}$. (Here, σ_p is a Frobenius element at p). Reducing this representation mod ℓ^n produces a finite extension $K_{\ell^n}/\mathbf{Q}$, namely the fixed field of the kernel of reduction mod ℓ^n. It is essentially these extensions to which the effective version of the Chebotarev density theorem is applied. Our deviation from the method of Serre starts with the observation (Section 3) that "on the average", the Lagarias-Odlyzko estimate can be significantly improved, if we assume Artin's conjecture on the holomorphy of Artin L-series at points $s \neq 1$. We then use ideas similar to Serre [16, Section 2.7] to reduce our problem to a case where we know Artin's conjecture to be true. The observation about averaging is well-suited for the estimation of $\pi_{f,a}(x)$ since in this problem, we have to sum over a relatively large number of conjugacy classes.

Finally, in Section 6, we apply our results to obtaining lower bounds for the largest prime divisor $P(a_p)$ (resp. $P(a_n)$) of a_p (resp. a_n). We show that for each $\epsilon > 0$,

$$P(a_p) \geq \exp((\log\log p)^{1-\epsilon})$$

holds for a set of primes p of density 1. A similar result is also obtained for $P(a_n)$. Of course, these can be improved if we assume *GRH*. In particular, $\log\log p$ can be replaced by $\log p$ in the above inequality.

The first two sections contain notations and preliminary observations. All implied constants in O-symbols are absolute and effective, unless otherwise specified. The letters p and ℓ will always be reserved for primes.

1. Group theoretic preliminaries. We collect some notation and facts about finite groups and their irreducible characters.

1.1. Let G be a finite group and C a conjugacy class. Let δ_C denote the characteristic function of C and $g_C \in G$ an element of C. By the orthogonality relations for characters,

$$\delta_C = \frac{|C|}{|G|} \sum_{\chi} \overline{\chi(g_C)}\chi$$

where the sum is over all irreducible characters χ of G. For an element $g \in G$, we denote by $C_G(g)$ its conjugacy class in G.

1.2. Let H be a subgroup of G, and s an element of H. Let $C_H = C_H(s)$ and $C = C_G(s)$. Let $\delta_{C_H} : H \to \{0, 1\}$ denote the characteristic function of C_H. We use the same letter to denote the extension by zero of δ_{C_H} to all of G. Now set

$$\varphi = \mathrm{Ind}_H^G \, \delta_{C_H}.$$

From the definition of induction, we see that $\varphi(g) = 0$ if $g \notin C$ and so $\varphi = \lambda\delta_C$. The value of λ is easily computed by Frobenius reciprocity:

$$\lambda \cdot \frac{|C|}{|G|} = \langle \varphi, 1_G \rangle = \langle \delta_{C_H}, 1_H \rangle = \frac{|C_H|}{|H|}.$$

Thus $\lambda = |C_H| \cdot |G| \, |H|^{-1} \, |C|^{-1}$.

1.3. Let $\mathbf{C}(G)$ denote the space of complex-valued class functions on G and let π be a linear function $\pi : \mathbf{C}(G) \to \mathbf{C}$.

PROPOSITION.

$$\sum \frac{1}{|C|} \left| \pi(\delta_C) - \frac{|C|}{|G|} \pi(1_G) \right|^2 = \frac{1}{|G|} \sum_{\chi \neq 1_G} |\pi(\chi)|^2$$

where the sum on the left is over conjugacy classes C of G, and the sum on the right is over the irreducible nonidentity characters.

Proof. By linearity

$$\pi(\delta_C) = \frac{|C|}{|G|} \sum_{\chi} \overline{\chi(g_C)}\pi(\chi).$$

Hence,

$$\pi(\delta_C) - \frac{|C|}{|G|}\pi(1_G) = \frac{|C|}{|G|} \sum_{\chi \neq 1_G} \overline{\chi(g_C)}\pi(\chi),$$

and

$$\sum_C \frac{1}{|C|} \left| \pi(\delta_C) - \frac{|C|}{|G|}\pi(1_G)\right|^2$$

$$= \frac{1}{|G|} \sum_{\chi_1,\chi_2 \neq 1_G} \pi(\chi_1)\overline{\pi(\chi_2)} \frac{1}{|G|} \sum_C |C| \overline{\chi_1(g_C)}\chi_2(g_C)$$

$$= \frac{1}{|G|} \sum_{\chi \neq 1_G} |\pi(\chi)|^2$$

using the orthogonality relations.

1.4. We recall some observations of Serre [16, pp. 136–140] which will be used in Section 3 and Section 4. Let L/K be a Galois extension of number fields, with group G. For each place w of L, let D_w and I_w denote the decomposition and inertia group at w (respectively). For each place v of k, let $\sigma_v \in D_w/I_w$ denote the Frobenius element at v. Let φ be a class function on G, and define

$$\tilde{\pi}_\varphi(x) = \sum_{Nv^m \leq x} \frac{1}{m}\varphi(\sigma_v^m).$$

(If v is ramified in L and w/v, $\varphi(\sigma_v^m)$ denotes the sum $|I_w|^{-1} \Sigma\, \varphi(g)$ over elements $g \in D_w$ whose image in D_w/I_w is σ_v^m).

We also define

$$\pi_\varphi(x) = \sum_{Nv \leq x} \varphi(\sigma_v)$$

where the sum is over places v of K unramified in L and where N denotes $\mathrm{Norm}_{K/\mathbf{Q}}$. If C is a conjugacy class and δ is its characteristic function, we shall often write $\pi_C(x)$ (resp. $\tilde{\pi}_C(x)$) for $\pi_\delta(x)$ (resp. $\tilde{\pi}_\delta(x)$). Let H be a subgroup of G and φ_H a class function on H. Let $\varphi = \mathrm{Ind}_H^G \varphi_H$. From the inductive property of L-functions, it follows that

$$\tilde{\pi}_\varphi(x) = \tilde{\pi}_{\varphi_H}(x).$$

If H is normal, let $\varphi_{G/H}$ be a class function on G/H, and let φ denote its pullback to G. Then

$$\tilde{\pi}_\varphi(x) = \tilde{\pi}_{\varphi_{G/H}}(x).$$

In particular, if $g \in G$ and $Hg \subseteq C_G(g)$ (= the conjugacy class of g in G), then the above applies with φ the characteristic function of $C_G(g)$ and $\varphi_{G/H}$ the characteristic function of the class of Hg in G/H.

Finally, let φ again be any class function on G, and let d_L denote the absolute value of the discriminant of L. Then

$$\pi_\varphi(x) = \tilde{\pi}_\varphi(x) + O\left(\|\varphi\| \left\{\frac{1}{|G|} \log d_L + [K:\mathbf{Q}]x^{1/2}\right\}\right)$$

where $\|\varphi\| = \sup_{s\in G}|\varphi(s)|$.

2. Growth of conductors and discriminants.

2.1. Let L/K be a finite extension of number fields, $\mathfrak{D}_{L/K}$ its different, and $\delta_{L/K}$ its discriminant. Thus, $\mathfrak{D}_{L/K}$ is an integral ideal of L and $\delta_{L/K} = \mathrm{Norm}_{L/K}(\mathfrak{D}_{L/K})$. Let v be a place of K and w a place of L dividing v. Let p_v denote the residue characteristic at v. The following estimate of Hensel is quite useful (see for example, Math. Ann. 55 (1902) 301–336).

LEMMA. *$w(\mathfrak{D}_{L/K}) = e_{w/v} - 1 + s_{w/v}$ where $e_{w/v}$ denotes the ramification index in L of the ideal $\mathfrak{p}_v$ of K attached to v, and $0 \le s_{w/v} \le w(e_{w/v})$.*

2.2. Hensel's lemma can be used to get an estimate for $\mathrm{Norm}_{K/\mathbf{Q}}(\delta_{L/K})$. Let $n_L = [L:\mathbf{Q}]$, $n_K = [K:\mathbf{Q}]$, $n = n_L/n_K = [L:K]$, and set $P(L/K) = \{p$: there is a prime $\mathfrak{p}$ of K with $\mathfrak{p}\,|\,p$ and $\mathfrak{p}$ is ramified in $L\}$. The following is [16, p. 128, Proposition 4].

PROPOSITION. Log $\mathrm{Norm}_{K/\mathbf{Q}}(\delta_{L/K}) \leq (n_L - n_K) \Sigma_{p \in P(L/K)} \log p + n_L(\log n)|P(L/K)|$. *Using this and the relation* $d_L = d_K^n \mathrm{Norm}_{K/\mathbf{Q}}(\delta_{L/K})$, *where* d_K *is the absolute value of the discriminant of* $K/\mathbf{Q}$ *and* $n = [L : K]$, *we get a bound for* $\log d_L$ *also*.

2.3. Now we assume that L/K is Galois with group G. In this case, the estimates of 2.2 can be slightly improved. The following is [16, p. 128, Proposition 5].

PROPOSITION. Log $\mathrm{Norm}_{K/\mathbf{Q}}(\delta_{L/K}) \leq (n_L - n_K) \Sigma_{p \in P(L/K)} \log p + n_L \log n$.

2.4. We now discuss an analogue of Proposition (2.3) for Artin conductors. Let χ be an irreducible character of G. We begin by recalling the definition of the Artin conductor $\mathfrak{F}_\chi$ of χ. For each finite place v of K, let w be a place of L dividing v and let G_i denote the i-th ramification group at w ($i \geq 0$). Thus, G_0 is the inertia group and we have a descending filtration

$$G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots .$$

Let V be a $\mathbf{C}$-vector space affording χ. Then we set

$$n(\chi, v) = \sum_{i=0}^{\infty} \frac{|G_i|}{|G_0|} \operatorname{codim}(V^{G_i}).$$

Implicit in this definition is that the right hand side does not depend on the choice of w over v, or on the choice of V. The Artin conductor is then defined by

$$\mathfrak{F}_\chi = \prod \mathfrak{p}_v^{n(\chi, v)}$$

where the product is over all finite v. Note that $n(\chi, v) = 0$ for all but finitely many v, so that the product is well-defined. The conductor-discriminant formula states

$$\mathrm{Norm}_{K/\mathbf{Q}}(\delta_{L/K}) = \prod_{\chi} (\mathrm{Norm}_{K/\mathbf{Q}} \mathfrak{F}_\chi)^{\chi(1)}$$

where the product is over all irreducible characters χ of G.

PROPOSITION 2.5. $\log(\mathrm{Norm}_{K/\mathbf{Q}} \mathfrak{F}_\chi) \leq 2\chi(1) n_K \{\Sigma_{p \in P(L/K)} \log p + \log n\}$.

Proof. Firstly, we observe that for each $i \geq 0$,

$$\dim V^{G_i} = \frac{1}{|G_i|} \sum_{a \in G_i} \chi(a),$$

and so for each finite v,

$$n(\chi, v) = \sum_i \frac{|G_i|}{|G_0|} \left(\chi(1) - \frac{1}{|G_i|} \sum_{a \in G_i} \chi(a)\right). \tag{2.5.1}$$

Denote by $\mathcal{O}_v$ (respectively $\mathcal{O}_w$) the ring of integers of K_v (resp. L_w). Define a function i_G on G by

$$i_G(g) = w(gx - x) = \max\{i : g \in G_{i-1}\}$$

where $\mathcal{O}_w = \mathcal{O}_v[x]$. Rearranging (2.5.1) gives

$$n(\chi, v) = \frac{\chi(1)}{|G_0|} \sum_i (|G_i| - 1) - \frac{1}{|G_0|} \sum_{1 \neq a \in G_0} \chi(a) i_G(a).$$

Applying this formula for χ the trivial character, and the character of the regular representation of G_0, we find that

$$\sum_{1 \neq a \in G_0} i_G(a) = \sum_i (|G_i| - 1) = w(\mathfrak{D}_{L/K}).$$

Hence,

$$\begin{aligned} n(\chi, v) &= \frac{1}{|G_0|} \sum_{1 \neq a \in G_0} i_G(a)(\chi(1) - \chi(a)) \\ &\leq \frac{2\chi(1) w(\mathfrak{D}_{L/K})}{e_{w/v}}. \end{aligned}$$

Now using Lemma (2.1), we deduce that (with $f_v =$ degree of the residue field of v over the prime field)

$$\begin{aligned} \log(\text{Norm } \mathfrak{F}_\chi) &\leq 2\chi(1) \sum \frac{1}{e_{w/v}} (e_{w/v} - 1 + s_{w/v}) f_v \log p_v \\ &\leq 2\chi(1) \left\{ \sum f_v \left(1 - \frac{e_v}{e_w}\right) \log p_v + \sum f_v \frac{e_v}{e_w} w(e_{w/v}) \log p_v \right\} \end{aligned}$$

where e_v, e_w denote the absolute ramification indices at v and w respectively and we have used $e_{w/v} = e_w/e_v$. Also, as $w(e_{w/v}) = e_w v_p(e_{w/v})$ and as L/K is Galois, $e_{w/v}$ divides n. Thus,

$$\log \text{Norm}\, \mathfrak{F}_\chi \leq 2\chi(1) n_K \Big\{ \sum_{p \in P(L/K)} \log p + \log n \Big\}.$$

Remark. The coefficient 2 has entered as we did not have a nontrivial estimate for the character sums $\Sigma_{a \in G_i} \chi(a)$.

3. Effective versions of the Chebotarev theorem. Let L/K be a finite Galois extension of number fields with group G. We retain the notation of Section 2.

3.1. Let C be a subset of G stable under conjugation, and define $\pi_C(x) = \#\{v$ a place of K unramified in L, $\text{Norm}_{K/\mathbf{Q}} \mathfrak{p}_v \leq x$ and $\sigma_v \subset C\}$. The Chebotarev density theorem asserts that

$$\pi_C(x) \sim \frac{|C|}{|G|} \pi_K(x)$$

where $\pi_K(x)$ denotes the number of primes of K of norm $\leq x$. Effective versions were given by Lagarias and Odlyzko [3].

PROPOSITION 3.2. *Suppose the Dedekind zeta function $\zeta_L(s)$ satisfies the Riemann Hypothesis. Then*

$$\pi_C(x) = \frac{|C|}{|G|} \operatorname{Li} x + O\left(\frac{|C|}{|G|} \cdot x^{1/2} (\log d_L + n_L \log x) \right).$$

(*This form of their result is due to Serre* [16, p. 133]).

PROPOSITION 3.3. *If* $\log x \gg n_L (\log d_L)^2$, *then*

$$\left| \pi_C(x) - \frac{|C|}{|G|} \operatorname{Li} x \right| \leqslant \frac{|C|}{|G|} \operatorname{Li}(x^\beta) + O(|\tilde{C}| x \exp(-c n_L^{-1/2} (\log x)^{1/2}))$$

where $|\tilde{C}|$ is the number of conjugacy classes contained in C and the term $|C| \operatorname{Li}(x^\beta)/|G|$ is suppressed if the exceptional zero β does not exist.

PROPOSITION 3.4. *We have*

$$\pi_C(x) \ll \frac{|C|}{|G|} \operatorname{Li} x$$

provided $\log x \gg (\log d_L)(\operatorname{loglog} d_L)(\operatorname{logloglog} e^{20} d_L)$.

This is due to Lagarias, Montgomery and Odlyzko [4].

3.5. These estimates can be significantly improved if we know Artin's conjecture on the holomorphy of L-series. The improvement is in the dependence of the error term on C. We shall only discuss Proposition (3.2) here. Let χ be a character of G and denote by $\pi(x, \chi)$ the function denoted $\pi_\chi(x)$ in Section 1.4. Let $\delta(\chi)$ denote the multiplicity of the trivial character in χ.

Set

$$A_\chi = d_K^{\chi(1)} \operatorname{Norm}_{K/\mathbf{Q}} \mathfrak{F}_\chi$$

and

$$\Lambda(s, \chi) = A_\chi^{s/2} \gamma(s) L(s, \chi)$$

where $\gamma(s)$ is a certain product of powers of π and Γ-functions (see for example, Martinet [6, p. 12] for a detailed description).

PROPOSITION. *Suppose that the Artin L-series* $L(s, \chi)$ *is analytic for all* $s \neq 1$ *and is nonzero for* $\operatorname{Re}(s) \neq 1/2$, $0 < \operatorname{Re}(s) < 1$. *Then*

$$\pi(x, \chi) = \delta(\chi) \operatorname{Li}(x) + O(x^{1/2}(\log A_\chi + \chi(1) n_K \log x))$$
$$+ O(\chi(1) n_K \log M(L/K))$$

where

$$M(L/K) = n d_K^{1/n_K} \prod_{p \in P(L/K)} p.$$

Proof. The argument proceeds along standard lines and so we just sketch it here. Artin [1] proved the functional equation

$$\Lambda(s, \chi) = W(\chi) \Lambda(1 - s, \bar{\chi}) \tag{3.5.1}$$

where $W(\chi) \in \mathbf{C}$, $|W(\chi)| = 1$ and $\bar{\chi}$ is the complex conjugate of χ. We know that $(s(s-1))^{\delta(\chi)}\Lambda(s, \chi)$ is entire and we have the Hadamard factorization

$$\Lambda(s, \chi) = e^{a(\chi)+b(\chi)s}\prod\left(1 - \frac{s}{\rho}\right)e^{s/\rho} \cdot (s(s-1))^{-\delta(\chi)} \tag{3.5.2}$$

where $a(\chi)$, $b(\chi) \in \mathbf{C}$, and the product runs over all zeroes ρ of $\Lambda(s, \chi)$ (necessarily, $0 \leq \operatorname{Re} \rho < 1$). From the equality

$$\overline{\Lambda(s, \chi)} = \Lambda(\bar{s}, \bar{\chi}) \tag{3.5.3}$$

we deduce the relation

$$\overline{\frac{\Lambda'}{\Lambda}(s, \chi)} = \frac{\Lambda'}{\Lambda}(\bar{s}, \bar{\chi}).$$

Moreover, (3.5.1) implies the relation

$$\frac{\Lambda'}{\Lambda}(s, \chi) = -\frac{\Lambda'}{\Lambda}(1 - s, \bar{\chi}).$$

From these two relations, we deduce that

$$\operatorname{Re} \frac{\Lambda'}{\Lambda}\left(\frac{1}{2}, \chi\right) = 0.$$

Now, (3.5.1) and (3.5.3) also imply that if ρ is a zero of $\Lambda(s, \chi)$, then so is $1 - \bar{\rho}$. Hence,

$$\operatorname{Re} \Sigma \left(\frac{1}{2} - \rho\right)^{-1} = 0$$

as is seen by grouping together the terms corresponding to ρ and $1 - \bar{\rho}$ in the absolutely convergent sum. Logarithmically differentiating (3.5.2) at $s = 1/2$, and taking real parts, we deduce that

$$\operatorname{Re}\left(b(\chi) + \Sigma \frac{1}{\rho}\right) = 0.$$

Hence,

$$\text{(3.5.4)}\quad \operatorname{Re}\frac{\Lambda'}{\Lambda}(s,\chi) = \sum_{\rho} \operatorname{Re}\left(\frac{1}{s-\rho}\right) - \delta(\chi)\operatorname{Re}\left(\frac{1}{s}+\frac{1}{s-1}\right).$$

Let $N(t, \chi)$ denote the number of zeroes $\rho = \beta + i\gamma$, $0 < \beta < 1$, $|\gamma - t| \leq 1$ of $L(s, \chi)$. Evaluating (3.5.4) at $s = 2 + it$, and observing that

$$\operatorname{Re}\left(\frac{1}{2+it-\rho}\right) = \frac{2-\beta}{(2-\beta)^2+(t-\gamma)^2}\begin{cases}\geq 0 & \text{for all } \rho\\ \geq 1/5 & \text{if } |t-\delta| \leq 1,\end{cases}$$

we deduce that

$$N(t,\chi) \ll \operatorname{Re}\frac{\Lambda'}{\Lambda}(2+it,\chi).$$

Since the Dirichlet series for $L(s, \chi)$ converges at $2 + it$, the right hand side is easily estimated, the essential contribution coming from $\log A_\chi$ and the number of Γ-factors. We get

$$\text{(3.5.5)}\qquad N(t,\chi) \ll \log A_\chi + \chi(1) n_K \log(|t|+5).$$

By developing an explicit formula as in [3] or [7], we find that

$$\sum_{Nv<x}{}' \chi(\sigma_v)\log Nv = \delta(x)x - \sum_{|\gamma|<x}\frac{x^\rho}{\rho} + O(\chi(1)n_K \log M(L/K))$$
$$+ O(x^{1/2}(\log x)(\log A_\chi + \chi(1)n_K\log x)),$$

where the prime on the sum indicates that we only include places v that are unramified in L. The sum over zeroes can be estimated by observing that

$$\sum_{|\gamma|<x}\frac{1}{\rho} \ll \sum_{j<x}\frac{N(j,\chi)}{j}$$

and using (3.5.5). The estimate for $\pi(x, \chi)$ can be deduced by partial summation.

PROPOSITION 3.6. *Suppose that all Artin L-series of the extension L/K are analytic at $s \neq 1$, and that GRH holds. Then*

$$(3.6.1) \quad \sum_C \frac{1}{|C|}\left(\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li} x\right)^2 \ll x n_K^2 (\log M(L/K)x)^2.$$

Proof. We first observe that

$$\sum_C \frac{1}{|C|}\left(\frac{|C|}{|G|}\pi(x, 1_G) - \frac{|C|}{|G|}\operatorname{Li} x\right)^2 = \frac{1}{|G|}(\pi(x, 1_G) - \operatorname{Li} x)^2.$$

Using the identity of Proposition (1.3), the left hand side is, therefore,

$$\leq \frac{1}{|G|}\left(\sum_{\chi \neq 1} |\pi(x, \chi)|^2 + (\pi(x, 1_G) - \operatorname{Li} x)^2\right)$$

where the sum is over the nontrivial irreducible characters of G. By Proposition (2.5) and Proposition (3.5),

$$\pi(x, \chi) - \delta(\chi)\operatorname{Li} x \ll \chi(1) n_K x^{1/2} \log(M(L/K)x).$$

The result follows on noting that $\Sigma\, \chi(1)^2 = |G|$.

COROLLARY 3.7. *Let D be a union of conjugacy classes. Under the same hypotheses as in Proposition* (3.6),

$$\pi_D(x) = \frac{|D|}{|G|}\operatorname{Li} x + O(|D|^{1/2} x^{1/2} n_K \log M(L/K)x).$$

Proof. We have

$$\pi_D(x) - \frac{|D|}{|G|}\operatorname{Li} x = \sum_C \left(\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li} x\right)$$

where the sum is taken over all conjugacy classes C contained in D. Now applying the Cauchy-Schwartz inequality gives

$$\sum_C \left|\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li} x\right| \ll \left(\sum_C |C|\right)^{1/2}\left(\sum_C \frac{1}{|C|}\left|\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li} x\right|^2\right)^{1/2}.$$

The result now follows from Proposition (3.6).

Remark 3.8. Using Proposition (2.3) we can write the error term in Proposition (3.2) as

$$O(|C|x^{1/2}n_K \log M(L/K)\,x).$$

Thus Artin's conjecture allows us to replace $|C|$ with $|C|^{1/2}$ in this estimate.

3.9. In some cases, Corollary (3.7) can be used to get refinements of Proposition (3.2) even without assuming Artin's conjecture. We give two such cases here.

PROPOSITION. *Let D be a union of conjugacy classes in G and let H be a subgroup of G satisfying*

(1) *Artin's conjecture is true for the irreducible characters of H*
(2) *H meets every class in D.*

Suppose the GRH holds. Then

$$\pi_D(x) = \frac{|D|}{|G|}\operatorname{Li} x + O\left(x^{1/2}\left(\sum_{C \subseteq D} \frac{|C|^2}{|C_H|}\right)^{1/2} n_K \log Mx\right).$$

where $M = M(L/K)$, and $C_H = C_H(\gamma)$ for some $\gamma \in H \cap C$.

COROLLARY 3.10. *Under the same hypotheses as above,*

$$\pi_D(x) = \frac{|D|}{|G|}\operatorname{Li} x + O\left(|D|^{1/2}x^{1/2}\left(\frac{|G|}{|H|}\right)^{1/2} n_K \log Mx\right)$$

where $|\tilde{D}|$ is the number of conjugacy classes contained in D.

The corollary follows from the proposition on noting that $|C|/|C_H| \leq |G|/|H|$.

3.11. *Proof of Proposition* 3.9. Firstly, from Section 1.4,

$$\pi_D(x) = \tilde{\pi}_D(x) + O\left(\frac{1}{|G|}\log d_L + n_K x^{1/2}\right). \tag{3.11.1}$$

Using Proposition (2.3), we find

$$\frac{1}{|G|}\log d_L \ll n_K \log Mx.$$

Also, from Section 1.4,

$$(3.11.2)\qquad \tilde{\pi}_D(x) = \sum_{C \subseteq D} \tilde{\pi}_C(x) = \sum_{C \subseteq D} \frac{|C|}{|G|} \cdot \frac{|H|}{|C_H|} \tilde{\pi}_{C_H}(x).$$

Now

$$\sum_{C \subseteq D} \frac{|C|}{|G|} \cdot \frac{|H|}{|C_H|} (\tilde{\pi}_{C_H}(x) - \pi_{C_H}(x))$$

$$\leq \frac{|H|}{|G|} \sum_{C \subseteq D} \frac{|C|}{|C_H|} \left(\sum_{\substack{Nv^m \leq x \\ m \geq 2}} \delta_{C_H}(\sigma_v^m) + \sum_{\substack{Nv \subseteq x \\ v \text{ ramified} \\ \text{in } L/K}} \delta_{C_H}(\sigma_v) \right)$$

$$\leq \frac{|H|}{|G|} \left(\max_{C \subseteq D} \frac{|C|}{|C_H|} \right) \cdot \left\{ \frac{|G|}{|H|} \cdot n_K x^{1/2} + \frac{2}{\log 2} \frac{1}{|H|} \log d_L \right\}$$

$$\leq \left(\max \frac{|C|}{|C_H|} \right) (n_K x^{1/2} + n_K \log Mx)$$

and this can be absorbed into the error term. Therefore, we can replace $\tilde{\pi}_{C_H}$ by π_{C_H} in (3.11.2). Now,

$$(3.11.3)\qquad \sum_{C \subseteq D} \frac{|C|}{|G|} \cdot \frac{|H|}{|C_H|} \cdot \pi_{C_H}(x) = \frac{|D|}{|G|} \operatorname{Li} x$$

$$+ O\left(\frac{|H|}{|G|} \cdot \sum_{C \subseteq D} \frac{|C|}{|C_H|^{1/2}} \frac{1}{|C_H|^{1/2}} \cdot \left| \pi_{C_H}(x) - \frac{|C_H|}{|H|} \operatorname{Li} x \right| \right)$$

Now applying the Cauchy-Schwartz inequality and using Proposition (3.6), we find that the O-term in (3.11.3) is

$$\ll \frac{|H|}{|G|} \cdot \left(\sum_{C \subseteq D} \frac{|C|^2}{|C_H|} \right)^{1/2} x^{1/2} \cdot n_K \frac{|G|}{|H|} \log M(L/K')x.$$

where K' is the fixed field of H. Combining this with (3.11.1)–(3.11.3) proves the Proposition, since $M(L/K') \ll M(L/K)$.

PROPOSITION 3.12. *Suppose the GRH holds. Let D be a nonempty union of conjugacy classes in G and let H be a normal subgroup of G such* *that Artin's conjecture is true for the irreducible characters of G/H, and $HD \subseteq D$. Then*

$$\pi_D(x) = \frac{|D|}{|G|}\operatorname{Li} x + O\left(\left(\frac{|D|}{|H|}\right)^{1/2} x^{1/2} n_K \log Mx\right),$$

where M is as in Proposition (3.9).

Proof. Let $\bar{D}$ be the image of D in G/H. It is a union of conjugacy classes in G/H and

$$\pi_{\bar{D}}(x) = \frac{|\bar{D}|\cdot|H|}{|G|}\operatorname{Li} x + O(|\bar{D}|^{1/2}\cdot x^{1/2} n_K \log M(K'/K)x)$$

where K' is the fixed field of H. As $HD \subseteq D$, $|\bar{D}|\cdot|H| = |D|$ and $\pi_D(x) = \pi_{\bar{D}}(x) + O((\log d_L)/|G|)$. Also, $M(K'/K) \ll M(L/K)$. The result follows.

3.13. Observe that the estimate above is *sharper* than that predicted by Artin's conjecture if $|H| > 1$. This leads us to ask what the true order of the error term should be. Let $\alpha(G)$ denote the number of conjugacy classes of G.

Question. Is it true that for any conjugacy set $D \subseteq G$,

$$\pi_D(x) = \frac{|D|}{|G|}\operatorname{Li} x + O\left(\left(\frac{|D|}{\alpha(G)}\right)^{1/2} x^{1/2} n_K \log Mx\right)?$$

This would be implied by (3.6.1) for example, if all the terms are of the same order. In the case $K = \mathbf{Q}$ and L/K is abelian, our question is a well-known conjecture of Montgomery.

4. Estimates for $\pi_{f,a}(x)$.

4.1. We return to the situation described in the Introduction. Thus, f is a cusp form of weight k for $\Gamma_0(N)$ with Fourier expansion

$$f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z}$$

at $i\infty$. Suppose that

(i) f is an eigenform for all the Hecke operators T_p, $p \nmid N$ and $a_1 = 1$
(ii) $k \geq 2$
(iii) f is not of CM type
(iv) the a_n are rational integers.

Let $h(T)$ be a polynomial in $\mathbf{Z}[T]$.

THEOREM 4.2. *Suppose GRH holds. Then*

$$\#\{p \leq x : a_p = h(p)\} \ll x^{4/5}(\log x)^{-1/5}.$$

where the implied constant depends only on f and the degree of h.

Remarks 4.3.

(i) If $h = 0$, the estimate can be improved to $x^{3/4}$. We do not discuss this further as Serre has also obtained $O(x^{3/4})$ in this case.

(ii) Assumption (iv) above is made only for convenience and is easily removed as in Serre [16, pp. 175–176].

(iii) If f has CM, we can obtain the sharper estimate $x^{1/2}(\log x)^2$ when $h \neq 0$.

(iv) Our proof of Theorem (4.2) uses only the mod ℓ reduction of the ℓ-adic representation and we vary ℓ. This should be compared with Serre's method where the (mod ℓ^n) reductions are used for all $n \geq 1$ (Cf. [16, p. 190, Remark (1)]).

The proof of Theorem (4.2) will require some preliminaries. For each prime p, let us denote by $\mathbf{Q}(\omega_p)$ the field $\mathbf{Q}(\sqrt{a_p^2 - 4p^{k-1}})$, and define for each ℓ,

$$\pi_h(x, \ell) = \#\{p \leq x : a_p = h(p) \text{ and } \ell \text{ splits in } \mathbf{Q}(\omega_p)\}.$$

Note that $\mathbf{Q}(\omega_p)$ is of degree ≤ 2 over $\mathbf{Q}$.

LEMMA 4.4. *Suppose for some $\epsilon > 0$, $y \geq u \geq y^{1/2}(\log y)^{1+\epsilon}(\log xy)$. Then assuming the GRH, we have*

$$\#\{p \leq x : a_p = h(p)\} \ll \max_{\ell \in I} \pi_h(x, \ell)$$

where the maximum is taken over primes ℓ in the interval

$$I = [y, y + u].$$

Proof. We have

$$\sum_{\ell \in I} \pi_h(x, \ell) = \sum_{\substack{p \le x \\ a_p = h(p)}} \pi_p(I) \tag{4.4.1}$$

where

$$\pi_p(I) = \#\{\ell \in I : \ell \text{ splits in } \mathbf{Q}(\omega_p)\}.$$

We know (for example, from Proposition 3.2) that *uniformly* in p,

$$\#\{\ell \le y : \ell \text{ splits in } \mathbf{Q}(\omega_p)\} = \begin{cases} \dfrac{1}{2}\pi(y) + O(y^{1/2}\log(py)) & \text{if } \omega_p \notin \mathbf{Q} \\ \pi(y) & \text{if } \omega_p \in \mathbf{Q} \end{cases}$$

Using the Riemann Hypothesis, and recalling that $u \geqslant y^{1/2}(\log y)^{2+\epsilon}$, we see that

$$\pi(y + u) - \pi(y) \gg \frac{u}{\log u}.$$

Therefore, if we take u as in the statement of the Lemma, we find that *uniformly* for each $p \le x$,

$$\#\{\ell : y \le \ell \le y + u, \ell \text{ splits in } \mathbf{Q}(\omega_p)\} \gg \pi(y + u) - \pi(y).$$

Hence, from (4.4.1),

$$\#\{p \le x : a_p = h(p)\} \ll \max_{\ell \in I} \pi_h(x, \ell).$$

4.5. Reducing the representation $\rho_{\ell,f}$ (see Introduction) mod ℓ gives (for large ℓ) a field K_ℓ which is Galois over $\mathbf{Q}$ with group

$$G = G_\ell = \{g \in GL_2(\mathbf{F}_\ell) : \det g \in (\mathbf{F}_\ell^\times)^{k-1}\}$$

(see, for example, Ribet [12]). We also know that $K_\ell/\mathbf{Q}$ is unramified outside ℓN.

Now, we fix ℓ and drop it from the subscripts. We consider the Borel subgroup

$$B = \left\{ g \in G : g = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}.$$

For $p \nmid N\ell$, the condition that ℓ splits in $\mathbf{Q}(\omega_p)$ means that σ_p is conjugate to an element of B.

Define a polynomial

$$H(U, T^{k-1}) = \prod_{w} (U - h(wT))$$

where the product is over the $(k-1)$st roots of unity. Then, as in Serre [16, p. 176], $a_p = h(p)$ implies that $H(\operatorname{tr} \sigma_p, \det \sigma_p) = 0$. Choose a maximal set Γ of elements γ in B which are nonconjugate in G, so that $H(\operatorname{tr} \gamma, \det \gamma) = 0$, and set

$$D = \cup\ C_G(\gamma) : \gamma \in \Gamma.$$

(Recall that $C_G(\gamma)$ is the conjugacy class of γ in G.) Our problem is to estimate $\pi_D(x)$.

4.6. *Proof of Theorem* (4.2). Let D_1 be the union of the singleton classes in D (i.e. the set of scalars in Γ) and set $D_2 = D - D_1$. Clearly, $|D_1| \leq (k-1)(1 + 2 \deg h) = \mu$(say), and so from Proposition (3.2),

$$\pi_{D_1}(x) \ll \frac{\mu}{\ell^4}\, \pi(x) + O(\mu x^{1/2} \log \ell N x).$$

It remains to estimate $\pi_{D_2}(x)$. For any nonscalar $\gamma \in B$, we observe that

$$|C_B(\gamma)|/|B| = \frac{1}{\ell^2} + O(1)$$

$$|C_G(\gamma)|/|G| = \frac{1}{\ell^2} + O(1).$$

Hence, from the discussion in Section 1.4,

$$\tilde{\pi}_{C_G(\gamma)}(x) = \tilde{\pi}_{C_B(\gamma)}(x).$$

Consider the subgroup

$$A = \left\{ g \in B : g = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}.$$

Then A is a normal subgroup of B with the property that if $\gamma \in B$ has distinct eigenvalues, $A\gamma = C_B(\gamma)$. Moreover,

$$A.(aI) = \left\{ \begin{pmatrix} a & * \\ 0 & a \end{pmatrix} \right\}$$

for $a \in \mathbf{F}_\ell^\times$.

Let $\varphi : B \to B/A$ be the quotient map, and $\bar{\gamma} = \varphi(\gamma)$. As B/A is abelian, $\varphi^{-1} C_{B/A}(\bar{\gamma}) = A\gamma$. Thus,

$$\pi_{D_2}(x) \leq \Sigma' \, \tilde{\pi}_{C_B(\gamma)}(x) \tag{4.6.1}$$

where the sum is over nonscalar elements $\gamma \in \Gamma$. As B/A is abelian, Artin's conjecture holds for the characters of B/A and so by Proposition (3.12),

$$\Sigma' \, \pi_{C_B(\gamma)}(x) \leq \frac{|\Gamma|}{(\ell - 1)^2} \operatorname{Li} x + O(|\Gamma|^{1/2} \cdot x^{1/2} \cdot \ell \cdot \log(\ell N x)) \tag{4.6.2}$$

Now, let φ be the characteristic function of the set

$$\cup \; C_B(\gamma) : \text{nonscalar } \gamma \in \Gamma.$$

Then the right hand side of (4.6.1) differs from the left hand side of (4.6.2) by

$$\tilde{\pi}_\varphi(x) - \pi_\varphi(x)$$

which by Section 1.4 and Proposition (2.3) is

$$\ll \ell \log(\ell N x) + \ell x^{1/2}.$$

Thus as $|\Gamma| \ll \ell$,

$$\pi_{D_2}(x) \ll \frac{1}{\ell}\pi(x) + O(\ell^{3/2}x^{1/2}\log(\ell Nx)).$$

Putting everything together, we deduce that

$$\pi_h(x, \ell) = \pi_D(x) \ll \frac{1}{\ell}\pi(x) + O(\ell^{3/2}x^{1/2}\log(\ell Nx)).$$

Now choosing $y = x^{1/5}(\log x)^{-4/5}$ and $\ell \in I$, and using Lemma (4.4), we find that

$$\#\{p \leq x : a_p = h(p)\} \ll x^{4/5}(\log x)^{-1/5}.$$

5. Lower bounds for a_p and a_n. Theorem (4.2) implies that (assuming *GRH*) for any $\epsilon > 0$,

$$|a_p| \geq p^{(1/5)-\epsilon}$$

holds for a set of primes of density 1. We now refine this lower bound.

THEOREM 5.1. *Suppose GRH is true. Then, we have for any* $\epsilon > 0$,

$$|a_p| \geq p^{(1/4)-\epsilon}$$

for a set of primes of density 1. *Unconditionally, there is a constant* $c > 0$ *such that*

$$|a_p| \geq (\log p)^c$$

holds for a set of primes of density 1.

Remark 5.2. It is clear that the factor $p^{-\epsilon}$ in the above bounds can be replaced by $1/F(p)$ where F is any real-valued function, tending monotonically and sufficiently rapidly to ∞. In fact, our proof produces such an F.

5.3. *Proof of Theorem* (5.1). We shall prove the first assertion. The second will follow from an unconditional version of Theorem (4.2).

We proceed as in the proof of Theorem (4.2). Let $z > 0$ and consider the sum

$$\sum_{|a|<z} \#\{p \leq x : a_p = a\}.$$

By Lemma (4.4), this sum is

$$\ll \max_{\ell \in I} \sum_{|a|<z} \pi_a(x, \ell)$$

where $I = (y, y + u)$ with y and u yet to be specified. For each a, we construct sets Γ_a and D_a as in Section 4.5. Thus, Γ_a is a maximal set of elements γ in B which are nonconjugate in G, so that $\operatorname{tr} \gamma = a$ and

$$D_a = \cup\, C_G(\gamma) : \gamma \in \Gamma_a.$$

Each D_a contains at most one scalar matrix, call it $D_1(a)$ and let $D_2(a) = D_a - D_1(a)$. From Proposition (3.2),

$$\text{(5.3.1)} \qquad \sum_{|a|<z} \pi_{D_1(a)}(x) \ll \frac{z}{\ell^4}\, \pi(x) + O(zx^{1/2} \log \ell N x).$$

For $D_2(a)$, we proceed as in Section 4.6 to get

$$\pi_{D_2(a)}(x) \leq \sum{}' \, \tilde{\pi}_{C_B(\gamma)}(x)$$

where the sum is over elements $\gamma \in \Gamma_a$. Note that $|\Gamma_a| \ll \ell$. Now summing over a we find

$$\text{(5.3.2)} \qquad \sum_{|a|<z} \pi_{D_2(a)}(x) \leq \sum_{|a|<z}\sum{}' \, \pi_{C_B(\gamma)}(x) + \tilde{\pi}_\varphi(x) - \pi_\varphi(x)$$

$$\ll \frac{z}{\ell}\, \pi(x) + O((zx)^{1/2} \ell^{3/2} \log \ell N x) + \ell \log(\ell N x) + \ell x^{1/2}$$

where φ is the characteristic function of

$$\bigcup_{|a|<z} \bigcup_{\gamma \in \Gamma_a} C_B(\gamma)$$

and we have used Corollary (3.7), Section 1.4 and Proposition (2.3). Therefore, from (5.3.1) and (5.3.2),

$$\sum_{|a|<z} \pi_a(x, \ell) \ll \frac{z}{\ell}\,\pi(x) + zx^{1/2}\log \ell Nx + (zx)^{1/2}\ell^{3/2}\log(\ell Nx) + \ell x^{1/2}.$$

We choose $y = z(\log x)^{\epsilon}$ and any u satisfying the conditions of Lemma (4.4). With these choices,

$$\max_{\ell\in I} \sum_{|a|<z} \pi_a(x, \ell)$$

$$\ll \left(\frac{z}{(\log x)^{3/4}} + 1\right)x^{3/4} + x^{1/2}(\log Nx)\left\{z + \frac{z^{1/2}x^{3/8}}{(\log x)^{3/8}}\right\}.$$

We choose $z = x^{1/4}(\log x)^{-(5/4)-\epsilon}(\log Nx)^{-2}$ and deduce that

$$\sum_{|a|<z} \#\{p \leq x : a_p = a\} \ll \frac{x}{(\log x)^{1+\epsilon}} = o(\pi(x)).$$

This proves that

$$|a_p| \geq p^{1/4}(\log p)^{-(5/4)-\epsilon}(\log pN)^{-2}$$

for a set of primes p of density 1. This proves the theorem.

5.4. We can use Theorem (5.1) to obtain a result about a_n also.

THEOREM. *Suppose that the GRH is true. Then there is a constant* $c > 0$ *such that the set*

$$\{n : a_n = 0 \text{ or } |a_n| > n^c\}$$

has density 1. *Unconditionally, the set*

$$\{n : a_n = 0 \text{ or } |a_n| > (\log n)^c\}$$

has density 1. (*The constant c is absolute and effectively computable.*)

Remark. Serre [16, Section 6] showed that the set $\{n : a_n \neq 0\}$ has a density $\alpha > 0$. Combined with the above Theorem, we see that the set $\{n : |a_n| > n^c\}$ has a density $\alpha > 0$.

The proof of Theorem (5.4) will require several lemmas. For a set of natural numbers S, we let $S(x)$ denote the number of elements of S which are $\leq x$.

LEMMA 5.5. *Let P denote a set of primes and set*

$$S_P = \{n : p \mid n \Rightarrow p \in P\}.$$

If P has density $\delta < 1$, then for any η satisfying $\delta < \eta < 1$,

$$S_P(x) = O(x/(\log x)^{1-\eta}).$$

Proof. If $\bar{P}$ denotes the set of primes *not* in P, then S_P consists of natural numbers not divisible by any prime $p \in \bar{P}$. The result now follows from Brun's sieve [13]:

$$S_P(x) \ll x \prod_{\substack{p \notin P \\ p \leq x}} \left(1 - \frac{1}{p}\right) \ll \frac{x}{(\log x)^{1-\eta}}.$$

5.6. Now let P denote any set of primes. We can then write every natural number n as a product $n = n_P \cdot m_P$ where n_P is not divisible by any prime in P and m_P has all its prime divisors in P.

LEMMA. *Let P be a set of primes satisfying $P(x) = O(x^\eta)$ with $\eta < 1$. Then for any δ satisfying $0 < \delta < 1$, the set*

$$S_\delta = \{n : n_p > n^\delta\}$$

has density 1.

Proof. Let $0 < \delta < 1$ and consider

$$\bar{S}_\delta = \{n : n_p \leq n^\delta\}.$$

Then

$$\bar{S}_\delta(x) \leq x^\alpha + \sum_{\substack{x^\alpha < n < x \\ n_p \leq x^\delta}} 1$$

for any α with $0 \leq \alpha < \delta$. Thus,

$$\bar{S}_\delta(x) \leq x^\alpha + \sum_{m_p \geq x^{\alpha-\delta}} \frac{x}{m_p}.$$

Since $P(x) = O(x^\eta)$ with $\eta < 1$, it follows that $\Sigma\, 1/m_p$ converges, and so the sum in the penultimate steps is $o(x)$. This proves the result.

Remark 5.7. The result holds with n^δ replaced by $n/F(n)$ where $F(x)$ is any function tending monotonically to ∞ as $n \to \infty$.

LEMMA 5.8. *There is an absolute and effectively computable constant $c > 0$ such that for each prime p, and each positive integer $m \neq 1, 2, 4$, we have*

$$a_{p^m} = 0 \quad \text{or} \quad |a_{p^m}| \geq p^{mc}.$$

If we assume GRH, then

$$\#\{p \leq x : \min(|a_p|, |a_{p^2}|, |a_{p^4}|) < p^{4c}\} \ll x^{1-\epsilon}.$$

Proof. The first assertion is proved in [10]. It follows from the proof of Theorem (5.1) that

$$\#\{p \leq x : |a_p| < p^{4c}\} \ll x^{1-\epsilon}$$

if c is smaller than $1/16$. For $m = 2, 4$, we consider

$$\rho_{\ell,m} : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \xrightarrow{\rho_\ell} GL_2(\mathbf{F}_\ell) \xrightarrow{\mathrm{Sym}^m} GL_{m+1}(\mathbf{F}_\ell)$$

where Sym^m denotes the m-th symmetric power of the standard representation. It then is easy to see that if $p \nmid \ell N$,

$$\mathrm{tr}\, \rho_{\ell,m}(\sigma_p) \equiv a_{p^m} (\mathrm{mod}\ \ell).$$

We proceed as in the proof of Theorem (5.1) for the field $K_{\ell,m}/\mathbf{Q}$ where $K_{\ell,m}$ is the fixed field of $\ker \rho_{\ell,m}$. This proves the lemma.

LEMMA 5.9. *There is an absolute and effectively computable constant $c > 0$ such that for each prime p, and each positive integer $m \geq 2$,*

$$|a_{p^m}| = 0 \quad \text{or} \quad |a_{p^m}| \geq (\log p^m)^c.$$

For $m = 1$, *we have*

$$|a_p| \geq (\log p)^c$$

for all but $O(x/(\log x)^{1+\gamma})$ *primes* $p \leq x$, *for some* $\gamma > 0$.

Proof. This is proved in [10] for $m \geq 2$. For $m = 1$ it follows from Theorem (5.1).

5.10. *Proof of Theorem* (5.4). Let D_A consist of prime powers p^m satisfying $|a_{p^m}| \geq p^{mc}$ where c is the constant of Lemma (5.8). Let D_B consist of prime powers p^m satisfying $|a_{p^m}| \geq (\log p^m)^c$ where c is the constant of Lemma (5.9). For any integer, write $n = n_1 n_2$ where

$$n_2 = \prod_{\substack{p^m \| n \\ p^m \in D_A}} p^m.$$

Then if $a_n \neq 0$,

$$|a_n| \geq |a_{n_2}| = \prod_{\substack{p^m \| n_2 \\ p^m \in D_A}} a_{p^m} \geq n_2^c$$

Now we apply Lemma 5.6 to $\bar{D}_A$ to find that for a set of integers n of density 1, $n_2 \geq n^\delta$. This proves the first part of Theorem (5.4). The second part is proved similarly using D_B instead of D_A in the above proof.

6. Large prime divisors of a_p and a_n.

6.1. Our third application is to large prime divisors of a_p and a_n. If $P(n)$ denotes the largest prime divisor of n, then, it was essentially shown in [8] that under *GRH*,

$$P(a_p) > \exp((\log p)^{1-\epsilon})$$

for any $\epsilon > 0$, and for a set of primes p of density 1. Now, we shall use a similar method to obtain an unconditional result. We suppress the details that overlap with [8]. The main difference in the argument here is in the handling of the Siegel zeros of the Dedekind zeta function.

THEOREM 6.2. *For any* $\epsilon > 0$, *and for almost all primes* p, *we have*

$$P(a_p) \geq \exp((\log\log p)^{1-\epsilon}).$$

Proof. As in [8], let us set

$$Z(x) = \#\{p \leq x : a_p = 0\}$$

and

$$\pi(x, \ell) = \#\{p \leq x : 0 \neq a_p \equiv 0 \pmod{\ell}\}.$$

Then by Proposition 3.3,

$$|\pi(x, \ell) - \delta(\ell)\mathrm{Li}\, x| \leq \delta(\ell)\mathrm{Li}(x^\beta) + O\left(\ell x \exp\left(-c_2\left(\frac{\log x}{4}\right)^{1/2}\right)\right)$$

where $\delta(\ell) = (1/\ell) + O(1/\ell^2)$. We have the estimate of Stark [17] that for any normal extension $L/\mathbf{Q}$ of degree n_L and discriminant d_L, there is an effectively computable absolute constant c_1 such that

$$\beta < \max\left(1 - \frac{1}{4 \log d_L}, 1 - \frac{c_1}{d_L^{1/n_L}}\right).$$

In our case, $\log d_L = O(\ell^4 \log \ell)$ so that for some absolute constant $c > 0$,

$$\beta < 1 - \frac{1}{\ell^c}.$$

Hence, for some small constant $c_2 > 0$, and $\ell \leq (\log x)^{c_2}$,

$$|\pi(x, \ell) - \delta(\ell)\mathrm{Li}\, x| \ll \ell x \exp(-c_2(\log x)^{1/2}/\ell^2).$$

It is then easy to see that if $\nu_y(n)$ denotes the number of prime factors of n which are $\leq y$, then for any $n^2 \leq (\log x)^{c_3}$,

$$\sum_{p<x} \{\nu_u(a_p) - \operatorname{loglog} n\}^2 = Z(x)(\operatorname{loglog} u)^2 + O(\pi(x)\operatorname{loglog} u),$$

by the methods of [8] and the effective estimate for $\pi(x, \ell)$ given above. As in [8], it is then straightforward to deduce that for almost all primes p,

$$P(a_p) \geq \exp((\operatorname{loglog} p)^{1-\epsilon}),$$

for any $\epsilon > 0$. This completes the proof.

Finally, we find an analogous result for a_n.

THEOREM 6.3. *The set*

$$\{n : a_n = 0 \text{ or } P(a_n) \geq \exp((\log n)^{1-\epsilon})\}$$

has density 1, *assuming GRH. The set*

$$\{n : a_n = 0 \text{ or } P(a_n) \geq \exp((\log\log n)^{1-\epsilon})\}$$

has density 1 *unconditionally*.

Proof. Let

$$E = \{p : p \text{ prime}, P(a_p) > \exp((\log\log p)^{1-\epsilon})\}.$$

We first show that almost all $n \leq x$ have a prime divisor $p > \exp((\log\log x)^{1-\epsilon})$, $p \in E$. Indeed, from Theorem (6.2), E has density 1 and so, by Brun's sieve, the number of $n \leq x$ not having any such prime divisor is

$$\ll x \prod_{\substack{p \in E \\ p < z}} \left(1 - \frac{1}{p}\right) \ll \frac{x}{(\log z)^{1-\eta}} = o(x)$$

where $z = \exp((\log x)^{1-\epsilon})$ and $\eta > 0$. If

$$F(x) = \{n \leq x : \exists p \in E, p^2 | n, p > \exp((\log x)^{1-\epsilon})\},$$

We find

$$|F(x)| \leq \sum_{z<p<x} \frac{x}{p^2} = o(x).$$

Therefore, for almost all $n \leq x$, there exists a prime divisor $p \in E$, $p \| n$ such that $p > \exp((\log x)^{1-\epsilon})$. Hence, for almost all n,

$$\begin{aligned} |a_n| \geq |a_p| &\geq \exp((\log\log p)^{1-\epsilon}) \\ &\geq \exp((\log\log n)^{1-\epsilon'}) \end{aligned}$$

for any $\epsilon' > 0$. The proof of the conditional assertion of the Theorem is similar.

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