Some Remarks on the Riemann Hypothesis*

M. Ram Murty

1 Pólya and Turán conjectures

The Liouville function $\lambda(n)$ is defined as $(-1)^{\Omega(n)}$ where $\Omega(n)$ is the total number of prime factors of n counted with multiplicity. It is a completely multiplicative function and it is easy to see that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} \tag{1.1}$$

for $\mathrm{Re}(s) > 1$. If we define

$$S(x) := \sum_{n \le x} \lambda(n) \tag{1.2}$$

then, by partial summation, we have

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = s \int_1^{\infty} \frac{S(t)}{t^{s+1}} dt. \tag{1.3}$$

Based on numerical data, Pólya [Po] conjectured that

$$S(x) \le 0$$

for all $x \ge 2$. It should be noted that Pólya's conjecture implies the Riemann hypothesis. Indeed, by a well-known theorem of Landau, the integral expression in (1.3) converges to the right of $\mathrm{Re}(s) > \sigma_0$ where σ_0 is the first real singularity of $\zeta(2s)/\zeta(s)$. For Landau's theorem, see for example, [EM, Theorem 10.4.2, p. 132], where the proof is given for Dirichlet series with non-negative coefficients. However, the proof also works, *mutatis mutandis*, for Dirichlet integrals of the form

$$\int_1^{\infty} \frac{S(t)}{t^{s+1}} dt,$$

*Research partially supported by NSERC and a Killam Research Fellowship.

where $S(t)$ is of fixed sign for t sufficiently large. In the case under discussion, $\zeta(s)$ has no real zeros in $1/2 \le s \le 1$, and so the first real singularity is at $s = 1/2$ coming from the pole of $\zeta(2s)$ on the numerator. Thus, $\zeta(2s)/\zeta(s)$ is regular for $\mathrm{Re}(s) > 1/2$ which implies that there are no zeros of $\zeta(s)$ in $\mathrm{Re}(s) > 1/2$ since $\zeta(2s)$ is regular and non-vanishing in that region.

Even if we have $S(x) \le 0$ for x sufficiently large, a similar argument allows us to deduce the Riemann hypothesis. Unfortunately, Haselgrove [Ha] has shown that $S(x)$ changes sign infinitely often and so the Pólya conjecture is false. The smallest counterexample is $x = 906,150,257$ for which $S(x) = 1$.

It is to be noted that the estimate

$$S(x) = O(x^{1/2+\epsilon}) \tag{1.4}$$

for any $\epsilon > 0$ (where the implied constant may depend on ϵ would also allow us to deduce the Riemann hypothesis. Indeed, (1.4) implies that the integral expression in (1.3) is regular for $\mathrm{Re}(s) > 1/2$. Thus, $\zeta(2s)/\zeta(s)$ is regular in that half-plane and by the same reasoning, we deduce the Riemann hypothesis. In fact, it is not hard to show that (1.4) is equivalent to the Riemann hypothesis.

Our goal in this paper is to formulate automorphic generalizations of the Pólya conjecture and (1.4) and then investigate when we can expect them to be true.

There is a related conjecture of Turán [T], namely that the sum

$$\sum_{n\le x} \frac{\lambda(n)}{n} \ge 0$$

for x sufficiently large. This too has been disproved by Haselgrove [H]. Below, we shall also investigate modular analogues of the Turán conjecture. In an appendix by Nathan Ng, we present some numerical evidence related to the modular versions of the Pólya and Turán conjectures.

Acknowledgement I would like to thank Michael Rosen for his comments on preliminary version of this paper. I also thank Nathan Ng for doing the computations recorded in the Appendices.

2 Modular analogues of Pólya's conjecture

Let f be a normalized eigenform of weight k and level N and trivial nebentypus. Let us write

$$f(z) = \sum_{n=1}^{\infty} a_f(n) n^{\frac{k-1}{2}} e(nz)$$

where $e(z) = e^{2\pi i z}$, as usual. Then,

$$a_f(m)a_f(n) = \sum_{d|m,n} a_f(mn/d^2).$$

It is easy to prove the following:

Lemma 2.1 *Let*

$$F(m,n) = \sum_{d|m,n} G(m/d, n/d).$$

Then

$$G(m,n) = \sum_{d|m,n} \mu(d) F(m/d, n/d)$$

and conversely.

We can apply Lemma 2.1 to deduce that

$$a_f(mn) = \sum_{d|m,n} \mu(d) a_f(m/d) a_f(n/d). \tag{2.1}$$

Now, let us observe that from (1.1),

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Then,

$$\sum_{n=1}^{\infty} a_f(n^2)/n^{2s} = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \Big(\sum_{d|n} \lambda(d) \Big)$$

by (2.2). Interchanging summations, using (2.1) and observing that λ is completely multiplicative, we find that

$$\sum_{n=1}^{\infty} a_f(n^2)/n^{2s} = \frac{1}{\zeta(2s)} L(s,f) L(s, f\lambda) \tag{2.3}$$

where $L(s,f)=\sum_{n=1}^{\infty} a_f(n)/n^s$ and $L(s,f\lambda)=\sum_{n=1}^{\infty} a_f(n)\lambda(n)/n^s$. Since

$$L(s,f)L(s,f\lambda)=L(2s,\mathrm{Sym}^2(f))/\zeta(2s), \tag{2.4}$$

as is easily seen by examining Euler factors, we deduce the identity

$$\zeta^2(s)\sum_{n=1}^{\infty} a_f(n^2)/n^s = L(s,\mathrm{Sym}^2(f)), \tag{2.5}$$

which is of independent interest. Thus, from the previous equation, we have

$$L(s,f\lambda)=\frac{L(2s,\mathrm{Sym}^2(f))}{\zeta(2s)L(s,f)}. \tag{2.6}$$

Now suppose that $a_f(n)$ are real and consider the hypothesis

$$\sum_{n\le x} a_f(n)\lambda(n)\ge 0. \tag{2.7}$$

Then, writing the left hand side of (2.6) as an integral via partial summation, we find that the right hand side of (2.6) converges for $\mathrm{Re}(s)>\sigma_0$ where σ_0 is the first real singularity of $L(2s,\mathrm{Sym}^2(f))/\zeta(2s)L(s,f)$. Since $L(s,f)$ has infinitely many zeros on $\mathrm{Re}(s)=1/2$, and because $L(2s,\mathrm{Sym}^2(f))/\zeta(2s)$ doesn't vanish in the half-plane $\mathrm{Re}(s)>1/2$, we deduce that this singularity must occur in the half-plane $\mathrm{Re}(s)\ge 1/2$. This leads to:

Theorem 2.2 *Suppose that $L(s,f)\neq 0$ for $1/2<s\le 1$ and that*

$$\mathop{ord}_{s=1/2} L(s,f)\le 1.$$

Then,

$$S_f(x):=\sum_{n\le x} a_f(n)\lambda(n)$$

changes sign infinitely often.

Proof Let us first consider the case $L(1/2,f)\neq 0$. If $S_f(x)$ is of constant sign for x sufficiently large, then

$$L(2s,\mathrm{Sym}^2(f))/\zeta(2s)L(s,f)$$

is regular for $\mathrm{Re}(s)>\alpha$ where α is the first real singularity of the right hand side of (2.6). By hypothesis, $L(s,f)$ does not vanish for any real s between $1/2$ and 1. Also, $\zeta(2s)$ has no real zeros between $1/4$ and 1 and the numerator is regular by a celebrated theorem of Shimura [Sh]. Thus, the right hand side of (2.6) is regular for $\mathrm{Re}(s)>\alpha$ with $\alpha<1/2$. We also know that $L(2s,\mathrm{Sym}^2(f))$ does not vanish on $\mathrm{Re}(s)=1/2$. Thus $L(s,f)$ has no zeros for $\mathrm{Re}(s)\ge 1/2$ which is a contradiction. This deals with the case $L(1/2,f)\neq 0$. If now, $L(1/2,f)=0$, and $s=1/2$ is a simple zero, then $\zeta(2s)L(s,f)$ is non-zero at $s=1/2$. Thus, $L(2s,\mathrm{Sym}^2(f))/\zeta(2s)L(s,f)$ is regular for $\mathrm{Re}(s)\ge 1/2$. But this is a contradiction since $L(s,f)$ has infinitely many zeros on $\mathrm{Re}(s)=1/2$.

□

It is easy to give examples of f which satisfy the hypothesis of Theorem 2.2.

Thus, the modular analogue of Pólya's conjecture is false in general. A necessary condition for it to be true is that $L(1/2,f)=0$ for then the right hand side of (2.6) will have a singularity at $s=1/2$.

It is quite possible that if E is an elliptic curve with large Mordell-Weil rank, then

$$S_E(x)=\sum_{n\le x} a(n)\lambda(n)/\sqrt{n}\ge 0$$

for all x sufficiently large.

Gonek [Go] and Hejhal [He] have independently conjectured that for Riemann zeta function, we should have

$$\sum_{|Im(\rho)|\le T}\frac{1}{|\zeta'(\rho)|^2}\ll T \tag{2.8}$$

where the summation is over zeros of the zeta function. If we suppose that all the zeros of $L(s,f)$ are simple (apart from the zero at $s=1/2$), then the analogue of the above is

$$\sum_{0<|Im(\rho)|\le T}\frac{1}{|L'(\rho)|^2}\ll T. \tag{2.9}$$

Murty and Perelli [MP] have shown that almost all zeros of $L(s,f)$ are simple if we assume the Riemann hypothesis for $L(s,f)$ and the pair correlation conjecture for it. For the discussion below, we do not need an estimate as strong as the above estimate. If r is the order of the zero at $s=1/2$, what is actually needed is that the order of every zero on the critical line have order $\le r-1$ and one would need a similar estimate for

$$\sum_{0<|Im(\rho)|<T}|\mathop{\mathrm{Res}}_{s=\rho}\frac{1}{L(s,f)}|^2\ll T. \tag{2.10}$$

In fact, one can prove the following.

Theorem 2.3 *Assume the Riemann hypothesis for $L(s,f)$ and suppose that $L(s,f)$ has a zero at $s = 1/2$ of order r. Suppose further that all zeros of $L(s,f)$ on $\mathrm{Re}(s) = 1/2$ are of order $\leq r-1$ apart from $s = 1/2$ and that the analogue of (2.10) is satisfied. Then,*

$$\sum_{n\leq x} a_f(n)\lambda(n) = x^{1/2}p_{r-2}(\log x) + O(x^{1/2}(\log x)^{3/2})$$

where p_{r-1} is a polynomial of degree $r-2$.

Here is an indication of the proof. For the sake of simplicity we shall suppose all zeros of $L(s,f)$ apart from $s = 1/2$ are simple. The sum

$$\sum_{n\leq x} a_f(n)\lambda(n)$$

can be written for $c > 1$,

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{L(2s,\mathrm{Sym}^2(f))x^s ds}{s\zeta(2s)L(s,f)} + O(x^c/T)$$

by Perron's formula. We will choose $T = T_j$ with $T_j \to \infty$ along an appropriate sequence that doesn't coincide with any ordinate of a zero of $L(s,f)$. Moving the line of integration to the left and picking up the residues arising from the zeros of $L(s,f)$, we obtain

$$S_f(x) = x^{1/2}p_{r-2}(\log x) + \sum_{|Im(\rho)|<T} \frac{L(2\rho,\mathrm{Sym}^2(f))}{\rho\zeta(2\rho)L'(f,\rho)} + \frac{1}{2\pi i}\int_C \frac{L(2s,\mathrm{Sym}^2(f))x^s}{s\zeta(2s)L(s,f)}$$

where C denotes the semi-rectangular path beginning at $c+iT_j$ to $a+iT_j$ and then to $a-iT_j$ ending at $c-iT_j$. The horizontal and vertical integrals are easily estimated by the functional equation. For the sum over zeros one can use

$$\sum_{0<|Im(\rho)|<T} \frac{1}{|L'(\rho,f)|^2} \ll T$$

or the more general (2.10), which is a modular analogue of a conjecture of Gonek [Go]. Breaking up the sum over the zeros into dyadic intervals of type $[U, 2U]$ we obtain an error term of

$$O(x^{1/2}(\log x)^{3/2}).$$

□

3 Modular analogues of the Turán conjecture

If we expect that

$$S_f(x) = \sum_{n\leq x} a_f(n)\lambda(n) \sim cx^{1/2}(\log x)^{r-2}$$

for $r \geq 4$, then by partial summation we deduce that

$$\sum_{n\leq x} \frac{a_f(n)\lambda(n)}{\sqrt{n}} = \int_1^x \frac{S_f(t)dt}{t^{3/2}} \sim c(\log x)^{r-1}$$

as $x \to \infty$, for some constant $c > 0$, so that the sums

$$T_f(x) = \sum_{n\leq x} \frac{a_f(n)\lambda(n)}{\sqrt{n}} \geq 0$$

for sufficiently large x. Unlike the Turán case, these sums are not partial sums of the corresponding series at the *edge* of the critical strip. They have the disadvantage of being the partial sums of the series at the center of the critical strip. It is not difficult to show that these series actually converge at the center of the critical strip (see for example, [KM]).

Thus, we see that if the modular analogue of the Pólya conjecture is true, then so is the modular analogue of the Turán conjecture.

4 Automorphic analogues

Let $L(s,\pi)$ be an automorphic L-function on $\mathrm{GL}(n)$. If π is self-dual, then it is reasonable to ask if

$$S_\pi(x) = \sum_{n\leq x} a_n(\pi)\lambda(n) \geq 0.$$

Certainly the Riemann hypothesis for $L(s,\pi)$ follows from

$$S_\pi(x) = O(x^{1/2+\epsilon})$$

since an easy calculation shows that

$$L(s,\pi)L(s,\pi\otimes\lambda) = \prod_p \prod_{i=1}^d (1-\alpha_{p,i}^2 p^{-2s})^{-1}.$$

The above reasoning suggests that if there is a high-order zero at $s = 1/2$, then the analogue of the Pólya conjecture should be true for a function which is "primitive" in the sense of Selberg. It would be interesting to test the conjecture for automorphic forms of higher dimension.

5 Certain sums of Fourier coefficients

In this section and the next, we indicate an approach to proving a quasi-Riemann hypothesis. To this end, we will need some estimates on averages of Fourier coefficients of modular forms. We use the notation $m \sim M$ to mean $M \leq m \leq 2M$. We will need to consider sums of the form

$$\sum_{m\sim M} a_f(mj)$$

for j fixed. We will prove that

Theorem 5.1 *We have*

$$\sum_{m\sim M} a_f(mj) = O(M^{1/3}j^{\epsilon}),$$

where the implied constant is independent of M.

Proof We have

$$\begin{aligned}\sum_{m\sim M} a_f(mj) &= \sum_{m\sim M}\sum_{d|m,j} \mu(d)a_f(m/d)a_f(j/d)\\ &= \sum_{d|j}\mu(d)a_f(j/d)\sum_{t\sim M/d} a_f(t)\end{aligned}$$

and the inner sum is by an estimate of Rankin [Ra], $O((M/d)^{1/3})$ from which we easily deduce the stated estimate.

□

The interest in knowing the asymptotics of such sums is due to the following:

Theorem 5.2 *Suppose that*

$$\sum_{k<X}\Big(\sum_{d|k,d\leq V}\mu(d)\Big)a_f(mk) = O(X^{1/2}m^{\epsilon}V^{\epsilon})$$

then $L(s,f)$ has no zeros for $\mathrm{Re}(s) > 3/4$.

Remark We say a few words about the hypothesis in Theorem 5.2. Firstly, if $V = 1$, then the hypothesis holds by Theorem 5.1. If V is bounded then the same is true. If $V = X$, then the sum is just $a_f(m)$ which is clearly m^{ϵ}. If we write $k = dt$ in the inner sum and interchange the sums, we

can estimate the inner sum by Theorem 5.1 to get an upper bound of $O(m^{\epsilon}X^{1/3}V^{2/3})$. This means that the hypothesis is satisfied for $V \leq X^{1/4}$. In fact, if even we can replace the above upper bound by $O(m^{\epsilon}X^{1/3}V^{2/3-\delta})$ for some small $\delta > 0$, then we will be able to deduce some quasi-Riemann hypothesis for $L(s,f)$. Thus, the hypothesised estimate (which can be viewed as a generalization of Theorem 5.1) seems to lie deeper. We make some further remarks about it in the final section.

6 Proof of Theorem 5.2

We will apply the method of Vaughan to study sums of the form

$$\sum_{n\leq x} a(n)\lambda(n)$$

where $a(n) = a_f(n)$. Vaughan's identity can be stated in the following way. It is based on the formal identity:

$$\begin{aligned}A/B &= (1-BG)(A/B) + AG\\ &= (F + (A/B - F))(1-BG) + AG\\ &= F + AG - BFG + (A/B - F)(1-BG).\end{aligned}$$

Suppose now we are given two Dirichlet series

$$A(s) = \sum_{n=1}^{\infty}\frac{a(n)}{n^s}, \quad B(s) = \sum_{n=1}^{\infty}\frac{b(n)}{n^s},$$

and write

$$\frac{A(s)}{B(s)} = \sum_{n=1}^{\infty}\frac{c(n)}{n^s}.$$

Set

$$F(s) = \sum_{n\leq U} c(n)/n^s, \quad G(s) = \sum_{n\leq V}\tilde{b}(n)/n^s.$$

Then, we have

$$c(n) = a_1(n) + a_2(n) + a_3(n) + a_4(n)$$

where

$$\begin{aligned} a_1(n) &= c(n) \quad \text{for } n \le U \\ &= 0 \quad \text{otherwise} \\ a_2(n) &= \sum_{d|n, d \le V} a(n/d)\tilde{b}(d) \\ a_3(n) &= -\sum_{et=n, e \le U} c(e)\Big(\sum_{df=t, f\le V} b(d)\tilde{b}(f)\Big) \\ a_4(n) &= -\sum_{de=n, d>U, e>V} c(d)\Big(\sum_{rs=e, s\le V} b(r)\tilde{b}(s)\Big) \end{aligned}$$

which is the essence of Vaughan's identity. In the case of interest, $A(s) = \zeta(2s)$ and $B(s) = \zeta(s)$ so that

$$\lambda(n) = a_1(n) + a_2(n) + a_3(n) + a_4(n),$$

where

$$\begin{aligned} a_1(n) &= \lambda(n) \quad \text{if } n \le U \\ &= 0 \quad \text{otherwise} \\ a_2(n) &= \sum_{\substack{h^2d=n \\ d\le V}} \mu(d), \\ a_3(n) &= -\sum_{\substack{mdr=n \\ m\le U, d\le V}} \lambda(m)\mu(d), \\ a_4(n) &= -\sum_{\substack{mk=n \\ m>U, k\ge V}} \lambda(m)\Big(\sum_{d|k, d\le V} \mu(d)\Big). \end{aligned}$$

Thus, we can write

$$\sum_{n\le x} a(n)\lambda(n)$$

as $S_1 + S_2 + S_3 + S_4$ with appropriate notation. We now suppose that the $a(n)$ are the coefficients (normalized) of our eigenform f. By Cauchy-Schwarz and Rankin-Selberg, we easily deduce that $S_1 \ll U$. We can write S_2 as

$$\sum_{n\le x}\Big(\sum_{\substack{h^2d=n \\ d\le V}} \mu(d)\Big)a(n) = \sum_{d\le V} \mu(d) \sum_{h\le (x/d)^{1/2}} a(h^2d).$$

The inner sum can be estimated trivially by $O((x/d)^{1/2})$. This gives $S_2 \ll x^{1/2+\epsilon}V^{1/2}$. For S_3, we have

$$S_3 = -\sum_{t\le UV}\Big(\sum_{md=t, m\le U, d\le V} \mu(d)\lambda(m)\Big)\sum_{r\le x/t} a(rt).$$

By Theorem 5.1, the inner sum is $O((x/t)^{1/3}t^{\epsilon})$, so we get easily $S_3 \ll x^{1/3}(UV)^{2/3+\epsilon}$. Finally, for S_4, we have

$$\sum_{V\le k\le x/U}\Big(\sum_{d|k, d\le V} \mu(d)\Big)\sum_{U<m<x/k} \lambda(m)a(mk).$$

this can be re-written as

$$\sum_{U<m<x/V} \lambda(m) \sum_{V<k<x/m}\Big(\sum_{d|k, d\le V} \mu(d)\Big)a(mk).$$

By hypothesis, the inner sum is $\ll (x/m)^{1/2}m^{\epsilon}$ so that we get $S_4 \ll x^{1+\epsilon}/\sqrt{V}$. We choose $V = x^{1/2}$ and $U = X^{\epsilon}$ to get a final estimate of $x^{3/4+\epsilon}$. Thus, $L(s, f)$ has no zeros for $\mathrm{Re}(s) > 3/4$.

7 Concluding remarks

It is clear that the obstacle in proving a quasi-Riemann hypothesis is really the estimation of the sum S_4. It is interesting to note that if the sum

$$\sum_{m<x} \lambda(m)a(mk)$$

are positive, then one can get the following estimate for S_4:

$$(x/U)^{\epsilon} \sum_{V<k\le x/U} \sum_{U<m<x/k} \lambda(m)a(mk)$$

which is

$$\ll (x/U)^{\epsilon} \sum_{U<m<x/V} \lambda(m) \sum_{V<k<x/m} a(mk)$$

which by Theorem 5.1 gives a final estimate of $x^{1+\epsilon}/V^{2/3}$ which would give a quasi Riemann hypothesis.

8 Appendices: by Nathan Ng

8.1 Modular analogues of Polya's conjecture

Let E be an elliptic curve. The coefficients of its L-series will be denoted $a(n)$. The normalized coefficients will be denoted $a_E(n)$ where $a_E(n) = a(n)/n^{\frac{1}{2}}$. The Liouville function is denoted $\lambda(n)$ where $\lambda(n) = (-1)^{\Omega(n)}$ and $\Omega(n)$ is the total number of prime factors of n (counted with multiplicity). Let $S_E(x) = \sum_{n \le x} a_E(n)\lambda(n)$ be the generalized Polya sum.

Note In the tables, only the integer part for S_E is given. We write S for $S_E(n \cdot 10^6)$ in the tables below.

8.1.1 $\mathbf{E1 : y^2 = x^3 + x^2 - 7x + 36}$ (rank(E1) = 4)

n	S	n	S	n	S	n	S	n	S
1	201404	2	322163	3	422250	4	511622	5	592659
6	669422	7	740673	8	807658	9	873727	10	935762
11	998750	12	1055369	13	1111007	14	1164917	15	1218562
16	1271467	17	1324716	18	1373508	19	1421993	20	1468089
21	1516194	22	1564940	23	1609313	24	1653517	25	1697040
26	1742414	27	1788221	28	1829214	29	1873512	30	1912127
31	1951990	32	1994299	33	2034881	34	2075782	35	2113478
36	2152129	37	2191081	38	2224929	39	2262398	40	2298416
41	2335326	42	2368912	43	2407780	44	2442943	45	2477384
46	2511918	47	2546599	48	2583300	49	2618861	50	2652814

8.1.2 $\mathbf{E2 : y^2 - 21y = x^3 + 67x^2 - 10x + 30}$ (rank(E2) = 5)

n	S	n	S	n	S	n	S	n	S
1	217561	2	353203	3	467854	4	570499	5	664760
6	752802	7	836816	8	916978	9	993251	10	1066276
11	1136854	12	1205474	13	1273073	14	1339060	15	1402266
16	1465722	17	1526688	18	1586506	19	1645289	20	1702981
21	1758113	22	1814534	23	1869888	24	1923348	25	1976276
26	2028424	27	2081935	28	2133258	29	2184795	30	2233014
31	2283240	32	2331103	33	2380388	34	2429313	35	2475573
36	2522469	37	2569446	38	2614393	39	2660464	40	2706789
41	2750564	42	2795057	43	2841453	44	2885226	45	2928576
46	2970948	47	3014348	48	3056984	49	3098133	50	3138632

8.1.3 $\mathbf{E3 : y^2 - 63y = x^3 + 351x^2 + 56x + 22}$ (rank(E3) = 6)

n	S	n	S	n	S	n	S	n	S
1	386697	2	645957	3	869445	4	1072938	5	1261476
6	1439449	7	1608641	8	1771245	9	1926524	10	2078573
11	2224311	12	2369104	13	2506776	14	2643033	15	2777310
16	2908091	17	3035366	18	3160920	19	3283870	20	3407035
21	3526513	22	3642749	23	3760472	24	3877013	25	3989843
26	4101297	27	4211884	28	4322482	29	4432330	30	4539339
31	4646646	32	4749538	33	4853587	34	4957684	35	5059171
36	5161085	37	5261785	38	5358391	39	5458689	40	5556704
41	5653294	42	5751511	43	5845392	44	5941619	45	6034557
46	6128691	47	6224164	48	6315399	49	6409947	50	6499323

8.1.4 $\mathbf{E4 : y^2 - 168y = x^3 + 1641x^2 + 161x - 8}$ (rank(E4) = 7)

n	S	n	S	n	S	n	S	n	S
1	594145	2	1015656	3	1385905	4	1725542	5	2043874
6	2346273	7	2634736	8	2914172	9	3183595	10	3445294
11	3699511	12	3948636	13	4191263	14	4430532	15	4663520
16	4893186	17	5118437	18	5341917	19	5560982	20	5776124
21	5989072	22	6197620	23	6406369	24	6612722	25	6814634
26	7014126	27	7213935	28	7410973	29	7604352	30	7796756
31	7987525	32	8177016	33	8362978	34	8549392	35	8733795
36	8918625	37	9099551	38	9279117	39	9458557	40	9636586
41	9813116	42	9989408	43	10161495	44	10332620	45	10503675
46	10675408	47	10847600	48	11016174	49	11182080	50	11350545

8.1.5 $\mathbf{E5 : y^2 - 2xy + 737y = x^3 + 531x^2 + 1262x - 110}$ (rank(E5) = 8)

n	S	n	S	n	S	n	S	n	S
1	746346	2	1295215	3	1782625	4	2234026	5	2658572
6	3063518	7	3453141	8	3830537	9	4194361	10	4549210
11	4896000	12	5234904	13	5568477	14	5892719	15	6213424
16	6529903	17	6837707	18	7142781	19	7444932	20	7740555
21	8035595	22	8326564	23	8611872	24	8896498	25	9176337
26	9456621	27	9731143	28	10004300	29	10276113	30	10542562
31	10810469	32	11073349	33	11331322	34	11591076	35	11847572
36	12104436	37	12360929	38	12611653	39	12861357	40	13109258
41	13357360	42	13602367	43	13847412	44	14090376	45	14332387
46	14571373	47	14810372	48	15048835	49	15282605	50	15515199

8.1.6 $\mathbf{E6 : y^2 + 3576y = x^3 + 9767x^2 + 425x - 2412}$ (rank(E6) = 9)

n	S	n	S	n	S	n	S	n	S
1	628669	2	1090005	3	1498764	4	1878154	5	2232601
6	2572880	7	2898629	8	3215406	9	3521342	10	3820162
11	4108589	12	4394015	13	4671069	14	4946030	15	5213297
16	5477051	17	5738393	18	5994435	19	6248832	20	6499274
21	6742563	22	6985878	23	7225992	24	7467909	25	7702909
26	7934087	27	8166383	28	8396313	29	8621645	30	8847970
31	9068998	32	9289189	33	9509889	34	9725722	35	9941257
36	10156603	37	10369435	38	10582542	39	10791065	40	11003125
41	11209192	42	11415744	43	11619274	44	11824137	45	12026375
46	12226343	47	12427274	48	12629308	49	12827095	50	13024838

8.1.7 $\mathbf{E7 : y^2 - 15336y = x^3 + 1461695x^2 - 1414x - 80334}$ (rank(E7) = 10)

n	S	n	S	n	S	n	S	n	S
1	863765	2	1518178	3	2103843	4	2650750	5	3167285
6	3661074	7	4138930	8	4601567	9	5049942	10	5490045
11	5918105	12	6337736	13	6750994	14	7154920	15	7552743
16	7945953	17	8332984	18	8714975	19	9092725	20	9463593
21	9832013	22	10197337	23	10556331	24	10913126	25	11265934
26	11616719	27	11961429	28	12304890	29	12645915	30	12983303
31	13318006	32	13653228	33	13983816	34	14311650	35	14638627
36	14963131	37	15283241	38	15604378	39	15923548	40	16237957
41	16551140	42	16863976	43	17174866	44	17485161	45	17789400
46	18095174	47	18400360	48	18702538	49	19001829	50	19300679

8.2 Modular analogues of Turan's conjecture

Let E be an elliptic curve. The coefficients of its L-series will be denoted $a(n)$. The normalized coefficients will be denoted $a_E(n)$ where $a_E(n) = a(n)/n^{\frac{1}{2}}$. The Liouville function is denoted $\lambda(n)$ where $\lambda(n) = (-1)^{\Omega(n)}$ and $\Omega(n)$ is the total number of prime factors of n (counted with multiplicity). Let $T_E(x) = \sum_{n \leq x} a_E(n)\lambda(n)/n^{\frac{1}{2}}$ be the generalized Turan sum.

Note In the tables, only the integer part for T_E is given. We write T for $T_E(n \cdot 10^6)$ in the tables below.

8.2.1 $\mathbf{E1 : y^2 = x^3 + x^2 - 7x + 36}$ (rank(E1) = 4)

n	T	n	T	n	T	n	T	n	T
1	347	2	608	3	709	4	773	5	821
6	859	7	920	8	945	9	967	10	987
11	1007	12	1024	13	1039	14	1054	15	1068
16	1082	17	1095	18	1106	19	1118	20	1128
21	1139	22	1149	23	1159	24	1168	25	1176
26	1185	27	1194	28	1202	29	1210	30	1218

8.2.2 $\mathbf{E2 : y^2 - 21y = x^3 + 67x^2 - 10x + 30}$ (rank(E2) = 5)

n	T	n	T	n	T	n	T	n	T
1	630	2	743	3	816	4	871	5	916
6	953	7	986	8	1016	9	1042	10	1066
11	1087	12	1108	13	1127	14	1145	15	1161
16	1177	17	1192	18	1207	19	1220	20	1233
21	1246	22	1258	23	1269	24	1280	25	1291
26	1302	27	1312	28	1322	29	1331	30	1340

8.2.3 $\mathbf{E3 : y^2 - 63y = x^3 + 351x^2 + 56x + 22}$ (rank(E3) = 6)

n	T	n	T	n	T	n	T	n	T
1	1034	2	1250	3	1392	4	1501	5	1591
6	1667	7	1733	8	1792	9	1846	10	1895
11	1940	12	1983	13	2022	14	2059	15	2094
16	2127	17	2159	18	2189	19	2217	20	2245
21	2272	22	2297	23	2321	24	2345	25	2368
26	2390	27	2412	28	2433	29	2453	30	2474

8.2.4 $\mathbf{E4 : y^2 - 168y = x^3 + 1641x^2 + 161x - 8}$ rank(E4) = 7)

n	T	n	T	n	T	n	T	n	T
1	1498	2	1848	3	2084	4	2266	5	2417
6	2546	7	2659	8	2761	9	2854	10	2939
11	3017	12	3091	13	3159	14	3224	15	3286
16	3344	17	3399	18	3453	19	3504	20	3553
21	3600	22	3645	23	3689	24	3731	25	3772
26	3811	27	3850	28	3888	29	3924	30	3960

8.2.5 $\mathbf{E5 : y^2 - 2xy + 737y = x^3 + 531x^2 + 1262x - 110}$ (rank(E5) = 8)

n	T	n	T	n	T	n	T	n	T
1	1821	2	2278	3	2589	4	2831	5	3031
6	3204	7	3357	8	3495	9	3620	10	3735
11	3842	12	3942	13	4037	14	4125	15	4209
16	4289	17	4365	18	4438	19	4508	20	4575
21	4641	22	4703	23	4763	24	4822	25	4879
26	4934	27	4988	28	5040	29	5091	30	5140

8.2.6 $\mathbf{E6 : y^2 + 3576y = x^3 + 9767x^2 + 425x - 2412}$ (rank(E6) = 9)

n	T	n	T	n	T	n	T	n	T
1	1548	2	1932	3	2192	4	2396	5	2563
6	2708	7	2836	8	2952	9	3057	10	3154
11	3243	12	3327	13	3405	14	3480	15	3550
16	3617	17	3682	18	3743	19	3802	20	3859
21	3913	22	3965	23	4016	24	4066	25	4113
26	4159	27	4204	28	4248	29	4290	30	4332

8.2.7 $\mathbf{E7 : y^2 - 15336y = x^3 + 1461695x^2 - 1414x - 80334}$ (rank(E7) = 10)

n	T	n	T	n	T	n	T	n	T
1	2060	2	2604	3	2977	4	3270	5	3514
6	3725	7	3913	8	4082	9	4236	10	4379
11	4511	12	4635	13	4751	14	4861	15	4966
16	5066	17	5161	18	5252	19	5340	20	5424
21	5506	22	5584	23	5660	24	5734	25	5805
26	5874	27	5941	28	6007	29	6071	30	6133

References

[EM] J. Esmonde and M. Ram Murty, *Problems in Algebraic Number Theory*, Graduate Texts in Mathematics, **190** (1999), Springer-Verlag.

[He] D. Hejhal, *On the distribution of* $\log|\zeta'(1/2+it)|$, in Number Theory, Trace Formulas, and Discrete Groups, (Oslo, 1987), 343–370, Academic Press, MA, 1989.

[Go] S. Gonek, *On negative moments of the Riemann zeta functions*, Mathematika, **36** 1 (1989), 71–88.

[Po] G. Pólya, *Verschiedene Bemerkungen zur Zahlentheorie*, Jahresbericht der deutschen Math.-Vereinigung **28** (1919), 31–40.

[Ha] C.B. Haselgrove, *A disproof of a conjecture of Pólya*, Mathematika **5** (1958), 141–145.

[KM] V. Kumar Murty, *Modular Elliptic Curves*, in Seminar on Fermat's Last Theorem, ed. K. Murty, CMS Conference Proceedings, **17** Amer. Math. Soc., 1995.

[MP] M. Ram Murty and A. Perelli, *The pair correlation of zeros of functions in the Selberg class*, Internat. Math. Res. Not. **10** (1999), 531–546.

[T] P. Turán, *On some approximative Dirichlet polynomials in the theory of the zeta function of Riemann*, Danske Vid. Selsk. Mat.-Fys. Medd., **24** 17, 36 (1948), (see also his Collected Papers.)