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RAMANUJAN AND THE ZETA FUNCTION

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ABSTRACT. Srinivasa Ramanujan wrote several papers related to the Riemann zeta function. These papers highlight two themes, the first being the special values of the zeta function at integer arguments and the second, its analytical theory as it pertains to the distribution of prime numbers. Apart from his published work, a good chunk of his contributions are also contained in his letters and notebooks (both lost and found) and it is the purpose of this article to highlight these contributions in light of modern developments. In particular, we show how Ramanujan's identities can be generalized to prove the non-vanishing of Artin L-series on $\Re(s) = 1$ and thus give a new proof of the Chebotarev density theorem. In the last section, we discuss its relation to the Rankin-Selberg method and special values of L-series attached to cusp forms.

1. INTRODUCTION

Without a doubt, the Riemann zeta function has two alluring aspects. The first is its relation to the distribution of prime numbers and the second is the determination of its special values. The goal of this article is to show that Srinivasa Aiyangar Ramanujan made significant contributions to both of these themes and this is not so widely known.

In fact, about a hundred years ago, Ramanujan published two papers [26] [27] in the Journal of the Indian Mathematical Society, one in 1911 and the second in 1913. The first was titled "Some properties of Bernoulli numbers" and the second, "Irregular numbers". Both papers focused on special values of the Riemann zeta function at positive even arguments. Clearly, these papers

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were written by Ramanujan before his historic trip to England to begin his epic collaboration with G.H. Hardy. A third paper entitled "Some formulas in the analytical theory of numbers," written in 1916 after his arrival in England, was to have a major impact in the study of the distribution of prime numbers. There is another paper [29] of Ramanujan devoted entirely to the Riemann zeta function. However, this paper derives some new integral expressions for Riemann's functions $\xi(s)$ and $\Xi(t)$, and is not directly related to the theme of this exposition.

Not so well-known is Ramanujan's work on the zeta function at odd arguments. This is contained in the celebrated notebooks (both lost and found) of Ramanujan. Indeed, in Entry 21(i) of Chapter 14 in Ramanujan's second notebook, we find the following unusual formula. If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if r is a positive integer, then

$$\begin{aligned}(4\alpha)^{-r}&\left(\frac{1}{2}\zeta(2r+1)+\sum_{m=1}^{\infty}\frac{1}{m^{2r+1}(e^{2m\alpha}-1)}\right)\\&-(-4\beta)^{-r}\left(\frac{1}{2}\zeta(2r+1)+\sum_{m=1}^{\infty}\frac{1}{m^{2r+1}(e^{2m\beta}-1)}\right)\\&=\sum_{k=0}^{r+1}\frac{(-1)^{k+1}B_{2k}B_{2r+2-2k}\alpha^{r+1-k}\beta^{k}}{(2k)!(2r+2-2k)!},\end{aligned}$$

where B_j denotes the j-th Bernoulli number. The case $\alpha = \beta = \pi$ is especially interesting. In this case, if r is even, we deduce

$$\sum_{k=0}^{r+1}(-1)^k\frac{B_{2k}B_{2r+2-2k}}{(2k)!(2r+2-2k)!} = 0.$$

If r is odd, we deduce the beautiful formula

$$\zeta(2r+1)+\sum_{m=1}^{\infty}\frac{1}{m^{2r+1}(e^{2\pi m}-1)} = (4\pi)^{2r+1}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}B_{2k}B_{2r+2-2k}}{(2k)!(2r+2-2k)!}$$

which apparently was discovered by Lerch [21] in 1901 and published in an obscure journal (see also [4] for a discussion on this). Several interesting corollaries emanate from this formula. For instance, using the transcendence of π and the rationality of the Bernoulli numbers, we infer that at least one of

$$\zeta(4r+3), \qquad \sum_{n=1}^{\infty}\frac{1}{n^{4r+3}(e^{2\pi n}-1)}$$

is transcendental. Another elegant identity is

$$\zeta(3) + 2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)} = \frac{7\pi^3}{180}.$$

What is evident in Ramanujan's notebooks is an awareness that special values of

$$\sum_{m=1}^{\infty} \frac{1}{m^r(e^{2\pi m}-1)}$$

are relevant to the study of the Riemann zeta function at odd arguments. Thus, he proceeds to investigate these sums and observes in Entry 13 (see page 261 of [6]) that for $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, we have

$$\alpha^r \sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\alpha}-1} - (-\beta)^r \sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\beta}-1} = (\alpha^r - (-\beta)^r)\frac{B_{2r}}{4r},$$

which is rather striking. Choosing $\alpha = \beta = \pi$ gives the unusual identity

$$\sum_{m=1}^{\infty} \frac{m^{4r+1}}{e^{2\pi m}-1} = \frac{B_{4r+2}}{8r+4},$$

which was discovered by Glaisher [11] in 1889.

Grosswald [13] (see also [14]) wrote several papers in the 1970's related to these formulas. (He may have been unaware of Ramanujan's work when he wrote the first paper since in the appendix of [13], he says that Siegel alerted him to the volumes of Ramanujan's notebooks published by the Tata Institute in 1957.) To state his theorem, let

$$F_r(z) = \sum_{n=1}^{\infty} \frac{\sigma_r(n)}{n^r} e^{2\pi i n z},$$

where

$$\sigma_r(n) = \sum_{d|n;\, d>0} d^r.$$

Then, Grosswald [13] proves that for any z with $\Im(z) > 0$,

$$F_{2r+1}(z) - z^{2r}F_{2r+1}(-1/z) = \frac{1}{2}\zeta(2r+1)(z^{2r}-1) +$$

$$\frac{(2\pi i)^{2r+1}}{2z} \sum_{k=0}^{r+1} z^{2r+2-2k} \frac{B_{2k}B_{2r+2-2k}}{(2k)!(2r+2-2k)!}.$$

Putting $z = i\beta/\pi$ with $\beta > 0$ gives the Ramanujan-Lerch formula above. The polynomial on the right hand side has been studied by the author with C. Smyth and R. Wang in [24]. In this context, the reader can also consult [12] as well as [19] and [20].

In [4], Berndt proves the following generalization: for any z with $\Im(z) > 0$, let $Vz = -1/z$. For any integer r, we have

$$\begin{aligned} & z^r(1+(-1)^r)\sum_{m=1}^{\infty}\frac{1}{m^{r+1}(e^{-2\pi imVz}-1)} \\ &= (1+(-1)^r)\sum_{m=1}^{\infty}\frac{1}{m^{r+1}(e^{-2\pi imz}-1)} + g(z,-r) \\ & -(2\pi i)^{r+1}\sum_{k=0}^{r+2}\frac{B_kB_{r+2-k}}{k!(r+2-k)!}z^{k-1}, \end{aligned} \tag{1.1}$$

where for $r \neq 0$,

$$g(z,-r) = (1-z^r)\zeta(r+1),$$

and for $r = 0$, $g(z, 0) = \pi i - \log(-z)$. From this, Berndt [4] deduces both Euler's formula for the zeta function at even arguments and Ramanujan's formula above for $\zeta(2r + 1)$. Indeed, if we put $r = 2n - 1$ in (1.1), then the left hand side vanishes as well as the first sum on the right hand side. An easy calculation now gives Euler's formula. To deduce the Ramanujan-Grosswald formula, we replace r with $2r$ ($r \neq 0$) and do the obvious simplifications. Thus, Euler's formula and Ramanujan's formula are special cases of Theorem 2.2 of Berndt [4]. In [7], Berndt discusses an unpublished manuscript of Ramanujan in which some hints are given of how Ramanujan may have derived his formulas for the special values of $\zeta(2r + 1)$. This unpublished paper is also part of a chapter in the forthcoming fourth volume by Andrews and Berndt [1] on Ramanujan's lost notebook.

As noted by the author, C. Smyth and R. Wang in [12], the function $F_r(z)$ is an example of an "Eichler integral" in the following sense. Let r be odd and

$$E_{r+1}(z) = \gamma_{r+1} + \sum_{n=1}^{\infty}\sigma_r(n)e^{2\pi irz}, \qquad \gamma_r = -\frac{B_{r+1}}{2(r+1)}$$

be the classical Eisenstein series of weight r for the full modular group. Then, for any odd $r > 1$, we have

$$F_r(z) = \frac{(2\pi i)^r}{(r-1)!}\int_{i\infty}^{z}[E_{r+1}(\tau)-\gamma_{r+1}](\tau-z)^{r-1}d\tau,$$

as is easily checked. This is called an "Eichler integral of the first kind" in the literature (see [12]).

In his 1916 paper, Ramanujan obtained more remarkable identities involving the Riemann zeta function and these identities have been used by Ingham [17] to give a new proof of the non-vanishing of the zeta function on the line

$\Re(s) = 1$ (a fact which is equivalent to the prime number theorem). More precisely, Ramanujan [28] showed that

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}. \tag{1.2}$$

Formula (1.2) was partly generalized by Chowla [9] in 1928. Shimura (see Lemma 1 of [33]) also generalized Ramanujan's identity and this is essentially our Theorem 5.2 below in different notation. Shimura seems to be unware of Ramanujan's work in this context since no reference is made to him in [33]. As Ingham [17] notes, a mild variation of the Ramanujan identity also leads to a proof of the non-vanishing of the Dirichlet L-functions on the line $\Re(s) = 1$ (which is again equivalent to the prime number theorem for arithmetic progressions). In this paper, we will discuss these two aspects of Ramanujan's work as well as generalizations and future directions. In particular, we show that a generalization of Ramanujan's identity leads to a proof of the non-vanishing of Artin L-series on $\Re(s) = 1$, a fact which is equivalent to the Chebotarev density theorem.

2. Special Values Of The Riemann Zeta Function

The Bernoulli numbers B_n, are defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}.$$

Euler's celebrated formula is

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = -\frac{(2\pi i)^{2k} B_{2k}}{2(2k)!},$$

for $k = 1, 2, \ldots$ Since the Riemann zeta function admits an Euler product, for $\Re(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product is over prime number numbers, Ramanujan observed in [26] that this gives us a rapid method of computing the Bernoulli numbers. Much of the discussion in Ramanujan's notebooks is replete with a study of the Bernoulli numbers and their relationship to special values of the Riemann zeta function. It is highly reminiscent of Euler [2] who, though he didn't have the concept of analytic continuation and did not view the zeta function as a function of a complex variable, managed to derive the functional equation for it and evaluate explicitly its special values at even arguments. (See for example, formula (K) of [2].) In the same spirit, Ramanujan seems to have derived the functional equation according to Berndt (see pages 153-154 in Chapter 7 of [5]). According to Ayoub (see the last line on page 1078 of [2]), Euler conjectured that $\zeta(2r+1)/\pi^{2r+1}$ is a "function of $\log 2$". This attribution is however, not quite accurate. According to the English translation of the paper [10], Euler was aiming to show that there are no integers a, b, c such that

$$a\zeta(3) + b(\log 2)^3 + c(\log 2)\pi^2 = 0,$$

and he seems to have conjectured that there was no such relation. If we interpret the "function of $\log 2$" to be a "polynomial with algebraic coefficients in $\log 2$, then it may be possible to show that $\zeta(2r+1)/\pi^{2r+1}$ is not a polynomial in $\log 2$ with algebraic coefficients, now with modern tools since it is generally believed that

$$\pi, \zeta(3), \zeta(5), ...$$

are all algebraically independent. In this context, Kohnen [18] has made general conjectures regarding special values of L-series attached to modular forms of weight $2k$ for the full modular group. These conjectures (when applied to the Eisenstein series) imply the transcendence of the numbers $\zeta(2r+1)/\pi^{2r+1}$. Of course, these conjectures say nothing about the relationship of these numbers to $\log 2$, but most probably there is no relation to it.

Returning to Ramanujan and his two papers related to the special values of the Riemann zeta function at even arguments, we find the curious recurrence for the Bernoulli numbers given as formula (34) of [26]:

$$B_{2n} = -\frac{2n(2n-1)}{4\pi^2} B_{2n-2} \prod_p \left(1 - \frac{p^2-1}{p^{2n}-1}\right),$$

where the product is over prime numbers. (Note that here we are writing B_{2n} adhering to the more modern notation, whereas Ramanujan writes B_n.) This recurrence is easily seen from writing the quotient $\zeta(2n)/\zeta(2n-2)$, applying Euler's formula for $\zeta(2n)$ and using the Euler product.

It is not difficult to make Ramanujan's asymptotic a bit more precise. Indeed,

$$\frac{|B_{2n}/B_{2n-2}|}{4\pi^2} = \prod_p \left(1 - \frac{p^2-1}{p^{2n}-1}\right) \leq 1,$$

from which we easily deduce

$$\frac{1}{\zeta(2n-2)} = \prod_p \left(1 - \frac{1}{p^{2n-2}}\right) \leq \frac{|B_{2n}/B_{2n-2}|}{n(2n-1)/2\pi^2} \leq 1.$$

Since

$$1 \geq \frac{1}{\zeta(2n-2)} \geq 1 - \sum_{m=2}^{\infty} \frac{1}{m^{2n-2}},$$

we deduce:

Theorem 2.1. *As n tends to infinity, we have*

$$\frac{|B_{2n}/B_{2n-2}|}{n(2n-1)/2\pi^2} = 1 + O(1/2^{2n}).$$

I suppose this is what Ramanujan meant when he wrote that B_{2n}/B_{2n-2} converges to the quantity above "very rapidly as n becomes greater and greater" (see [26, (35)]).

3. Special Values Of Related Dirichlet Series

In his 1913 paper on "irregular numbers", Ramanujan evaluates the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{2k}}$$

where $a_n = 1$ if n has an odd number of prime divisors counted with multiplicity. Indeed, let $\Omega(n)$ be the number of prime divisors of n counted with multiplicity. We define the Liouville function $\lambda(n)$ as $(-1)^{\Omega(n)}$. Since $\lambda(n)$ is a multiplicative function of n (that is, $\lambda(mn) = \lambda(m)\lambda(n)$ whenever m and n are coprime), it is easy to see that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \left(1 + \frac{1}{p^s}\right)^{-1} = \frac{\zeta(2s)}{\zeta(s)}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{2k}} = \frac{1}{2}\left(\zeta(2k) - \frac{\zeta(4k)}{\zeta(2k)}\right) = \frac{\zeta(2k)^2 - \zeta(4k)}{2\zeta(2k)}$$

which is the main formula in section 4 of [27] (in slightly different notation). Later in the same paper, Ramanujan writes [27]

$$\sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^{2k}} = \prod_p \left(1 + \frac{1}{p^{2k}}\right) = \frac{\zeta(2k)}{\zeta(4k)},$$

where μ denotes the Möbius function. Thus,

$$\frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^{2k}} - \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2k}}\right) = \sum_{n=1}^{\infty} \frac{b_n}{n^{2k}} = \frac{1}{2}\left(\frac{\zeta(2k)}{\zeta(4k)} - \frac{1}{\zeta(2k)}\right) = \frac{\zeta(2k)^2 - \zeta(4k)}{2\zeta(2k)\zeta(4k)},$$

where $b_n = 1$ if n contains an odd number of distinct prime divisors and zero otherwise. This is formula (12) of [27].

Finally,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} - \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^{2k}} = \sum_{n=1}^{\infty} \frac{c_n}{n^{2k}} = \zeta(2k) - \frac{\zeta(2k)}{\zeta(4k)} = \frac{\zeta(2k)(\zeta(4k)-1)}{\zeta(4k)},$$

where $c_n = 1$ if n is not squarefree and zero otherwise. So in this way, Ramanujan evaluates the Dirichlet series supported at "irregular numbers" in terms of Bernoulli numbers, first by writing them as functions of the Riemann zeta function and then noting that the arguments of the zeta function are all even.

4. Special Values Of The Riemann Zeta Function At Odd Arguments

The Ramanujan-Grosswald formula allows us to write the special value $\zeta(2r+1)$ in terms of special values of $F_{2r+1}(z)$ and the Ramanujan polynomials

$$R_{2r+1}(z) = \sum_{k=0}^{r+1} z^{2r+2-2k} \frac{B_{2k}B_{2r+2-2k}}{(2k)!(2r+2-2k)!}.$$

Indeed, we have [13]

$$\frac{1}{2}\zeta(2r+1)(z^{2r}-1) = F_{2r+1}(z) - z^{2r}F_{2r+1}(-1/z) - \frac{(2\pi i)^{2r+1}}{2z}R_{2r+1}(z).$$

We can choose an algebraic value of z lying in the upper half-plane so that the Ramanujan polynomial vanishes. Indeed As a consequence of the work in [24], it follows that for $r \geq 4$, there is an algebraic α lying in the upper half-plane, with $|\alpha| = 1$, which is not a $2r$-th root of unity such that

$$\zeta(2r+1) = \frac{2}{\alpha^{2r}-1}(F_{2r+1}(\alpha) - \alpha^{2r}F_{2r+1}(-1/\alpha)).$$

The functions $F_{2r+1}(z)$ are special cases of Eichler integrals (integrals of modular forms). This expression for $\zeta(2r+1)$ motivates the study of the function

$$G_{2r+1}(z) = \frac{1}{z^{2r}-1}(F_{2r+1}(z) - z^{2r}F(-1/z))$$

and study its special values. Thus, the author, C. Smyth and R. Wang showed in [12] the following striking theorem:

Theorem 4.1. *The set*

$$\{G_{2r+1}(z) : \Im(z) > 0, z \, \text{algebraic}, \, z^{2r} \neq 1\}$$

contains at most one algebraic number.

Presumably, there are no algebraic numbers in the set described by the theorem, for otherwise, $\zeta(2r+1)$ would be an algebraic linear combination of 1 and π^{2r+1}, which is highly unlikely in view of Kohnen's conjectures and the prevailing philosophy.

5. Ramanujan And Prime Number Theory

In his twelve lectures on Ramanujan delivered at Harvard University in 1936, G.H. Hardy [15] relates how Ramanujan wrote about explicit formulas for the number of primes less than x and how most of his assertions were in error. Ramanujan's first two letters to Hardy contained several striking results, some of which were new, and some had been discovered by Riemann and Gram earlier. In his first letter of January 16, 1913, he wrote saying he had several interesting formulas for the number of primes up to x but does not write them down explicitly. In his second letter of 27 February, 1913 (this date is given incorrectly as 29 February 1913 in Hardy's book), Ramanujan gives three formulas for the number of primes less than x. These are:

$$\int_0^\infty \frac{y^t}{t\zeta(t+1)\Gamma(t+1)}dt, \qquad y = \log x;$$

$$\frac{2}{\pi}\left\{\frac{2}{B_2}\left(\frac{\log x}{2\pi}\right) + \frac{4}{3B_4}\left(\frac{\log x}{2\pi}\right)^3 + \frac{6}{5B_6}\left(\frac{\log x}{2\pi}\right)^5 + \cdots\right\};$$

$$\sum_{m=1}^\infty \frac{\mu(m)}{m}\int_c^{x^{1/m}} \frac{dt}{\log t}, \qquad c = 1.45136380...$$

Of these formulas, the third occurs in Riemann's 1860 paper and a series related to (but not identical with) the second formula appears in the work of Gram. But the first one was new and Hardy [15] comments that it "had never, as far as I know, appeared before; and in any case I am sure (for reasons which I will state later) that Ramanujan had found all three functions for himself." He then proceeds to show how the assertions that Ramanujan made in his letters are wrong. (In this context, the reader may also read further letters between Hardy and Ramanujan discussed in the book by Berndt and Rankin [8].)

In the fourth of his twelve lectures, Hardy [15] elaborates on how Ramanujan's identity (1.2) was used by Ingham [17] to deduce the prime number theorem, that the number of primes up to x is asymptotic to $x/\log x$ as x tends to infinity. As is well-known via Tauberian theory, this is equivalent to the non-vanishing of the Riemann zeta function on the line $\Re(s) = 1$. Ramanujan's identity is used in conjunction with a famous lemma of Landau, which we reproduce below, for the sake of completeness. Recall that if a Dirichlet series

$$g(s) = \sum_{n=1}^\infty \frac{c_n}{n^s},$$

converges at $s = s_0$ (say), then it converges for any s with $\Re(s) > \Re(s_0)$ and the function thus defined is holomorphic in this region. This fact is easily established using partial summation and is a standard fact in the theory of

Dirichlet series. Thus, given a Dirichlet series $g(s)$ as above, the set of values of s for which the series for $g(s)$ converges contains a maximal open half-plane of the form $\Re(s) > \sigma_0$ and the value of σ_0 is called its *abscissa of convergence*.

Lemma 5.1. (Landau) *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series with non-negative coefficients. Then $f(s)$ has a singularity at its abscissa of convergence.

Ingham's proof of the non-vanishing of the Riemann zeta function using Ramanujan's identity (1.2) now proceeds as follows. Suppose that $\zeta(1+it_0) = 0$ with $t_0 \in \mathbb{R}$ and $i = \sqrt{-1}$. Put $a = it_0$ and $b = -it_0$ in (1.2). Thus,

$$\sum_{n=1}^{\infty} \frac{|\sigma_{it_0}(n)|^2}{n^s} = \frac{\zeta(s)^2\zeta(s-it_0)\zeta(s+it_0)}{\zeta(2s)}. \tag{5.1}$$

Since $\zeta(1+it_0) = 0$, we see that $\zeta(1-it_0) = 0$. Thus, the double pole at $s = 1$ of $\zeta(s)^2$ is cancelled by the double zero of $\zeta(s-it_0)\zeta(s+it_0)$ at $s = 1$. Thus, the right hand side of the identity (5.1) is analytic for $\Re(s) > 1/2$. By Landau's theorem, the Dirichlet series in (5.1) has a singularity at its abscissa of convergence which must be $\leq 1/2$ by what we have just said above. However, the right hand side of (5.1) has a zero at $s = 1/2$ since the denominator has a pole there. But the left hand side being a Dirichlet series with non-negative coefficients is ≥ 1 (corresponding to the term $n = 1$) as $s \to \frac{1}{2}^+$. This contradiction proves that $\zeta(1+it_0) \neq 0$. Hardy discusses this on page 60 of his celebrated twelve lectures on Ramanujan [15].

In his paper, Ingham [17] observes that a minor change in Ramanujan's identity allows one to deduce the non-vanishing of Dirichlet's L-functions $L(s,\chi)$ on the line $\Re(s) = 1$. It is well-known that this again is equivalent to the prime number theorem for arithmetic progressions. Ingham's idea is best formulated in the following theorem which can be viewed as a generalization of the "general theorem" in [17]. To state this, let us introduce some notation. Given two arithmetical function f_1, f_2, we define the Dirichlet convolution $f_1 * f_2$ by

$$(f_1 * f_2)(n) = \sum_{d|n, d>0} f_1(d) f_2(n/d).$$

We say an arithmetical function h is *completely multiplicative* if $h(mn) = h(m)h(n)$ for all natural numbers m, n. We say a function h is *multiplicative* if $h(mn) = h(m)h(n)$ whenever m and n are coprime. It will also be convenient to introduce the associated Dirichlet series for any arithmetical function h,

$$L(s,h) := \sum_{n=1}^{\infty} \frac{h(n)}{n^s}.$$

Here is the generalization of the Ramanujan identity (1.2).

Theorem 5.2. *Let f_1, f_2, f_3, f_4 be completely multiplicative arithmetical functions. Then*

$$L(s,(f_1 * f_2)(f_3 * f_4)) = \frac{L(s,f_1f_3)L(s,f_1f_4)L(s,f_2f_3)L(s,f_2f_4)}{L(2s,f_1f_2f_3f_4)}.$$

Proof. Since f_1, f_2, f_3, f_4 are completely multiplicative, it is easy to see that $f_1 * f_2$ and $f_3 * f_4$ are multiplicative functions. It now follows that the series on the left hand side can be expressed as an Euler product. Indeed, noting the identity

$$x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n = \frac{x^{n+1} - y^{n+1}}{x - y}, \tag{5.2}$$

we see that the p-th Euler factor is given by

$$\sum_{n=0}^{\infty} \left(\frac{f_1(p)^{n+1} - f_2(p)^{n+1}}{f_1(p) - f_2(p)} \right) \left(\frac{f_3(p)^{n+1} - f_4(p)^{n+1}}{f_3(p) - f_4(p)} \right) p^{-ns}. \tag{5.3}$$

Here if $f_1(p) = f_2(p)$, we interpret the term in brackets as

$$f_1(p)^n + f_1(p)^{n-1}f_2(p) + \cdots + f_1(p)f_2(p)^{n-1} + f_2(p)^n$$

and the same if $f_3(p) = f_4(p)$. Setting $T = p^{-s}$, we see that the sum (5.3) is a sum of four geometric series and the final result becomes

$$\frac{1 - f_1(p)f_2(p)f_3(p)f_4(p)T^2}{(1 - f_1(p)f_3(p)T)(1 - f_1(p)f_4(p)T)(1 - f_2(p)f_3(p)T)(1 - f_2(p)f_4(p)T)},$$

from which the result is now evident. □

Taking $f_1 = 1$ and $f_2(n) = n^a$, $f_3 = 1$ and $f_4(n) = n^b$, we deduce Ramanujan's identity (1.2). Let now χ, ψ be two Dirichlet characters (mod q). Choosing $f_1 = 1$, $f_2(n) = \chi(n)n^a$, $f_3 = 1$ and $f_4(n) = \psi(n)n^b$, then with

$$\sigma_{a,\chi}(n) := \sum_{d|n} \chi(d)d^a,$$

we deduce Ramanujan's formula for L-series (as given in [3]):

$$\sum_{n=1}^{\infty} \frac{\sigma_{a,\chi}(n)\sigma_{b,\psi}(n)}{n^s} = \frac{\zeta(s)L(s-a,\chi)L(s-b,\psi)L(s-a-b,\chi\psi)}{L(2s-a-b,\chi\psi)}.$$

To deduce the non-vanishing of $L(s,\chi)$ on $\Re(s)=1$, we choose $\psi=\overline{\chi}$, $b=\overline{a}$, $a=it_0$ with $t_0\in\mathbb{R}$ and argue as before. Indeed, with these choices we have

$$\sum_{n=1}^{\infty}\frac{|\sigma_{a,\chi}(n)|^2}{n^s}=\frac{\zeta(s)L(s-it_0,\chi)L(s+it_0,\overline{\chi})L(s,\chi_0)}{L(2s,\chi_0)},$$

where χ_0 is the trivial character (mod q). If $L(1-it_0,\chi))=0$, then $L(1+it_0,\overline{\chi})=0$ which would cancel the poles arising from $\zeta(s)$ and $L(s,\chi_0)$ at $s=1$. Thus, the right hand side is analytic for $\Re(s)\geq 1/2$. Thus the abscissa of convergence is $<1/2$. But the right hand side has a zero at $s=1/2$ (arising from the pole at $s=1/2$ of the denominator) and the left hand side is ≥ 1 as $s\to\frac{1}{2}^+$.

6. The Chebotarev Density Theorem Via Ramanujan's Identity

Just as Ramanujan's identity was used to prove the non-vanishing of Dirichlet L-series on $\Re(s)=1$, one can apply a similar method to deduce non-vanishing of Artin L-series on the line $\Re(s)=1$. This is equivalent to the Chebotarev density theorem, which can be viewed as the natural generalization of Dirichlet's theorem. Since the background is formidable to present in a short expository paper such as this, we will content ourselves to give the main idea of how this can be carried out. We refer the reader to [22] for further details and background of the theory.

Brauer's induction theorem reduces the problem of non-vanishing of Artin L-series on $\Re(s)=1$ to the case of Hecke L-series attached to a number field. We then need to show that such an L-series does not vanish on $\Re(s)=1$. Any Hecke L-series attached to a number field F is of the form

$$L(s,\chi)=\sum_{\mathfrak{a}\neq 0}\frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s},$$

where the summation is over non-zero ideals $\mathfrak{a}$ of F and χ is a Hecke character attached to a generalized ideal class group. For $\Re(s)>1$, this L-series has an Euler product:

$$L(s,\chi)=\prod_{\mathfrak{p}}\left(1-\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}.$$

Hecke (see [22]) proved that each $L(s,\chi)$ extends to an entire function (for χ non-trivial) and satisfies a suitable functional equation. When χ is trivial, this is essentially the Dedekind zeta function of F, denoted $\zeta_F(s)$, multiplied by

$$\prod_{\mathfrak{p}|\mathfrak{f}_\chi}\left(1-\frac{1}{N(\mathfrak{p})^s}\right),$$

where $\mathfrak{f}_\chi$ is the conductor of χ. Thus, when $\chi = \chi_0$ is the trivial character, then $L(s,\chi_0)$ is analytic everywhere except at $s=1$ where it has a simple pole.

Our presentation of Ramanujan's identity in Theorem 5.2 was completely formal and it is easily seen that if we define

$$\sigma_{a,\chi}(\mathfrak{a}) = \sum_{\mathfrak{b}|\mathfrak{a}} \chi(\mathfrak{b}) N(\mathfrak{b})^a,$$

then

$$\sum_{\mathfrak{a}\neq 0} \frac{|\sigma_{a,\chi}(\mathfrak{a})|^2}{N(\mathfrak{a})^s} = \frac{\zeta_F(s) L(s-a-\overline{a},\chi_0) L(s-a,\chi) L(s-\overline{a},\overline{\chi})}{L(2s-a-\overline{a},\chi_0)},$$

where χ_0 is the trivial character corresponding to the associated generalized ideal class group. As before, if $L(1+it_0,\chi)=0$, we choose $a = it_0$ and proceed along the earlier line of argument.

7. Ramanujan's Identity And The Rankin-Selberg Method

Had Ramanujan lived another forty years, he would have been elated to see the essential role his identity would play in the determination of the special values of L-series attached to cusp forms, as well as Rankin's spectacular 1939 paper [30] in which he makes inroads towards the Ramanujan conjecture regarding the estimate on his τ-function. More generally, if f is a normalized cuspidal eigenform of weight k for the full modular group, then we can associate the L-series

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where the a_n's are the Fourier coefficients of f at the cusp $i\infty$. Hecke showed that $L(s,f)$ extends to an entire function and satisfies a suitable functional equation relating its value at s to its value at $k-s$. In [31], Rankin proved the following theorem. Set

$$\Lambda_f(s) := (2\pi)^{-s}\Gamma(s)L(s,f),$$

where $\Gamma(s)$ denotes the classical Γ-function of Euler. For any even integer j with $k/2+2 \le j \le k-4$, we have

$$\Lambda_f(j)\Lambda_f(k-1) = (-1)^{j/2} 2^{j-1} (f, E_j E_{k-j}),$$

where $(\cdot,\cdot)$ denotes the Petersson inner product and E_j denotes the Eisenstein series of weight j we met earlier. Surprisingly, the fact that such an explicit formula for the special values holds in greater generality was noted by Shimura [33] much later in 1976. Here is a brief description of the essential ideas.

Let k be a positive integer and χ a Dirichlet character (mod N) such that $\chi(-1) = (-1)^k$. Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 (\mathrm{mod}\, N) \right\}$$

and $M_k(N, \chi)$ the vector space of all holomorphic modular forms $f(z)$ satisfying

$$f(\gamma z) = \chi(d)(cz+d)^k f(z), \qquad \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

and

$$\gamma z := \frac{az+b}{cz+d}.$$

The subspace of modular forms vanishing at all the cusps (see [34] for terminology) is denoted $S_k(N, \chi)$. We put

$$M_k(N) = \underset{\chi \mathrm{mod}\, N}{\oplus} M_k(N, \chi), \qquad S_k(N) = \underset{\chi \mathrm{mod}\, N}{\oplus} S_k(N, \chi),$$

where the direct sums run over all Dirichlet characters (mod N). Clearly, these spaces consist of modular forms and cusp forms of weight k with respect to the group

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 (\mathrm{mod}\, N) \right\}.$$

Every element f of $M_k(N)$ has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

and we have its associated L-series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

For two elements f, h in $M_k(N)$ such that fh is a cusp form, we define the Petersson inner product (f, h) by

$$(f, h) := \frac{3}{\pi [SL_2(\mathbb{Z}) : \Gamma_1(N)\{\pm 1\}]} \int_{D_N} \overline{f(z)} h(z) y^{k-2} dx dy, \qquad z = x + iy,$$

where D_N denotes a fundamental domain for the action of $\Gamma_1(N)$ on the upper half-plane.

Given $f \in S_k(N, \chi)$ and $g \in M_j(N, \psi)$ with Fourier expansions

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}, \quad g(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z},$$

we define the Rankin-Selberg L-series

$$D(s,f,g) := \sum_{n=1}^{\infty} \frac{a_n b_n}{n^s}.$$

If f, g are nomalized Hecke eigenforms, the coefficients a_n, b_n are multiplicative and so the Euler factor can be computed using Theorem 5.2. In the special case that $g(z)$ is a classical Eisenstein series, the result is particularly striking. If f has level 1, we find from Theorem 5.2, that

$$\sum_{n=1}^{\infty} \frac{a_n \sigma_r(n)}{n^s} = \frac{L(s,f)L(s-r,f)}{\zeta(2s-r-k+1)}.$$

Rankin showed in the level one case that special values of the left hand side can be expressed as a rational multiple of the Peterssson inner product of f with products of classical Eisenstein series. About twenty-five years later, Shimura [33] generalized this to the higher level case. Following Shimura [33], we put

$$D_N(s,f,g) := L(2s+2-k-j, \chi\psi) D(s,f,g),$$

where $L(s,\omega)$ denotes the Dirichlet L-function attached to the character ω and write $f_*(z) := \overline{f(-\overline{z})}$ which has Fourier expansion

$$\sum_{n=1}^{\infty} \overline{a}_n e^{2\pi i n z}.$$

Then, $f_*(z) \in S_k(N, \overline{\chi})$ and it is easily checked that

$$\int_0^1 \overline{f_*(z)} g(z) dx = \sum_{n=1}^{\infty} a_n b_n e^{-4\pi n y}, \quad z = x+iy.$$

By taking the Mellin transform of the right hand side we see that

$$\int_0^{\infty} y^{s-1} \int_0^1 \overline{f_*(z)} g(z) dx dy = (4\pi)^{-s} \Gamma(s) D(s,f,g),$$

which is valid for $\Re(s)$ sufficiently large. Now let $\lambda \geq 0$ be an integer and ω a Dirichlet character (mod N), with $\omega(-1) = (-1)^{\lambda}$. Following Hecke [16], we define the Eisenstein series

$$E_{\lambda}^*(z,s,\omega) = \sum_{\gamma \in R} \omega(d)(cz+d)^{-\lambda} |cz+d|^{-2s}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where R is a complete set of representatives for $\Gamma_{\infty} \backslash \Gamma_0(N)$ and

$$\Gamma_{\infty} = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}.$$

It is easy to see that the series for $E_{\lambda}^*(z,s,\omega)$ is absolutely convergent for $\Re(s) > 2 - \lambda$. We suppose that $k \geq j$ and transform the integral above by the standard unfolding method (see for example p. 54 of [23]) to deduce with $\lambda = k - j$,

$$(4\pi)^{-s}\Gamma(s)D(s,f,g) = \int_{D_N} \overline{f_*(z)} g(z) E_\lambda^*(z, s+1-k, \chi\psi) y^{s-1} dxdy.$$

If we put

$$E_\lambda(z,s,\omega) = \sum_{(m,n)\neq(0,0)} \omega(n)(mNz+n)^{-\lambda}|mzN+n|^{-2s},$$

where the summation is over all $(m,n) \in \mathbb{Z}^2$ with $(m,n) \neq (0,0)$. It is then not too difficult to show that

$$E_\lambda(z,s,\omega) = 2L(2s+\lambda,\omega)E_\lambda^*(z,s,\omega)$$

so that

$$2(4\pi)^{-s}\Gamma(s)D_N(s,f,g) = \int_{D_N} \overline{f_*(z)} g(z) E_\lambda(z, s+1-k, \chi\psi) y^{s-1} dxdy.$$

Hecke [16] studied the function $E_\lambda(z,s,\omega)$. He proved the following. Put $\mathcal{E}(s) = \Gamma(s+\lambda)E_\lambda(z,s,\omega)$. If $\lambda \neq 0$ or ω is non-trivial, then $\mathcal{E}(s)$ extends to an entire function. If $N = 1$, and $\lambda = 0$, then $\mathcal{E}(s)$ is regular everywhere except for simple poles at $s = 0, 1$; if $N > 1$, $\lambda = 0$ and ω is trivial, then $\mathcal{E}(s)$ is regular everywhere except for a simple pole at $s = 1$. In addition, Hecke [16] proved that both $E_\lambda^*(z,0,\omega)$ and $E_\lambda(z,0,\omega)$ belong to $M_\lambda(N,\overline{\omega})$ except when $\lambda = 2$ and ω is trivial. In the latter case, each of these functions is a constant times y^{-1} plus a holomorphic function in z. In addition, if we define the differential operators δ_λ and $\delta_\lambda^{(r)}$ by

$$\delta_\lambda = \frac{1}{2\pi i}\left(\frac{\lambda}{2iy} + \frac{\partial}{\partial z}\right), \qquad \lambda > 0,$$

$$\delta_\lambda^{(r)} = \delta_{\lambda+2r-2}\cdots\delta_{\lambda+2}\delta_\lambda, \qquad r \in \mathbb{Z}, \quad r \geq 0,$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

with the understanding that $\delta_\lambda^{(0)}$ is the identity operator, then it is easily verified that if $h \in M_\lambda(N,\omega)$, then $\delta_\lambda^{(r)}h$ has the same automorphic property as the elements of $M_{\lambda+2r}(N,\omega)$. Shimura [33] proved that if $f \in S_k(N,\chi)$ and $g \in M_j(N,\psi)$ and $j + 2r < k$ with r a non-negative integer, then

$$D(k-1-r,f,g) = c\pi^k(f_*, g\delta_\lambda^{(r)}E_\lambda^*(z,0,\chi\psi)),$$

where $\lambda = k - j - 2r$ and

$$c = \frac{\Gamma(k-j-2r)}{\Gamma(k-1-r)\Gamma(k-j-r)}\frac{(-1)^r 4^{k-1}N}{3}\prod_{p|N}\left(1+\frac{1}{p}\right),$$

where the product is over the prime divisors p of N. In the level one case, this was also proved by Zagier (see [35, Proposition 6]) by different methods. Zagier's technique also allows him to prove the following theorem which would definitely have interested Ramanujan. If f is a cusp form of weight k for the full modular group as above, and

$$D_f(s) = \frac{\zeta(2s-2k+2)}{\zeta(s-k+1)} \sum_{n=1}^{\infty} \frac{a_n^2}{n^s},$$

then Rankin [30] and Selberg [32] independently showed that $D_f(s)$ extends as a meromorphic function to the entire complex plane and satisfies the functional equation

$$D_f^*(s) = 2^{-s}\pi^{-3s/2}\Gamma(s)\Gamma((s-k+2)/2)D_f(s) = D_f^*(2k-1-s).$$

Zagier [35] shows that the values of $D_f(s)/\pi^{2s-k+1}$ at $s = k, k+2, ..., 2k-2$ are all algebraic multiples of the inner product (f, f). In this context, Ramanujan would have been intrigued by the following connection of the special value $\zeta(3)$ to the Petersson inner product discovered by Petersson [25]. If we let Γ be the group generated by the matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$\theta_3(z) = \sum_{m=-\infty}^{\infty} e^{\pi i m^2 z},$$

$$\theta_0(z) = \sum_{m=-\infty}^{\infty} (-1)^m e^{\pi i m^2 z},$$

$$\theta_2(z) = \sum_{m=-\infty}^{\infty} e^{\pi i (m+1/2)^2 z},$$

be the classical theta functions, then $\theta_3^8, (\theta_0\theta_2)^4$ are modular forms of weight 4 for the group Γ. Petersson [25] proved that

$$\zeta(3) = \frac{\pi^3}{7}(\theta_3^8, (\theta_0\theta_2)^4).$$

Undoubtedly, the future of the study of special values of the Riemann zeta function has modular connections, as is revealed by the early work of Ramanujan.

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