ON THE PARITY OF THE FOURIER COEFFICIENTS
OF $j$-FUNCTION

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Abstract. Klein’s modular $j$-function is defined to be

$$j(z) = E_4^3 / \Delta(z) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n)q^n$$

where $z \in \mathbb{C}$ with $\Im(z) > 0$, $q = \exp(2i\pi z)$, $E_4(z)$ denotes the normalized Eisenstein series of weight 4 and $\Delta(z)$ is the Ramanujan’s Delta function. In this short note, we show that for each integer $a \geq 1$, the interval $(a, 4a(a+1))$ (respectively, the interval $(16a-1, (4a+1)^2)$) contains an integer $n$ with $n \equiv 7 \pmod{8}$ such that $c(n)$ is odd (respectively, $c(n)$ is even).

1. Introduction

Let $z$ be a complex number with $\Im(z) > 0$ and $q = e^{2\pi iz}$. The modular invariant $j$-function defined as

$$j(z) = \frac{E_4^3(z)}{\Delta(z)}$$

where

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^24$$

is the Ramanujan’s Delta function and

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n$$

is the normalized Eisenstein series of weight 4. The Fourier expansion for $j(z)$ is

$$j(z) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n)q^n$$

where $c(n)$ are integers.

It is well known that $c(n)$ is even whenever $n \not\equiv 7 \pmod{8}$. Indeed, a result of J. P. Serre implies that for almost all integers $n \not\equiv 7 \pmod{8}$, one has $c(n) \equiv 0 \pmod{2^t}$ for any integer $t \geq 1$. Later, Ono and Taguchi [4] proved that for any
For all \( a \geq 1 \), there exists an integer \( n \in (a, 4a(a + 1) - 1] \) with \( n \equiv 7 \pmod{8} \), such that \( c(n) \) is an odd integer. In particular, there are infinitely many odd integers \( n \equiv 7 \pmod{8} \) for which \( c(n) \) is an odd integer.

Note that when \( a = 1 \) in Theorem 1.1, we get that the interval \([1, 7]\) contains an integer \( n \equiv 7 \pmod{8} \) such that \( c(n) \) is odd. This must be \( n = 7 \). Indeed, \( c(7) = 44656994071935 \), which is an odd integer.

**Corollary 1.2.** For all \( x \geq 8 \), we have

\[
\{1 \leq n \leq x : c(n) \text{ is odd}\} = \{n \leq x : n \equiv 7 \pmod{8} \text{ and } c(n) \text{ is odd}\} \geq c_0 \log \log x,
\]

for some positive constant \( c_0 \).

**Theorem 1.3.** For all \( a \geq 1 \), there exists an integer \( n \in [16a - 1, (4a + 1)^2 - 1] \) with \( n \equiv 7 \pmod{8} \) such that \( c(n) \) is even. In particular, there exist infinitely many integers \( n \equiv 7 \pmod{8} \) for which \( c(n) \) is even.

When \( a = 1 \) in Theorem 1.3, we get that 15 and 23 lie in the interval \([15, 24]\). Note that \( c(15) \) and \( c(23) \) are even integers.

**Corollary 1.4.** For all \( x \geq 15 \), we have

\[
\#\{1 \leq n \leq x : n \equiv 7 \pmod{8} \text{ and } c(n) \text{ is even}\} \geq c_1 \log \log x,
\]

for some positive constant \( c_1 \).

**Corollary 1.5.** For a given residue class \( \epsilon \pmod{2} \), there exist infinitely many \( n \) such that \( c(n) \equiv \epsilon \pmod{2} \).

In their paper, Ono and Ramsey [3] mention that it is expected that for half of the \( n \equiv 7 \pmod{8} \), we should have \( c(n) \) odd.
We shall start with the following lemma.

**Lemma 2.1.** For all integer \( n \geq 1 \), we have
\[
\sum_{m \geq 0} c \left( n - (2m + 1)^2 \right) \equiv 0 \pmod{2}.
\]

**Proof.** The well-known Jacobi identity says that
\[
\prod_{n=1}^{\infty} (1 - q^n)^3 = \sum_{k=0}^{\infty} (-1)^k (2k + 1)q^{k(k+1)/2}.
\]
Since \((x + y)^{2m} \equiv x^{2m} + y^{2m} \pmod{2}\), we use (2.2) in (1.2) to write
\[
\Delta(z) \equiv q \prod_{n=1}^{\infty} (1 - q^{8n})^3 \equiv q \sum_{n=0}^{\infty} q^{8n(n+1)/2} \equiv \sum_{n=0}^{\infty} q^{(2n+1)^2} \pmod{2}.
\]
By (1.3), we have \( E_4(z) \equiv 1 \pmod{2} \). Therefore, (1.1) becomes
\[
j(z)\Delta(z) \equiv 1 \pmod{2}.
\]
From (1.4) and (2.3), we have
\[
j(z)\Delta(z) \equiv \left( \sum_{n=-1}^{\infty} c(n)q^n \right) \left( \sum_{n=0}^{\infty} q^{(2n+1)^2} \right) \pmod{2}.
\]
Therefore, we get
\[
1 \equiv \sum_{n=0}^{\infty} \sum_{k \geq 0} c \left( n - (2k + 1)^2 \right) q^n \pmod{2}.
\]
Now by comparing the coefficients of \( q^n \) on both sides, we get the required congruence. \( \square \)

**Proof of Theorem 1.1.** Let \( a \geq 1 \) be a given integer. Assume that \( c(m) \) is even for every \( m \in (a, 4a(a + 1) - 1] \). Put \( n = 4a(a + 1) \) in (2.1). We get
\[
\sum_{k \geq 0} c \left( 4a(a + 1) - (2k + 1)^2 \right) = \sum_{k \geq 0} c \left( 4a(a + 1) - 4k(k + 1) - 1 \right) \equiv 0 \pmod{2}.
\]
In the above congruence, the term corresponding to \( k = a \) is \( c(-1) \) which is indeed 1 and hence \( c(-1) \not\equiv 0 \pmod{2} \). When we put \( k = a - j \), we get
\[
4a(a + 1) - 4(a - j)(a - j + 1) - 1 = 8ja - 4j^2 + 4j - 1 = 4j(2a - j + 1) - 1.
\]
If we vary \( j = 1, 2, \ldots, a - 1 \), then we see that
\[
4j(2a - j + 1) - 1 \geq 4(2a - (a - 1) + 1) - 1 = 4(a + 2) - 1 > a
\]
for all \( a \geq 1 \). Therefore, if
\[
c \left( 4a(a + 1) - 4k(k + 1) - 1 \right)
\]
are all even for all \( k = 1, 2, \ldots, a - 1 \) and \( k = a \), the above integer is odd. Therefore, their sum cannot be even, which is a contradiction. Hence there is an integer \( n \in (a, 4a(a + 1) - 1] \) for which \( c(n) \) is an odd integer.
we get
\[ j(z) \equiv \frac{1}{\prod_{n=1}^{\infty} (1 - q^{8n})^3} \equiv \sum_{k=-1}^{\infty} b(k)q^{8k+7} \pmod{2}, \]
where \( b(k) \equiv 0, 1 \pmod{2} \), by comparing the Fourier coefficients on both sides, we get if \( n \not\equiv 7 \pmod{8} \), we have \( c(n) \equiv 0 \pmod{2} \) and if \( c(n) \) is odd, then \( n \equiv 7 \pmod{8} \). Therefore the integer \( n \in (a, 4a(a+1) - 1) \) (for which \( c(n) \) is odd) must be an odd integer and \( n \equiv 7 \pmod{8} \).

**Proof Corollary 1.2.** We want to count \( n \leq x \) for which \( c(n) \) is odd. For that we define \( a_0 = 1, a_1 = 7, \) for every \( k \geq 2 \)
\[ a_k = 4a_{k-1}(a_{k-1} + 1) - 1 = 4a_{k-1}^2 + 4a_{k-1} - 1. \]
Then, we partition the interval
\[ [1, x] = [1, 7] \cup (7, a_2) \cup [a_2, a_3) \cup \cdots \cup [a_{\ell-1}, a_{\ell}) \cup [a_{\ell}, x] \]
where \( \ell \) is the largest integer \( k \) such that \( a_k \leq x \). By Theorem 1.1, we know each interval \([a_{k-1}, a_k]\) contains at least one integer \( n \equiv 7 \pmod{8} \) for which \( c(n) \) is odd. Hence, the number of \( n \leq x \) with \( n \equiv 7 \pmod{8} \) for which \( c(n) \) is odd is at least \( \ell \) and it is remains to find the value of \( \ell \) as a function of \( x \). Since
\[ a_k = 4a_{k-1}^2 + 4a_{k-1} - 1 < 8a_{k-1}^2 \]
for all \( k \geq 0 \),
we get
\[ a_k \leq 8^k a_1^{2^k-1} \leq 8^k \text{ for all } k \geq 0. \]
Since \( a_\ell \leq x \), we see that \( \ell \geq c_0 \log x \) which proves the corollary.

**Proof of Theorem 1.3.** For every \( a \geq 1 \), we denote the interval
\[ I_a := [16a - 1, (4a + 1)^2 - 1]. \]
We need to prove that \( I_a \) contains an integer \( n \equiv 7 \pmod{8} \) for which \( c(n) \) is even.
Suppose we assume that \( c(n) \) is odd for every integer \( n \equiv 7 \pmod{8} \) and \( n \) lies in the interval \( I_a \). Put \( n = (4a + 1)^2 - 1 \) in (2.1) and we get
\[ \sum_{k \geq 0} c((4a + 1)^2 - 1 - (2k + 1)^2) \equiv 0 \pmod{2}. \]
Note that the argument of \( c \) in the summands is \((4a + 1)^2 - 1 - (2k + 1)^2 \equiv -1 \pmod{8} \) and \((4a + 1)^2 - 1 - (2k + 1)^2 \in I_a \) for all \( k = 0, 1, \cdots, 2a - 1 \). When we put \( j = 2a \), we get \( c(-1) \) which is an odd integer. By assumption, we get \( 2a \) number of \( 1 \)'s and \( c(-1) \) add up to \( 0 \pmod{2} \), which is a contradiction as \( c(-1) \) is odd. Thus, there exists \( n \in I_a \) with \( n \equiv 7 \pmod{8} \) such that \( c(n) \) is an even integer.

**Proof Corollary 1.4.** We want to count \( n \leq x \) with \( n \equiv 7 \pmod{8} \) for which \( c(n) \) is even. Since we know \( c(15) \) and \( c(23) \) are even integers, we define \( a_0 = 1, a_1 = 15, \) for every \( k \geq 2 \) as
\[ a_k = (4a_{k-1} + 1)^2 - 1. \]
Then, we see that the disjoint union of the following intervals
\[ [1, 15] \cup (15, 25) \cup [a_1, a_2) \cup \cdots \cup [a_{\ell-1}, a_{\ell}) \cup [a_{\ell}, x] \subset [1, x] \]
where \( \ell \) is the largest integer \( k \) such that \( a_k \leq x \). By Theorem 1.3, we know each interval \([a_{k-1}, a_k]\) contains at least one integer \( n \equiv 7 \pmod{8} \) for which \( c(n) \) is
even. Hence, the number of \( n \leq x \) and \( n \equiv 7 \pmod{8} \) for which \( c(n) \) is even is at least \( \ell \). Since \( a_k \leq 32a_{k-1}^2 \) for all \( k \geq 0 \), we get,

\[
a_k \leq 32^k a_1^{2^k-1} \leq 32^{2^k} \text{ for all } k \geq 0
\]

and hence we get the result. \( \square \)

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