Some Reflections on the Partition Function

M. Ram Murty

Dedicated to the memory of Professor M.V. Subbarao

Abstract In 1966, M.V. Subbarao conjectured that there are infinitely many numbers n in a given arithmetic progression such that the partition function $p(n)$ is in a specified residue class (mod 2). With the work of Ono in 1996 completed by Radu in 2012, we now know the truth of this conjecture. However, there are related questions that Subbarao's conjecture raises which are still open. We will present some reflections around this topic and suggest future directions for research.

Keywords Partition functions · Ramanujan τ-function · ℓ-adic representations · Chebotarev density theorem

AMS 2020 Subject Classification 11P82, 11P83, 11F03, 11F80, 11N05

1 Introduction

A *partition* of n is a division of n into any number of positive integral parts. The number of ways of doing this is denoted $p(n)$. Thus,

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

and so $p(4) = 5$. Any partition of n can be written as

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k, \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1.$$

M. Ram Murty (✉)
Department of Mathematics and Statistics, Queen's University, Kingston, ON, Canada
e-mail: murty@queensu.ca

B. Balasubramanyam et al. (eds.), *Some Contributions to Number Theory and Beyond*, Fields Institute Communications 90,
https://doi.org/10.1007/978-3-032-10837-1_4

We say k is the *length* of the partition. The descending array of numbers λ_i is abbreviated as simply λ and one writes $\lambda \vdash n$ to indicate that λ is a partition of n.

Euler seems to have been the first person to have studied this function. In 1741, he discovered the generating function:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{-1}. \tag{1}$$

He also discovered the famous pentagonal number theorem (see page 6 of [14]):

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)/2}. \tag{2}$$

Notice that the summation goes from $-\infty$ to $+\infty$.

A beautiful application of Euler's pentagonal number theorem to the study of the parity of the partition function was discovered by Oddmund Kolberg as late as 1959. He proved that $p(n)$ takes both even and odd values infinitely often. Indeed, multiplying the power series of (1) and (2), we have for $n \geq 1$,

$$\sum_{k}(-1)^k p\left(n - \frac{k(3k-1)}{2}\right) = 0, \tag{3}$$

where the summation is over all k such that

$$n - \frac{k(3k-1)}{2} \geq 0.$$

If $p(n)$ is even say for all $n \geq a$, then taking $n = \frac{a(3a-1)}{2}$ in (3) gives

$$p\left(\frac{a(3a-1)}{2}\right) - p\left(\frac{a(3a-1)}{2} - 1\right) - \cdots \pm p\left(\frac{a(3a-1)}{2} - \frac{(a-1)(3a-4)}{2}\right)$$
$$= \mp p(0) = \mp 1. \tag{4}$$

Noting that

$$\frac{a(3a-1)}{2} - \frac{(a-1)(3a-4)}{2} = 3a - 2 > a,$$

and that all the arguments of the partition function on the left hand side are greater than a, we have a contradiction since the left hand side is even and the right hand side is odd. This proves that $p(n)$ is odd infinitely often.

Now suppose that $p(n)$ is odd for all $n \geq b$. The number of terms in (3) is equal to the number of integers k such that

$$n - \frac{k(3k-1)}{2} \geq 0.$$

Separating the solution $k = 0$ from the others, it is easy to see that the problem is equivalent to counting the positive $k > 0$ such that

$$n - \frac{k(3k \pm 1)}{2} \geq 0.$$

We choose $n = b(3b+1)/2$ and a simple calculation shows that the number of such k is exactly $2b$. Together with $k = 0$, we get a total of $2b+1$ permissible values of k. In other words, for this choice of n, the number of summands on the left hand side of (3) is $2b + 1$. Moreover, the non-zero arguments of the partition function in the summation are all greater than or equal to b. If $p(n)$ is odd for all $n \geq b$, then the left hand side of (3) is an odd number of odd numbers which must be odd and this is not the case because the right hand side is zero. This contradiction proves the assertion.

This argument can be refined to derive some quantitative estimates. Indeed, the number of $n \leq x$ such that $n = a(3a-1)/2$ is $\gg \sqrt{x}$. This leads to a lower bound of $\gg \sqrt{x}$ for the number of $n \leq x$ for which $p(n)$ is even. A similar estimate holds for the number of $n \leq x$ for which $p(n)$ is odd.

This is essentially the extent of our knowledge of the parity of $p(n)$. In 1967, Parkin and Shanks [20] conjectured that the natural density of n such that $p(n)$ is even equals 1/2. Their conjecture is still (surprisingly) open.

Partly inspired by Kolberg's paper, M.V. Subbarao [25] conjectured in 1966 that there are infinitely many numbers n in a given arithmetic progression such that the partition function $p(n)$ is in a specified residue class (mod 2). In other words, given A and B, there are infinitely many m and n such that $p(Am + B)$ is even and $p(An + B)$ is odd. This conjecture is now a theorem and was settled in two stages. In 1996, Ono [18] proved that for any A, B, $p(An + B)$ is even for infinitely many n. This settled the even case of Subbarao's conjecture. He then proved that if there is one n_0 such that $p(An_0 + B)$ is odd, then there are infinitely many n such that $p(An + B)$ is odd. That such an n_0 always exists was finally shown by Radu [21] in 2012, thus giving a complete solution to the Subbarao conjecture. This latter work used some deep results on schemes of modules of elliptic curves due to Deligne and Rapoport [4]. Thus, the final resolution of Subbarao's conjecture involved an elegant mélange of the theory of modular forms and elliptic curves.

One may be tempted to say that $p(An+B)$ should be equidistributed in arithmetic progressions (mod m) for any number m. But this is hasty. Ramanujan discovered that

$$p(5n+4) \equiv 0 \pmod 5, \qquad p(7n+5) \equiv 0 \pmod 7 \quad p(11n+6) \equiv 0 \pmod{11}$$

for all natural numbers n.

By using these celebrated congruences, Kolberg [12] showed that $p(n) \equiv c \pmod{d}$ holds for infinitely many n for the pairs $(c, d) = (0, 10), (5, 10), (0, 14), (7, 14)$.

The importance of Subbarao's conjecture and subsequent developments reveal how simple questions can lead to rapid advances in the field. The conjecture is fertile because more simple conjectures emerge that still defy simple answers. Like the enigmatic sphinx of Giza, the conjecture gazes towards greater mysteries. For example, these results raise the following question. Given a natural number m, does $p(n)$ represent all the residue classes (mod m) infinitely often? In 1960, Newman [17] conjectured that this is always so and extended Kolberg's results for more progressions. But Newman's conjecture is still open.

In this paper, we will give an outline of the proof of Subbarao's conjecture in the hope that our exposition will inspire further research into these questions.

2 The Ramanujan τ-Function

To understand the partition function, we must also understand Ramanujan's τ-function because they both fit into a larger view. Ramanujan seems to have known this. Indeed, as he was fond of congruences for the partition function, he was equally enamoured by congruences for the τ-function. Recall that this function is defined by the q-series

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Ramanujan discovered some special congruences for the τ-function. Let $\sigma_k(n) = \sum_{d|n} d^k$. Then,

$$\tau(n) \equiv \sigma_{11}(n) \pmod{2^8},\ n \text{ odd}; \qquad \tau(n) \equiv n^2 \sigma_7(n) \pmod{3^3};$$

$$\tau(n) \equiv n\sigma_9(n) \pmod{5^2}; \qquad \tau(n) \equiv n\sigma_3(n) \pmod{7}; \tag{5}$$

$$\tau(n) \equiv 0 \pmod{23} \text{ if } (n/23) = -1; \quad \tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

Here $(\cdot/23)$ denotes the Legendre symbol.

For Ramanujan, the partition function and the τ-function had some similarities in that both can be viewed as being generated by infinite products of the form

$$\prod_{n=1}^{\infty} (1 - q^n)^{\alpha}$$

with $\alpha = -1$ generating $p(n)$ and $\alpha = 24$ generating $\tau(n)$. Thus, the study of both functions leads one inevitably into the world of modular forms.

3 The Theory of ℓ-Adic Representations

The year 1967 stands out as a Himalayan peak in the landscape of time for number theory because it was then that three transformative ideas emerged. The first is Grothendieck's theory of motives. The second is Langlands program and the third is the development of the theory of ℓ-adic representations that seems to connect both the Langlands program and motivic theory. Though Taniyama discovered the notion of an ℓ-adic representation ten years earlier in 1957, it was only in 1967 that Serre and Deligne recognized its central role in number theory and connected it to the theory of modular forms. Serre's famous book [23] "mon cours à McGill" underlines this realization and his 1967 paper "Une interprétation des congruences relative à la fonction τ de Ramanujan" initiates a new theme of number theory linking congruences and modular forms via the theory of ℓ-adic representations. In fact, it was Serre who first conjectured the existence of a Galois representation ρ_ℓ (for each prime ℓ) such that

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_\ell)$$

with $\mathrm{tr}\,\rho_\ell(\mathrm{Frob}_p) = \tau(p)$ and $\det \rho_\ell(\mathrm{Frob}_p) = p^{11}$, $(p \neq \ell)$ where Frob_p denotes the Frobenius automorphism at the prime p. The existence of this representation was proved by Deligne [3] in 1969.

The primes 2, 3, 5, 7, 23 and 691 are precisely the primes where the representation ρ_ℓ ramifies and this led Swinnerton-Dyer to explain all of Ramanujan's congruences in light of this representation. The remarkable epiphany that was felt by Serre at these developments is evident in the opening sentence of his Séminaire Bourbaki talk describing Swinnerton-Dyer's work. In my free translation, he says "Diverse arithmetic functions are defined as *coefficients* of modular functions. Notably,

$$\tau(n), \quad \text{coef. of } q^n \text{ in } \Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} \quad \text{function of Ramanujan,}$$

$$c(n) \qquad \text{coef. of } q^n \text{ in the modular invariant } j = q^{-1} + 744 + \cdots,$$

$$p(n) \qquad \text{coef. of } q^n \text{ in } \prod_{n=1}^{\infty}(1-q^n)^{-1}$$

$$\sigma_k(n) \qquad \text{coef. of } q^n \text{ in the Eisenstein series } E_{k+1}$$

$$\zeta(-k) \qquad \text{constant term of } 2E_{k+1}(k \text{ odd } > 1).$$

These functions are linked by numerous congruences and it is not possible to summarize them in a single exposition." This opened the door to view all congruences of these functions through the lens of the ℓ-adic representation.

4 The Chebotarev Density Theorem and Divisibility of Fourier Coefficients of Modular Forms

Serre [24] seems to have been the first person to realise that methods of analytic number theory can be applied to the systematic study of congruences. Using the Serre - Deligne ℓ-adic representation attached to any Hecke eigenform and alluded to above, he deftly applied the Chebotarev density theorem to deduce the following theorem (see Theorem 4.7 of [24]):

Theorem 1 *Let f be a modular form of integral weight $k \geq 1$ belonging to a congruence subgroup of $SL_2(\mathbb{Z})$. Suppose that the Fourier coefficients a_n of f belong to the ring of integers O_F of a finite extension F of $\mathbb{Q}$ and let $\mathfrak{m}$ be a non-zero ideal of O_F. Then:*

(i) There exists $\alpha(\mathfrak{m}) > 0$ such that

$$\#\{n \leq x : a_n \not\equiv 0 \pmod{\mathfrak{m}}\} = O(x/\log^{\alpha(\mathfrak{m})} x).$$

(ii) If f is of type (k, ε) on $\Gamma_0(N)$ and an eigenfunction of the Hecke operators T_p ($p \not| N$) and the U_p (for $p|N$) with $a_1 = 1$ and for $\mathfrak{m}$ a prime ideal, then:

(A) if the characteristic of $O_F/\mathfrak{m}$ is not 2, or there exists a $p \not| 2N$ such that $a_p \not\equiv 0 \pmod{\mathfrak{m}}$, then one has an asymptotic development

$$\#\{n \leq x : a_n \not\equiv 0 \pmod{\mathfrak{m}}\} = \frac{x}{\log^{\alpha(\mathfrak{m})} x}(c_0 + c_1/\log x + \cdots)$$

with $0 < \alpha(\mathfrak{m}) < 1$ and $c_0 > 0$.

(B) if the characteristic of $O_F/\mathfrak{m}$ is 2, and if $a_p \equiv 0 \pmod{\mathfrak{m}}$ for all $p \not| 2N$, then there is a $c > 0$ such that

$$\#\{n \leq x : a_n \not\equiv 0 \pmod{\mathfrak{m}}\} \sim cx^{1/2}.$$

In particular, given any modular form f with integer coefficients, we have $a_n \equiv 0 \pmod{m}$ for almost all n (in the sense of natural density for the positive integers). The proof of this theorem involved a combination of the Chebotarev density theorem [16] and the method of contour integration in analytic number theory [13].

5 Ono's Proof of Subbarao's Conjecture

Ono's proof of Subbarao's conjecture uses Serre's theorem in an essential way. The first step is to connect the partition function to modular forms of integral weight. This is easily done by noting that

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{-1} \equiv \prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{2n}} \pmod 2. \tag{6}$$

Recall that the Dedekind η-function

$$\eta(z) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n), \qquad q = e^{2\pi i z},$$

is a modular form of weight $1/2$ on the upper half-plane. In terms of the Dedekind η-function, we find from (6) that

$$\frac{\eta(24z)}{\eta(48z)} = \frac{1}{q}\prod_{n=1}^{\infty}\frac{1-q^{24n}}{1-q^{48n}} \equiv \sum_{n=0}^{\infty} p(n)q^{24n-1} \pmod 2.$$

Observe that the eta-quotient is a modular function (weight zero) with a pole at infinity. Let us now recall Jacobi's triple product identity (see for example, Theorem 1.2.1 of [14]):

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}x)\left(1+\frac{q^{2n-1}}{x}\right) = \sum_{n=-\infty}^{\infty} q^{n^2}x^n. \tag{7}$$

If we change q to q^4 and put $x = q^4$, we find that the product becomes

$$\prod_{n=1}^{\infty}(1-q^{8n})(1+q^{8n})(1+q^{8n-8}).$$

Combining the factors $(1-q^{8n})$ and $(1+q^{8n-8})$, the product reduces to

$$2\prod_{n=1}^{\infty}(1-q^{16n})(1+q^{8n}) = 2\prod_{n=1}^{\infty}\frac{(1-q^{16n})^2}{1-q^{8n}}.$$

Thus, with these substitutions, multiplying through by q we find that the right hand side of (7) is

$$2\sum_{n=0}^{\infty} q^{(2n+1)^2}.$$

Thus, an application of Jacobi's triple product identity shows that

$$\frac{\eta^2(16z)}{\eta(8z)} = q\prod_{n=1}^{\infty}\frac{(1-q^{16n})^2}{1-q^{8n}} = \sum_{n=0}^{\infty} q^{(2n+1)^2}.$$

As $(1-X)^2 \equiv 1 - X^2 \pmod 2$, we have

$$\begin{aligned}\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} &= q\prod_{n=1}^{\infty}\frac{(1-q^n)^{32}}{(1-q^n)^8}\\ &\equiv q\prod_{n=1}^{\infty}\frac{(1-q^{16n})^2}{1-q^{8n}} \equiv \sum_{n=0}^{\infty} q^{(2n+1)^2} \pmod 2.\end{aligned}$$

Incidentally, this congruence shows that $\tau(n)$ is odd if and only if n is an odd square, a key fact used to treat some cases of the Lang-Trotter conjecture by the authors in [15].

Now consider

$$f_{t,A}(z) := \frac{\eta(24z)}{\eta(48z)}\Delta^A(24tz)$$

with $A > t/24$ being a power of 2. As Δ is a cusp form that vanishes at infinity, taking a suitably large A ensures that $f_{t,A}$ is holomorphic at all the cusps. By standard theorems of modular forms, it is quickly verified that $f_{t,A}$ is a cusp form of weight $12A$ and level 1152t and Nebentypus $(2/\cdot)$. Serre's theorem shows that almost all the Fourier coefficients of $f_{t,A}$ are even. We can write

$$f_{t,A}(z) = \sum_{n=0}^{\infty} a_t(n)q^{24n-1} \equiv \left(\sum_{n=0}^{\infty} p(n)q^{24n-1}\right)\left(\sum_{n=0}^{\infty} q^{24At(2n+1)^2}\right) \pmod 2.$$

We can compare coefficients of both sides of this congruence. In particular,

$$a_t(Atj^2+n) \equiv \sum_{i\geq 1, i \text{ odd}} p(Atj^2+n-Ati^2) \equiv \sum_{i\geq 1, i \text{ odd}} p(At(j^2-i^2)+n) \pmod 2.$$

Now suppose that for every $N \geq n_0$ with $N \equiv r \pmod t$, we have $p(N)$ odd. Choosing $n \equiv r \pmod t$ in the above congruence, we find every summand on the right is odd. Choosing $j \equiv 1 \pmod 4$, we can arrange to have an odd number of summands. Thus, the right hand side is odd. In other words, $a_t(Atj^2 + n)$ is odd for every $n \equiv r \pmod t$. This gives a positive density of such numbers which contradicts Serre's theorem. Hence there are infinitely many n with $n \equiv r \pmod t$ for which $p(n)$ is even. This proves the even case of Subbarao's conjecture.

To treat the odd case, we need to recall some general facts about modular forms (see Lemma 2 of [19]). Let N be a natural number. We let $M_k(N)$ denote the space of modular forms of weight k on $\Gamma_1(N)$. If χ is a Dirichlet character modulo N, we denote by $M_k(N, \chi)$ the space of modular forms of weight k on $\Gamma_0(N)$ with Nebentypus χ. We denote by $S_k(N)$ and $S_k(N, \chi)$ the corresponding subspaces of cusp forms in $M_k(N)$ and $M_k(N, \chi)$ respectively.

If $f \in M_k(N, \chi)$ with Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n,$$

we let

$$f_{r,t}(z) := \sum_{n \equiv r \pmod{t}} a(n)q^n.$$

If $d = (r, t)$, then it is known that $f_{r,t} \in M_k(Nt^2/d)$. Thus,

$$f_{24r-1,24t}(z) = \sum_{n \equiv 24r-1 \pmod{24t}} a_t(n)q^n \in S_{12A}(1152t^3/d).$$

It is then easily seen that reducing mod 2, we have

$$f_{24r-1,24t}(z) \equiv \left(\sum_{n \equiv r \pmod{t}} p(n)q^{24n-1} \right) \left(\sum_{n=0}^{\infty} q^{24At(2n+1)^2} \right) \pmod 2.$$

If now $p(n)$ is odd for only finitely many $n \equiv r \pmod t$, then the right hand side of the congruence is of the form

$$\sum_{1 \le i \le s} \sum_{n=0}^{\infty} q^{24At(2n+1)^2 + b_i} \pmod 2, \tag{8}$$

for certain positive integers $b_1, ..., b_s$. The assumption that there exists one $n \equiv r \pmod t$ ensures the right hand side is not identically zero. An easy verification shows that only finitely many of the infinite sequence of exponents on the right hand side can coincide. Thus, infinitely many of the coefficients of $f_{24r-1,24t}$ are non-zero (mod 2). Thus, $\mathrm{ord}_2(f_{24r-1,24t}) < \infty$.

But $f_{24r-1,24t}$ is a cusp form and so for each prime p coprime to the level, we may apply the Hecke operator T_p to it to get another cusp form of the same weight and level. The effect of the Hecke operator on the Fourier coefficients is easily determined and well-known from the following.

Given $F \in M_k(N, \chi)$ with Fourier expansion $F(z) = \sum_{n=0}^{\infty} a(n)q^n$, we have

$$(F|T_p)(z) = \sum_{n=0}^{\infty}(a(pn) + \chi(p)p^{k-1}a(n/p))q^n,$$

where the second term $a(n/p)$ appears only when $p|n$. We will choose below $p \equiv 1 \pmod{N}$ so that $\chi(p) = 1$ and consequently

$$(F|T_p)(z) = \sum_{n=0}^{\infty}(a(pn) + p^{k-1}a(n/p))q^n.$$

Applying this fact to our form $F = f_{24r-1,24t}$ gives

$$(F|T_p)(z) = \sum_{1\le i\le s}\ \sum_{n=0,\, p|24At(2n+1)^2+b_i}^{\infty} q^{(24At(2n+1)^2+b_i)/p}$$

$$+ p^{k-1}\sum_{1\le i\le s}\ \sum_{n=0,\, p|24At(2n+1)^2+b_i}^{\infty} q^{(24At(2n+1)^2+b_i)p} \pmod 2.$$

Again, only finitely many exponents on the right hand side can coincide and if we let $h = \mathrm{ord}_2(F|T_p)$, we must have $h = (24At(2n+1)^2 + b_i)/p$ for some n and i. If $C < p < x$, the number of such pairs is at most $O(\sqrt{x})$. Allowing for all the possible $h < C$, we still get $O(\sqrt{x})$ as an estimate for the number of such primes. But as the only restriction on p is that $p \equiv 1 \pmod{N}$, and the number of such primes $p < x$ is $\gg x/\log x$ by Dirichlet's theorem on primes in arithmetic progressions, we get a contradiction by choosing a suitable p. Thus, this contradicts our hypothesis that $p(n)$ is odd for only finitely many $n \equiv r \pmod{t}$. Notice that we needed the existence of at least one such n_0 so that the right hand side of our congruences does not vanish identically.

In the following sections, we will describe the connection of the partition function to the representation theory of the symmetric group and indicate how this led to further developments ultimately leading to Ramanujan's conjectural theory of mock modular forms.

6 Partitions and Young Diagrams

It is well-known that $p(n)$ is the number of irreducible representations of the symmetric group S_n and one has an elegant description of these representations based on the theory of Young diagrams [8].

Each partition of $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ can be visualised as a "Young diagram" (sometimes called a "Ferrers diagram") as an arrangement of dots with λ_1 dots in the first row, λ_2 dots in the second row and so on:

$$\begin{array}{ll} \bullet\ \bullet \quad \cdots \bullet & \lambda_1 \text{ nodes} \\ \bullet\ \bullet\ \cdots\ \bullet & \lambda_2 \text{ nodes} \\ \vdots\ \ \vdots\ \ \vdots & \\ \bullet \cdots\ \bullet & \lambda_k \text{ nodes} \end{array} \tag{9}$$

To each dot of a Young diagram, we can associate a *hook length* which is the number of dots to the right and below that dot including the dot. There is a canonical construction of the irreducible representation of S_n using this diagram that the reader can find in many books (such as on pages 45–46 of Fulton and Harris [8]). The dimension of the associated irreducible representation turns out to be

$$\frac{n!}{\text{product of all the hook lengths}}.$$

The Durfee square of the diagram (9) is the largest square of nodes in the upper left hand corner of the diagram. This divides the partition into a perfect square and two "smaller" partitions whose number of parts does not exceed the side length of the Durfee square. If we denote by $a_m(n)$ the number of partitions of n that do not exceed m, then the generating function for $a_m(n)$ is clearly

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} = \sum_{n=0}^{\infty} a_m(n) q^n.$$

If $b_m(n)$ is the number of partitions of n with Durfee square of size m, then by similar reasoning,

$$\frac{q^{m^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^m)^2} = \sum_{n=0}^{\infty} b_m(n) q^n.$$

Summing over m gives the combinatorial identity

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^{-1} = 1 + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^m)^2}. \tag{10}$$

The structural elegance of the identity (10) awakens further questioning. Before his arrival in England, Ramanujan had empirically discovered the following varia-

tions of (10):

$$\prod_{n=1}^{\infty}(1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1} = 1+\sum_{m=1}^{\infty}\frac{q^{m^2}}{(1-q)(1-q^2)\cdots(1-q^m)};$$

$$\prod_{n=1}^{\infty}(1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1} = 1+\sum_{m=1}^{\infty}\frac{q^{m(m+1)}}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

The left hand side of the first of these can be seen to be the generating function of partitions using only numbers congruent to 1 or 4 (mod 5) and an analogous assertion holds for the second. Ramanujan had no proofs for either of these but in 1917, while in England and sitting in the library of Trinity College, he stumbled upon some old volumes of the Proceedings of the London Mathematical Society where he found an 1894 paper of Rogers in which these identities were proved. Hardy later recalled, "I can remember very well his surprise and admiration which he expressed for Rogers' work. A correspondence [between them] followed in the course of which Rogers was led to a considerable simplification of his original proof." These identities are now called the Rogers-Ramanujan identities.

7 The Parity of $p(n)$ Revisited

If we reflect the Young diagram along the main diagonal, we obtain another Young diagram whose corresponding partition λ' is called the conjugate partition. If $\lambda = \lambda'$, we say the partition is *self-conjugate*. If we let $p^*(n)$ be the number of self-conjugate partitions of n, we immediately see that $p(n) \equiv p^*(n) \pmod 2$ so that the parity of $p(n)$ is the same as the parity of $p^*(n)$.

The generating function of $p^*(n)$ is easily determined. Given any self-conjugate partition, we can obtain a partition of n into distinct *odd* summands as follows. Looking at the diagonal hook lengths of a self-conjugate diagram, we have $n = \mu_1 + \mu_2 + \cdots + \mu_k$ where μ_i is the hook length of the dot in the (i,i)-th position. Each μ_i is odd and we have $\mu_1 > \mu_2 > \cdots > \mu_k$. Thus, we obtain a partition of n into distinct odd summands. Conversely, given such a partition, we can construct a self-conjugate Young diagram with $(\mu_1 - 1)/2$ dots in the first row and column, and so on. This gives the well-known identity (see page 270 of Hardy and Wright [11]):

$$\sum_{n=0}^{\infty} p^*(n)q^n = \prod_{n=0}^{\infty}(1+q^{2n+1}).$$

An argument analogous to the one we used above to derive (10) gives:

$$\sum_{n=0}^{\infty} p^*(n)q^n = \prod_{n=0}^{\infty}(1+q^{2n+1}) = 1+\sum_{m=1}^{\infty}\frac{q^{m^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2m})},$$

a formula due to Euler (see page 277 of Hardy and Wright [11]). As we are interested in the parity of $p^*(n)$, we find that

$$\sum_{n=0}^{\infty} p^*(n)q^n \equiv f(q) \pmod 2,$$

where

$$f(q) = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^m)^2}$$

is a mock theta function of order 3 that Ramanujan had written down in his last letter to G.H. Hardy (see page xii of [2]). In order to explain the Ramanujan congruences for the partition function, Dyson [6] defined the *rank* of a partition λ to be $\lambda_1 - k$, which is the difference between the largest part and the number of parts. He then defined

$$R(n, m; M) := \#\{\lambda \vdash n : \text{rank}(\lambda) \equiv m \pmod M\}$$

which enumerates the partitions of n whose rank lies in the arithmetic progression $m \pmod M$. In 1944, he conjectured that

$$p(5n+4) = 5R(5n+4, m, 5), \qquad p(7n+5) = 7R(7n+5, m, 7).$$

In other words, these equations make Ramanujan's congruences self-evident and also give combinatorial meaning to them.

Let us write

$$f(q) = 1 + \sum_{n=1}^{\infty} \alpha(n)q^n.$$

If $N_e(n)$ (resp. $N_o(n)$) is the number of partitions of n with even rank (resp. odd rank) then,

$$\alpha(n) = N_e(n) - N_o(n),$$

which alternate in sign [1]. This becomes evident from the rank generating function: if we let $N(m, n)$ be the number of partitions of n with rank m, then (see page 229 of [2])

$$\sum_{m,n\geq 0} N(m, n)\zeta^m q^n = (1-\zeta)\left(\prod_{n=1}^{\infty}(1-q^n)^{-1}\right)\left(\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n(3n+1)/2}}{1-\zeta q^n}\right).$$

This can be seen as a generalization of the Euler pentagonal number theorem which corresponds to the case $\zeta = 0$.

Is $R(n, m; 5)$ random (mod 5)? Is $R(n, m; 7)$ random (mod 7)? Bringmann [2] has proved recently that this is so.

8 A Formula for $p^*(n)$

Given a partition λ of n, we denote by $\ell(\lambda)$ its number of parts and call it the length. The weight $w(\lambda)$ is simply n. The generating function for these two functions is simply

$$F(q, u) := \prod_{n=1}^{\infty}(1 - uq^n)^{-1} = \sum_{\lambda} u^{\ell(\lambda)} q^{w(\lambda)}.$$

Putting $u = -1$, we get

$$\prod_{n=1}^{\infty}(1 + q^n)^{-1} = \sum_{n=0}^{\infty}(p_e(n) - p_o(n))q^n, \tag{11}$$

where $p_e(n)$ is the number of partitions of n with even length and $p_o(n)$ is the number with odd length. On the other hand,

$$\prod_{n=1}^{\infty}(1 + q^n)^{-1} = \prod_{n=1}^{\infty}\frac{1 - q^n}{1 - q^{2n}}$$

and the right hand side can be viewed as "sieving out" even exponents from the numerator. Thus,

$$\prod_{n=1}^{\infty}(1 + q^n)^{-1} = \prod_{n=0}^{\infty}(1 - q^{2n+1}).$$

Changing q to $-q$ gives

$$\prod_{n=0}^{\infty}(1 + q^{2n+1}) = \prod_{n=1}^{\infty}(1 + (-q)^n)^{-1},$$

so that we get on comparing with (11):

Theorem 2

$$p^*(n) = (-1)^n\{p_e(n) - p_o(n)\}.$$

We would expect $p_e(n) \sim p_o(n) \sim \frac{1}{2}p(n)$. This is indeed the case as we will show in the next section. But $p^*(n)$ is substantially smaller.

9 Hardy-Ramanujan Asymptotics

As is well-known, Hardy and Ramanujan [10] derived the asymptotic formula for $p(n)$ in a fundamental paper that formally introduced the circle method to the world. They proved that

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}, \tag{12}$$

as n tends to infinity. In the same paper, at the end, they remark that one can derive by the circle method the asymptotic formula (see page 304 of [22])

$$p^*(n) \sim \frac{d}{dn} J_0\Big(i\pi\sqrt{\frac{1}{6}(n-\frac{1}{24})}\Big),$$

where J_0 is the classical Bessel function. A more amenable asymptotic was derived by Hagis [9]:

$$p^*(n) \sim \frac{\sqrt{6}}{(24n)^{3/4}} \exp\left(\frac{\pi}{12}\sqrt{24n-1}\right).$$

By way of comparison, we have for a suitable constant $C > 0$,

$$p^*(n)/p(n) \sim Cn^{1/4}\exp(-\sqrt{n/6}).$$

In other words, the proportion of self-conjugate partitions is $O(e^{-c\sqrt{n}})$ for some $c > 0$ which goes to zero as n tends to infinity.

10 Erdös's Elementary Proof of the Hardy-Ramanujan Formula

In [5], Dewar and I gave a new proof of (12) using an algebraic formula due to Bruinier and Ono. This proof avoids the circle method but cannot be deemed "elementary." What is perhaps not so well-known is that Erdös [7] gave an elementary proof of the Hardy-Ramanujan theorem using a simple recursion. This recursion is derived as follows. We list all the partitons of n:

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k.$$

There are $p(n)$ such formulas and so the tally from the left hand side is $np(n)$. To tally the right hand side, we count how often a given h appears. Clearly, the number of partitions in which h appears exactly once is $p(n-h)-p(n-2h)$. The number in which h appears exactly twice is $p(n-2h)-p(n-3h)$ and so on. Thus, the contribution of h to all the right hand sides is

$$h[p(n-h)-p(n-2h)]+2h[p(n-2h)-p(n-3h)]+\cdots=\sum_{hk\leq n} hp(n-hk).$$

Summing over all h gives:

Theorem 3

$$np(n)=\sum_{hk\leq n} hp(n-hk)=\sum_{r\leq n}\sigma(r)p(n-r),$$

where $\sigma(r)$ is the sum of the divisors of r.

Erdös [7] used this recursion to give an elementary proof of the Hardy-Ramanujan formula.

11 Subbarao's Conjectures on $s_r(n)$

At the end of his paper, Subbarao [25] introduced a new function $s_r(n)$ which he defined as

$$s_r(n):=\sum_{\lambda\vdash n}\ell(\lambda)^r.$$

In other words, $s_r(n)$ is the sum of the r-th power of the lengths of all the partitions. In particular, $s_0(n)=p(n)$ and wrote $s(n)$ for $s_1(n)$. He noted that one could derive generating functions for each $s_r(n)$ and wrote these down in the cases $r=1$ and $r=2$. In fact, recall the function $F(q,u)$ introduced earlier:

$$F(q,u):=\prod_{n=1}^{\infty}(1-uq^n)^{-1}=\sum_{\lambda}u^{\ell(\lambda)}q^{w(\lambda)}.$$

Differentiating with respect to u gives,

$$F_u(q,u):=\frac{\partial}{\partial u}F(q,u)=\sum_{\lambda}\ell(\lambda)u^{\ell(\lambda)-1}q^{w(\lambda)},$$

so that setting $u = 1$ gives the generating function of $s(n)$:

$$\frac{\partial}{\partial u} F(q,u)\Big|_{u=1} = \sum_{n=1}^{\infty} s(n) q^n.$$

On the other hand, differentiating with respect to u, the logarithm

$$\log F(q,u) = -\sum_{n=1}^{\infty} \log(1-uq^n) = \sum_{m,n\geq 1} \frac{u^m q^{mn}}{m},$$

gives:

$$\frac{F_u(q,u)}{F(q,u)} = \sum_{m,n\geq 1} u^{m-1} q^{mn}$$

so that setting $u = 1$ gives

$$\sum_{n=1}^{\infty} s(n) q^n = \left(\sum_{n=0}^{\infty} p(n) q^n\right)\left(\sum_{n=1}^{\infty} d(n) q^n\right), \tag{13}$$

where $d(n)$ is the number of positive divisors of n. We therefore get:

Theorem 4

$$s(n) = \sum_{m=1}^{n} p(n-m) d(m).$$

Since the divisor function $d(m)$ is odd if and only if m is a perfect square, we deduce:

Corollary 1

$$s(n) \equiv \sum_{j=1}^{[\sqrt{n}]} p(n-j^2) \pmod 2.$$

But from (13), we have

$$\left(\sum_{n=1}^{\infty} s(n) q^n\right) \prod_{n=1}^{\infty} (1-q^n) = \sum_{n=1}^{\infty} d(n) q^n,$$

so that by Euler's pentagonal theorem, we get the following remarkable formula:

Theorem 5 (Subbarao [25])

$$\sum_k (-1)^k s\left(n - \frac{k(3k \pm 1)}{2}\right) = d(n),$$

where the sum on the left is over those positive k for which the argument is non-negative.

The formula is beautiful in that it connects the divisor function and the partition function. Subbarao uses it to deduce that $s(n)$ is even (resp. odd) for infinitely many values of n by noting that by choosing n to be a perfect square, we have

$$\sum_k s\left(n - \frac{k(3k \pm 1)}{2}\right) \equiv 1 \pmod 2.$$

He then adapted the proof of Kolberg to deduce the result. He also conjectured that $s(An + B)$ is even (resp. odd) for infinitely many values of n. In fact, he conjectured this for every $s_r(An + B)$, but it suffices to solve this for $s(An + B)$ since $s_r(n) \equiv s_1(n) \pmod 2$ for every r. This opens the door for further research.

Acknowledgments I thank Sonika Dhillon, Anup Dixit, Rashi Lunia, Siddhi Pathak, Akshaa Vatwani and the referee for their careful reading and corrections of an earlier draft of this paper.

Research of the author was partially supported by an NSERC Discovery grant.

References

1. G.E. Andrews and R. Lewis, The ranks and cranks of partitions modulo 2, 3 and 4, Journal of Number Theory, **85** (2000), no. 1, 74–84.
2. K. Bringmann, A. Folsom, K. Ono, and L. Rolen, Harmonic Maass Forms and Mock Modular Forms: Theory and Applications, American Mathematical Society Colloquium Publications, Volume 64, American Mathematical Society, Providence, Rhode Island, 2017.
3. P. Deligne, Formes modulaires et représentations ℓ-adiques, Séminaire Bourbaki, exposé 355 (février 1969), Lecture Notes **179**, Springer-Verlag, 1971.
4. P. Deligne and M. Rapoport, Les schémas de modules de courbes elliptiques, in: Modular functions of one variable II, P. Deligne and W. Kuyk, eds., Lect. Notes Math. 349, Springer-Verlag, Berlin (1973), 143–316.
5. M. Dewar and M. Ram Murty, A derivation of the Hardy-Ramanujan formula from an arithmetic formula, Proceedings of the American Math. Society, **141** (2013), no. 6, 1903–1911.
6. F. Dyson, Some guesses in the theory of partitions, Eureka, **8** (1944), 10–15.
7. P. Erdös, On an elementary proof of some asymptotic formulas in the theory of partitions, Annals of Math., **43** (1942), 437–450.
8. W. Fulton and J. Harris, Representation Theory, A First Course, Springer, 1991.
9. P. Hagis Jr., Partitions into odd and unequal parts, American Journal of Mathematics, **86**(1964), No. 2, 317–324.
10. G.H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, Proceedings of the London Mathematical Society, **17** no. 2, (1918), 75–115.

11. G. H. Hardy and E.M. Wright, An introduction the theory of numbers, Fifth edition, Clarendon Press, Oxford, 1979.
12. O. Kolberg, Note on the parity of the partition function, Math. Scand. **7** (1959), 377–378.
13. M. Ram Murty, Problems in Analytic Number Theory, Graduate Texts in Mathematics, **206**, Second Edition, Springer, 2008.
14. M. Ram Murty, M. Dewar, and H. Graves, Problems in the Theory of Modular Forms, Institute of Mathematical Sciences Lecture Notes 1, Hindustan Book Agency, 2015.
15. M. Ram Murty and V. Kumar Murty, Odd values of Fourier coefficients of certain modular forms, International Journal of Number Theory, **3** (2007), no. 3, 455–470.
16. M. Ram Murty, V. Kumar Murty and N. Saradha, Modular forms and the Chebotarev density theorem, American Journal of Mathematics, **110** (1988), 253–281.
17. M. Newman, Periodicity modulo m and divisibility properties of the partition function, Transactions of the American Math. Society, **97** (1960), 225–236.
18. K. Ono, On the parity of the partition function in arithmetic progressions, J. reine angew. Math. **472** (1996), 1–15.
19. K. Ono, Parity of the partition function, Electronic Research Announcements of the American Mathematical Society, Volume 1, Issue 1, 1995.
20. T. R. Parkin and D. Shanks, On the distribution of parity in the partition function, Math. Comp. 21 (1967), 466–480.
21. C.-S. Radu, A proof of Subbarao's conjecture, J. reine angew. Math. 672 (2012), 161–175.
22. S. Ramanujan, Collected Papers, edited by G.H. Hardy, P.V. Seshu Aiyar and B.M. Wilson, AMS Chelsea Publishing, American Mathematical Society, Providence, Rhode Island, 1962.
23. J.-P. Serre, Abelian ℓ-adic representations and elliptic curves, W.A. Benjamin Inc., New York, Amsterdam, 1968.
24. J.-P. Serre, Divisibilité de certaines fonctions arithmétiques, L. Enseignement Math., **22** (1976), 227–260.
25. M. V. Subbarao, Remarks on the partition function, Amer. Math. Monthly 73 (1966), no. 8, 851–854.