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DISTINGUISHING HECKE EIGENFORMS

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ABSTRACT. Let f_1, f_2 be two distinct normalized Hecke eigenforms of weights k_1 and k_2 with at least one of them not of CM type and with p-th Hecke eigenvalues given by $a_p(f_1)p^{(k_1-1)/2}$ and $a_p(f_2)p^{(k_2-1)/2}$ respectively and p being prime. If $a_p(f_1) = a_p(f_2)$ for a set of primes with positive upper density, then we show that $f_1 = f_2 \otimes \chi$ for some Dirichlet character χ.

1. INTRODUCTION

Given two normalized Hecke eigenforms f_1 and f_2 of weights k_1, k_2 and levels N_1, N_2 respectively, let

$$f_i(z) = \sum_{n=1}^{\infty} a_n(f_i) n^{(k_i-1)/2} q^n, \qquad q = e^{2\pi i z}, \qquad i = 1, 2,$$

be the Fourier expansions at infinity. Our goal is to prove the following:

Theorem 1.1. *Suppose that at least one of f_1, f_2 is not of CM type. If*

$$\limsup_{x\to\infty} \frac{\#\{p \le x : a_p(f_1) = a_p(f_2)\}}{x/\log x} > 0,$$

then $f_1 = f_2 \otimes \chi$ for some Dirichlet character χ.

If both f_1, f_2 are of CM type, then the theorem is not necessarily true since one can easily construct counterexamples. A variant of our theorem had been proved earlier by Rajan [14] using Galois theoretic methods. More precisely, he showed that if $a_p(f_1)p^{(k_1-1)/2} = a_p(f_2)p^{(k_2-1)/2}$ on a set of primes p of positive upper density, then $f_1 = f_2 \otimes \chi$ for some Dirichlet character χ. Thus, our theorems agree in the case of equal weights. Our interest in this work was partially motivated by a recent work of Kulkarni, Patankar and Rajan [9] who showed using Galois theoretic methods that if E_1 and E_2 are two elliptic curves defined over a number field K, with at least one of them not of CM type, such that

$$\#E_1(\mathbb{F}_p) = \#E_2(\mathbb{F}_p),$$

for a set of primes of positive lower density, then E_1 and E_2 become isogenous after base change. For elliptic curves over $\mathbb{Q}$, our theorem also implies this result thanks

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to the celebrated work on the modularity of elliptic curves over $\mathbb{Q}$ due to Wiles [18], Breuil, Conrad, Diamond and Taylor [3].

The methods of this paper are analytic. If we write

$$a_p(f_i) = 2\cos\theta_p^{(i)}, \qquad i = 1, 2, \qquad \theta_p^{(i)} \in [0, \pi],$$

then the angles $\theta_p^{(i)}$ satisfy the Sato-Tate distribution law if the f_i are not of CM type. This has recently been proved by Barnet-Lamb, Geraghty, Harris and Taylor (see Theorem B of [2]).

If f_i is of CM type, the corresponding equidistribution theorem goes back to Hecke. In the course of their proof in the non-CM case, the authors of [2] show that if π_i is the automorphic representation associated to f_i, then for all positive integer values of m, $\mathrm{Sym}^m(\pi_i)$ is potentially automorphic. More precisely, they prove that there is a finite totally real Galois extension F (depending on m) such that $\mathrm{Sym}^m(\pi_i)$ becomes automorphic over F. To prove our theorem, we refine a strategy outlined in a paper by Murty and Rajan [13].

2. Preliminaries

In this section, we collect the relevant facts that will be needed in various stages of our proof.

Proposition 2.1. *If f_1, f_2 are normalized Hecke eigenforms, with at least one not of CM type, such that $f_1 \neq f_2 \otimes \chi$ for any Dirichlet character χ, then for any positive integers m, n,*

$$\sum_{p \le x} \frac{\sin(m+1)\theta_p^{(1)}}{\sin\theta_p^{(1)}} \frac{\sin(n+1)\theta_p^{(2)}}{\sin\theta_p^{(2)}} = o(x/\log x),$$

as x tends to infinity. Here, the summation is over primes.

Proof. This is essentially Theorem 2.4 of [7] combined with the standard Tauberian theorem. However, for the sake of completeness, we give an outline of the proof in section 4. The exposition in [7] is a bit confusing in that Theorem 2.4 is stated in the text as being conditional, but in the abstract on the first page of the paper, the author has stated that due to recent developments, the theorem is no longer conditional. □

Proposition 2.2. *Let $0 < \delta < \pi$. Let $f_\delta(x)$ be the "tent" function defined on $[-\pi, \pi]$ given by*

$$f_\delta(x) = \begin{cases} 1 - |x|/\delta & \text{if } |x| \le \delta, \\ 0 & \text{if } |x| > \delta. \end{cases}$$

Then, for any $M \ge 1$, we have

$$f_\delta(x) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{M} \frac{1 - \cos n\delta}{\pi n^2 \delta} \cos nx + O\left(\frac{1}{M\delta}\right),$$

where the implied constant is absolute.

Proof. It is easily seen that $f_\delta(x)$ has a Fourier series expansion given by

$$\frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1 - \cos n\delta}{\pi n^2 \delta} \cos nx,$$

by an easy computation. If we truncate the series for $1 \le n \le M$, we see that the tail estimate is $O(1/M\delta)$. ☐

3. Proof of the main theorem

Let $\pi > \delta > 0$ and take $f_\delta(x)$ as in Proposition 2.2. Clearly,

$$\#\{p \le x : \theta_p^{(1)} = \theta_p^{(2)}\} \le \sum_{p \le x} \left(f_\delta(\theta_p^{(1)} - \theta_p^{(2)}) + f_\delta(\theta_p^{(1)} + \theta_p^{(2)}) \right).$$

By Proposition 2.2, the right hand side is equal to

$$\frac{\delta \pi(x)}{\pi} + 4 \sum_{n=1}^{M} \frac{1 - \cos n\delta}{\pi n^2 \delta} \sum_{p \le x} \cos n\theta_p^{(1)} \cos n\theta_p^{(2)} + O\left(\frac{\pi(x)}{M\delta}\right)$$

upon using the trigonometric identity

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B.$$

The inner sum corresponding to $n = 1$ is

$$\sum_{p \le x} \cos \theta_p^{(1)} \cos \theta_p^{(2)}$$

which by the Rankin-Selberg theory is $o(\pi(x))$ (see for example Lemma 5 of [10] where a stronger result is stated for f_1, f_2 of the same weight, though the result extends for eigenforms of different weight also). To treat $n \ge 2$, we use the identity

$$2\cos n\theta = \frac{\sin(n+1)\theta}{\sin\theta} - \frac{\sin(n-1)\theta}{\sin\theta},$$

so that we can rewrite our sum as

$$\sum_{n=2}^{M} \frac{1-\cos n\delta}{\pi n^2 \delta} \sum_{p \le x} \left(\frac{\sin(n+1)\theta_p^{(1)}}{\sin\theta_p} - \frac{\sin(n-1)\theta_p^{(1)}}{\sin\theta_p^{(1)}} \right) \left(\frac{\sin(n+1)\theta_p^{(2)}}{\sin\theta_p^{(2)}} - \frac{\sin(n-1)\theta_p^{(2)}}{\sin\theta_p^{(2)}} \right).$$

Dividing by $\pi(x)$ and taking lim sup as x tends to infinity, we obtain upon applying Proposition 2.1, that the inner sums go to zero. Thus, we obtain

$$\limsup_{x \to \infty} \frac{\#\{p \le x : \theta_p^{(1)} = \theta_p^{(2)}\}}{\pi(x)} \le \frac{\delta}{\pi} + O\left(\frac{1}{\delta M}\right).$$

Letting M tend to infinity, we see that this density can be made arbitrarily small since δ is arbitrary. This contradicts our hypothesis. This completes the proof.

4. Joint Sato-Tate distribution for two Hecke eigenforms

In this section, we outline for the convenience of the reader, the joint equidistribution theorem alluded to in Proposition 2.1. There are already several readable expositions of the proof of the Sato-Tate conjecture deduced from the potential automorphy of symmetric power L-functions (see, for example, [7] and section 6 of Chapter 12 of [12]). What has not been explicitly presented in the literature is that the joint Sato-Tate distribution holds for two Hecke eigenforms, provided that one is not the Dirichlet twist of the other. Here is an outline of the argument.

Let ρ_1 and ρ_2 be the associated (ℓ-adic) Galois representations of f_1, f_2 respectively. By the work of [2], both $\mathrm{Sym}^n(\rho_1)$ and $\mathrm{Sym}^m(\rho_2)$ are potentially automorphic over a totally real Galois extension F over $\mathbb{Q}$. By the Arthur-Clozel theory of base change [1], we see that for any subfield F_1 of F with F/F_1 solvable, both

$\mathrm{Sym}^m(\rho_1)|_{F_1}$ and $\mathrm{Sym}^n(\rho_2)|_{F_1}$ are also automorphic over F_1. Indeed, both of the representations are Galois invariant over $\mathbb{Q}$ and hence over F_1. As F/F_1 is solvable, there is a chain of fields $F \supset F_m \supset F_{m-1} \cdots \supset F_1$ such that F/F_m and F_j/F_{j-1} for $2 \le j \le m$ are all cyclic of prime degree. By [1], the automorphic induction map exists from F to F_m and successively from F_j to F_{j-1} for $2 \le j \le m$ which at the final stage is F_1. The essential point here is that both the Galois and automorphic representations obtained by descent are Galois invariant at every step (see, for example, the comment at the bottom of page 11 in [4]). Now let $G = \mathrm{Gal}(F/\mathbb{Q})$. By Brauer induction, we can write

$$1 = \sum_i a_i \mathrm{Ind}_{H_i}^G \psi_i,$$

where the a_i's are integers and ψ_i's are one-dimensional characters of nilpotent subgroups H_i of G. Thus,

$$L(s, (\mathrm{Sym}^m(\rho_1) \otimes \mathrm{Sym}^n(\rho_2)) \otimes 1) = \prod_i L(s, (\mathrm{Sym}^m \rho_1 \otimes \mathrm{Sym}^n \rho_2) \otimes \mathrm{Ind}_{H_i}^G \psi_i)^{a_i}.$$

By Frobenius reciprocity,

$$(\mathrm{Sym}^m \rho_1 \otimes \mathrm{Sym}^n \rho_2) \otimes \mathrm{Ind}_{H_i}^G \psi_i = \mathrm{Ind}_{H_i}^G (((\mathrm{Sym}^m \rho_1) \otimes (\mathrm{Sym}^n \rho_2))|_{F^{H_i}} \otimes \psi_i).$$

Since $(\mathrm{Sym}^m \rho_1)|_{F^{H_i}}$ and $(\mathrm{Sym}^n \rho_2)|_{F^{H_i}}$ are both automorphic over F^{H_i}, and ψ_i is a Hecke character of F^{H_i} by Artin reciprocity, we can form the Rankin-Selberg convolution:

$$L(s, (\mathrm{Sym}^m(\rho_1))|_{F^{H_i}} \otimes (\mathrm{Sym}^n(\rho_2))|_{F^{H_i}} \otimes \psi_i) \tag{1}$$

which by our hypothesis on f_1 and f_2 is analytic and non-vanishing for $\Re(s) \ge 1$. To elaborate, there are theorems of Cogdell and Michel (in the trivial Nebentypus case) and Rajan (in the non-trivial Nebentypus case) that give a "multiplicity one theorem" for symmetric powers (see Proposition 5.1 of [6] and Corollary 5.1 of [15]). These theorems say that if the L-series attached to the m-th symmetric powers of the Galois representations associated with f_1 and f_2 are equal, then f_1 is a Dirichlet twist of f_2. Thus, as $f_1 \ne f_2 \otimes \chi$ for any Dirichlet character χ, we have that

$$\pi_1 := (\mathrm{Sym}^m(\rho_1))|_{F^{H_i}} \quad \text{and} \quad \pi_2 := (\mathrm{Sym}^n(\rho_2))|_{F^{H_i}} \otimes \psi_i$$

are such that $\pi_2 \not\simeq \pi_1 \otimes |\det|^{it}$ for any real number t. By standard Rankin-Selberg theory (see for example p. 69 or 225 of [5]), the L-function (1) is analytic and non-vanishing in the region $\Re(s) \ge 1$. (The non-vanishing is derived from the work of Shahidi [16].) Thus,

$$L(s, (\mathrm{Sym}^m \rho_1) \otimes (\mathrm{Sym}^n \rho_2))$$

extends to an analytic function to $\Re(s) \ge 1$ and is non-vanishing there. By the classical Tauberian theorem (see, for example, [11] or [17]) applied to the logarithmic derivative of this L-function, we deduce Proposition 2.1.

5. Concluding remarks

Our argument extends easily to imply a corresponding result for any two modular forms f_1 and f_2 over a totally real field since the results of [2] apply in this context also. As noted in [13], one expects that the set of primes $p \le x$ for which $a_p(f_1) = a_p(f_2)$ is $O(x^\theta)$ for some $0 \le \theta < 1$, if we assume the analog of the Riemann hypothesis for the L-functions under consideration. It would be interesting to obtain error terms, but since the fields obtained in [2] are not effective, it seems difficult (at present) to move forward in this direction.

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