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ON THE TRANSCENDENCE OF CERTAIN PETERSSON INNER PRODUCTS

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To Professor Freydoon Shahidi, on the occasion of his 70th birthday

Abstract. We show that for all normalized Hecke eigenforms f with weight one and of CM type, the number (f, f) where $(\cdot, \cdot)$ denotes the Petersson inner product, is a linear form in logarithms and hence transcendental.

Keywords: CM modular forms, Petersson inner product, transcendence.
MSC(2010): Primary: 11J89; Secondary: 11M35.

1. Introduction

In earlier works [2, 3], we studied the value $L(1, \chi)$ with χ being a character of the ideal class group (or more generally of any ray class group) of an imaginary quadratic field. We showed that the value $L(1, \chi)/\pi$ is a Baker period, that is, an algebraic linear combination of logarithms of algebraic numbers. By Baker's theorem and the non-vanishing of $L(1, \chi)$, we deduced that $L(1, \chi)/\pi$ is transcendental. Presumably, $L(1, \chi)$ is itself transcendental but this is not known at present. If one assumes the celebrated Schanuel's conjecture (or even the weak Schanuel conjecture), this would follow. Recall that Schanuel's conjecture predicts that if $x_1, ..., x_n$ are complex numbers that are linearly independent over the rationals, then the transcendence degree of the field

$$\mathbb{Q}(x_1, ..., x_n, e^{x_1}, ..., e^{x_n})$$

is at least n. For many applications, the weak Schanuel conjecture (which is still open) will be sufficient to deduce transcendence. This says that if $\alpha_1, ..., \alpha_n$ are non-zero algebraic numbers such that

$$\log \alpha_1, ..., \log \alpha_n$$

are linearly independent over $\mathbb{Q}$, then they are algebraically independent. This conjecture was formulated in [1].

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In this paper, we study $L(s, \mathrm{Sym}^2(f))$ where f is a normalized Hecke eigenform of CM type, and $\mathrm{Sym}^2(f)$ denotes the symmetric square of the Galois representation associated to f. Our main theorem is:

Theorem 1.1. *If f is a normalized Hecke eigenform of CM type and of level N and weight 1, then (f, f) is a non-zero Baker period, and hence transcendental. (Here $(\cdot, \cdot)$ denotes the Petersson inner product.)*

In [1], the authors studied the transcendence of the Petersson inner product in some cases. They showed (see [1, Theorem 6.1]) that if f is a normalized cuspidal eigenform on $\Gamma_0(N)$ with weight one, with rational Fourier coefficients, then, (f, f) is transcendental under the weak Schanuel conjecture.

This theorem is now unconditional if f has CM-type by our main theorem. Moreover, we do not assume anything about the rationality of the Fourier coefficients of f.

The transcendental nature of Petersson inner products is an interesting topic in its own right. Indeed, the study is intimately related to the transcendence of $\zeta(3)$. Curiously, in 1949, Petersson [4] showed that

$$\zeta(3) = \frac{\pi}{7}(\vartheta_3^8, \vartheta_0^4\vartheta_2^4),$$

where

$$\vartheta_0(z) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i n^2 z},$$

$$\vartheta_2(z) = \sum_{n=-\infty}^{\infty} e^{\pi i (n+1/2)^2 z},$$

and

$$\vartheta_3(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z},$$

are the classical theta functions. Here, it is to be noted that the inner product of two modular forms can be defined via the usual integral formula as long as one of them is a cusp form and this is the case with ϑ_3.

2. **Proof of main theorem**

Recall that a newform

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

of level N and weight 1 is said to have CM if there is an imaginary quadratic number field K such that $a_p = 0$ for all primes p which are inert in K. It is well-known (see [5]) that there is a Hecke character χ of finite order, such that

$$L(s, f) = L(s, \chi).$$

If we write

$$a_p = \alpha_p + \beta_p$$

then

$$L(s, \mathrm{Sym}^2(f)) = \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p \beta_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p^2}{p^s}\right)^{-1}.$$

If π_f is the automorphic representation attached to f, then the Rankin-Selberg L-function, $L(s, \pi_f \otimes \pi_f)$ factors as $\zeta(s)L(s, \mathrm{Sym}^2(f))$. Since the Rankin-Selberg L-function has a simple pole at $s = 1$ with residue

$$\frac{2\pi^2}{N^2}\phi(N)(f, f)$$

(see for example, [6, p. 90]), we deduce that

$$L(1, \mathrm{Sym}^2(f)) = \frac{2\pi^2}{N^2}\phi(N)(f, f). \tag{2.1}$$

On the other hand, it is not difficult to see that

$$L(s, \pi_f \otimes \pi_f) = \zeta(s)L(s, \chi^2)L(s, \chi_D),$$

where χ_D is the quadratic character associated to the CM field K. Indeed, if p is inert in K, then $\{\alpha_p, \beta_p\} = \{1, -1\}$ so that

$$(\alpha_p^2, \alpha_p\beta_p, \alpha_p\beta_p, \beta_p^2) = (1, -1, -1, 1).$$

If p splits in K as $p = \mathfrak{p}\mathfrak{p}'$, then

$$(\alpha_p^2, \alpha_p\beta_p, \alpha_p\beta_p, \beta_p^2) = (\chi(\mathfrak{p})^2, 1, 1, \chi(\mathfrak{p}')^2).$$

Grouping the components for the symmetric square L-function, the zeta function and $L(s, \chi_D)$ gives us the result. It is now evident that

$$L(1, \mathrm{Sym}^2(f)) = L(1, \chi^2)L(1, \chi_D).$$

By Dirichlet's class number formula, we have

$$L(1, \chi_D) = \frac{2\pi h}{w\sqrt{|D|}}$$

together with the results from [2] and [3], we deduce from (2.1) the desired result.

3. Concluding remarks

We expect that in general, if f is a normalized Hecke eigenform of weight k and level N, then (f, f) is a transcendental number. This paper gives some evidence for the truth of this general conjecture. Perhaps more is true. If f and g are such that they have algebraic Fourier coefficients and at least one of them is a cusp form, is it true that (f, g) is transcendental? Petersson's example for $\zeta(3)$ certainly falls into this rubric.

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