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Ramanujan Graphs: An Introduction

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Dedicated to the memory of Srinivasa Ramanujan

Let us expand our consciousness
And learn from the one who knows infinity,
Decipher the language of symbol and metaphor
And feel the fabric that covers divinity.

Abstract. We give a short survey of the theory of Ramanujan graphs. Beginning with basic notions, we introduce Cayley graphs and describe how group representations can be used to determine the eigenvalues of Cayley graphs. We include a short history of character theory. We then discuss a theorem discovered by Noga Alon and Ravi Bopanna that motivates the search for Ramanujan graphs. The Ihara zeta function of a regular graph is invoked to amplify the analogies with number theoretic themes. The explicit construction of $(p+1)$-regular Ramanujan graphs for p prime given by Lubotzky, Phillips and Sarnak is described. Finally, we highlight the recent work of Marcus, Spielman and Srivastava showing the existence of infinite families of k-regular Ramanujan graphs for every degree k. We conclude with some open questions suggested by this study. The article is largely self-contained and accessible to the graduate student.

Keywords:

1. Introduction

The power of a symbol and its meaning lies at the heart of mathematical discovery. For example, the discovery of zero and the number system marks a significant event

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in the dawn of human civilization. It is said that the language of the cosmos is mathematics, and we must discover this language which seems to be infinite in its variety of symbols, syntax, vocabulary, grammar and meaning.

The mathematical concept of a graph is as profound as the discovery of zero. It enables us to understand complicated processes in the world around us at a single glance. It is now used routinely to model communication networks, and many theorems of graph theory are applied to resolve practical concerns of optimization. Recently, there have been attempts to model the human brain using graph theory, with neurons being represented as vertices and synapses being the edges [59]. What can be more mysterious and complicated than the human brain?

Though the discipline now has wide application, graph theory as a proper subject of mathematics is relatively of recent origin and its beginning can be traced back to 1736 when Euler solved the famous Königsberg bridge problem using basic ideas of graph theory. Though this work was important, it was not seen at that time as such and was viewed as a chapter in recreational mathematics. In fact, much of the earlier work in graph theory was seen through this lens. The 1852 letter of de Morgan to Hamilton asking about the empirical observation of cartographers that any map can be colored using only four colors such that no two adjacent countries are colored using the same color, can be said to be the first significant awareness that suggested there is a new subject here to be studied in its own right. In 1878, Cayley presented this empirical observation as a problem for research to the London Mathematical Society. This became known as the four color conjecture. In 1879, Kempe published an incorrect proof of the conjecture and it was only in 1890 that Heawood identified the mistake in Kempe's proof. Heawood's correction led to the "five color theorem" and with this episode, we have the birth of a bona fide subject called graph theory. Its rapid development is a twentieth century phenomenon. In 1976, Appel and Haken finally announced a computer-assisted proof of the four color conjecture. It is desirable, from an aesthetical mathematical standpoint, to have a "pure thought" proof of this theorem.

With the four color conjecture as an inspiring muse, graph theory blossomed into a thousand petaled lotus, full of color, beauty and a remarkable range of applicability. The topic of this paper, Ramanujan graphs, has a surprising number of applications in our modern digital age. We will mention some of them below. The reader can find, for example, applications to the construction of optimal communication networks in a paper of Bien [5]. An earlier survey [44] of mine was written almost twenty years ago. In this survey, we will present a snapshot of some recent developments.

2. Basic Notions

A *graph* X is a collection of points called *vertices* together with a collection of *edges* that may or may not be directed. More precisely, X is a pair (V, E) with V being the *vertex set* and E being the *edge set* (which can be viewed as a subset or a multi-subset of $V \times V$). The number of edges emanating from a vertex x is called the *outdegree* and the number of edges coming into x is called the *indegree*. The sum of the outdegree and the indegree of x is simply called the *degree* of the vertex x. If the degree of each vertex of the graph has the same value k, we say the graph is a *k-regular graph*. We say the graph is *finite* if the vertex set V is a finite set. In this paper, we will be studying a subset of finite regular graphs called *Ramanujan graphs* (to be defined below).

If we label the vertices of the graph $v_1, \ldots, v_n$, it is possible to represent the graph by an $n \times n$ matrix, called the *adjacency matrix* A of X. The (i, j)-th entry of this matrix, denoted a_{ij}, equals the number of edges emanating from v_i into v_j. It is clear that given the adjacency matrix A, we can reconstruct the graph X and in this way, we can transfer the study of the graph X to the study of the matrix A. When one speaks of the eigenvalues of X, we really mean the eigenvalues of the adjacency matrix A of X. When two vertices u, v are adjacent, we sometimes write $u \sim v$ and say that u and v are *neighbors*.

For the most part, graphs in this paper are undirected. Therefore, the adjacency matrix is a real symmetric matrix. In addition, our graphs will be loopless, so that the diagonal entries of the adjacency matrix will be zero. By familiar theorems of linear algebra, a real symmetric matrix has all eigenvalues real and we can order the eigenvalues as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

The (i, j)-th entry of A^r is evidently the number of paths of length r from i to j. Thus, the trace of A^r tabulates the number of *closed* paths of length r in X. On the other hand, by linear algebra, this number is equal to

$$\lambda_1^r + \cdots + \lambda_n^r.$$

A closed path is also called a *cycle* in the literature. When a graph has no cycles, we say it is *acyclic*. For any acyclic graph, $\operatorname{tr}(A^r) = 0$ for every r. In this way, we see an intimate connection between the eigenvalues of A and properties of X, which is a dominant theme of *spectral graph theory*. There are some excellent textbooks on spectral graph theory that the student may consult such as [10, 24] and [7].

The *complete graph*, denoted $\mathscr{K}_n$, is a graph on n vertices in which any two distinct vertices are adjacent. It is not difficult to calculate the eigenvalues of the adjacency matrix of $\mathscr{K}_n$ (and we present one method in Section 5). The eigenvalues are n with multiplicity 1 and -1 with multiplicity $n - 1$.

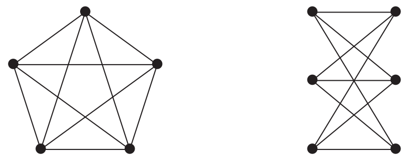

Figure 1. $\mathcal{K}_5 : 4^{(1)}, -1^{(4)}$; $\mathcal{K}_{3,3} : 3^{(1)}, 0^{(4)}, -3^{(1)}$

A set S of vertices of X is said to be *independent* if no two vertices of S are adjacent in X. A graph X is called *bipartite* if the vertex set V of X can be written as a disjoint union of independent sets. With suitable labelling, it is clear that the adjacency matrix of a bipartite graph has the form

$$\begin{pmatrix} \mathbf{0} & B \\ B^T & \mathbf{0} \end{pmatrix},$$

where $\mathbf{0}$ represents a suitable zero matrix and B^T is the transpose of B.

The *complete bipartite graph* $\mathcal{K}_{r,s}$ is the graph consisting of two independent vertex sets V_1 and V_2 of sizes r and s respectively, and every vertex in V_1 is adjacent to every vertex in V_2. It is easy to calculate the eigenvalues of the complete bipartite graph. Indeed, the matrix evidently has rank 2 and so the eigenvalue 0 has multiplicity $n-2$ where $n = r+s$. Letting λ_1, λ_2 be the two non-zero eigenvalues, we see that $\lambda_1 + \lambda_2 = 0$ since the trace of the adjacency matrix is zero. Moreover, $\lambda_1^2 + \lambda_2^2$ is equal to the number of closed walks of length 2 which is $2rs$ since any such trail goes from a vertex to an adjacent vertex and immediately backtracks to its origin. So the two non-zero eigenvalues are $\pm\sqrt{rs}$. In particular, the complete bipartite graph $\mathcal{K}_{k,k}$ has two non-zero eigenvalues equal to $\pm k$.

Here are some small examples of graphs along with the eigenvalues (together with their multiplicities):

3. Bounds for Eigenvalues of Trees

We say a graph is connected if for any two vertices, there is a path connecting them. A *tree* is an acyclic connected graph. In particular, it has no closed cycles of odd length, and therefore can be shown to be bipartite. In fact, a finite graph is bipartite if and only if it has no odd cycle (see for example, Theorem 1.5.1 of [12]). If T is a tree, a vertex of T is called a *leaf* if it has degree one. A *rooted tree* distinguishes one vertex r (say) as a *root*. Any vertex of the tree can be taken as a root. Using a root r, we see that as the graph is connected and acyclic, there is a unique path joining the root r to every other vertex v. We let $\ell(v)$ be the length of this path from r to v. Sometimes, we refer to $\ell(v)$ as the length of v (which, of course, depends on the choice of the root r). An important role is played by the following theorem in the theory of Ramanujan graphs.

Theorem 1. *If T is a tree with maximum degree k, then all the eigenvalues λ of T satisfy $|\lambda| \leq 2\sqrt{k-1}$ for $k \geq 2$.*

Proof. Let us fix a root r of the tree, Let A be the adjacency matrix of T. For any invertible diagonal matrix D, the eigenvalues of $B := DAD^{-1}$ are the same as A. With $\delta > 0$ to be chosen later, we let the i-th diagonal entry of D to be $\delta^{\ell(i)}$. Then a routine calculation shows

$$b_{ij} = a_{ij}\delta^{\ell(i)-\ell(j)}.$$

Let λ be a non-zero eigenvalue of B. If $\mathbf{x} = (x_1, \ldots, x_n)$ is a non-zero eigenvector of B corresponding to the eigenvalue λ, we have

$$\lambda x_i = \sum_j a_{ij}\delta^{\ell(i)-\ell(j)}x_j.$$

Since $\mathbf{x}$ is a non-zero vector, we can choose i so that $x_i \neq 0$ and $|x_i|$ is of maximal absolute value among the components of $\mathbf{x}$. If i corresponds to the root vertex v, then $\ell(i) = 0$ and every vertex j adjacent to i has length 1. As the maximum degree of T is k, we find

$$|\lambda x_i| \leq |x_i|k/\delta \qquad \Longrightarrow |\lambda| \leq k/\delta.$$

If i is not a root, then precisely one neighbor (occuring in the unique path from the root to i) has length $\ell(i) - 1$. Any other neighbor of i (which is not a root or a leaf) has length $\ell(i) + 1$. As the degree of i is at most k, this gives

$$|\lambda x_i| \leq |x_i|\left(\delta + \frac{k-1}{\delta}\right) \qquad \Longrightarrow |\lambda| \leq \delta + \frac{k-1}{\delta}.$$

Finally, if i is a leaf, it is adjacent to only one vertex which has length $\ell(i) - 1$. Thus, in this case,

$$|\lambda x_i| \leq |x_i|\delta \qquad \Longrightarrow |\lambda| \leq \delta.$$

The middle case suggests we choose $\delta = \sqrt{k-1}$ to optimise our estimates, and we find $|\lambda| \leq 2\sqrt{k-1}$ in all cases. $\square$

4. Regular Graphs

For a k-regular graph X, with adjacency matrix A, it is evident that

$$A\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = k\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix},$$

since each vertex has degree k. Thus, k is an eigenvalue of the adjacency matrix of any k-regular graph. Any eigenvalue with absolute value k is called a "trivial eigenvalue" of A. We call all other eigenvalues as "non-trivial". For a k-regular graph X, we define

$$\lambda(X) := \max_{\lambda_i \neq \pm k} |\lambda_i|.$$

A *Ramanujan graph* is a k-regular graph all of whose non-trivial eigenvalues lie in the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$. That is, $\lambda(X) \leq 2\sqrt{k-1}$. The reason for this strange interval will be made clear later and its significance to optimization questions will be discussed at the end of the paper. But for the moment, it is important to observe that the complete bipartite graph $\mathscr{K}_{k,k}$ is Ramanujan as all the non-trivial eigenvalues are zero.

It is not difficult to show that for a k-regular graph X the multiplicity of the eigenvalue k is equal to the number of its connected components. Thus, if X is a k-regular connected graph, the eigenvalue k has multiplicity one. Also, $-k$ is an eigenvalue if and only if X is bipartite and in fact, the eigenvalues are symmetric about the origin, counted with multiplicity (see for example, Theorem 4.3.2 of [12]).

In this paper, our concern is mainly with connected k-regular graphs. For a finite connected graph X, we can define a metric on X as follows. Given two vertices $u, v \in X$, we let $d(u,v)$ be the length of the shortest path from u to v. The *diameter* of X is then defined as the maximum value of this distance function. Thus, the diameter is a measure of how fast one can go from one vertex in a graph to another. There is a significant theorem of Chung [9] which gives a bound for the diameter of a k-regular graph in terms of $\lambda(X)$. A refinement due to van Dam and Haemers [15] (see also [51]) is presented next.

Theorem 2. *If X is a connected k-regular graph with n vertices, the diameter of X is bounded by*

$$1 + \frac{\log 2n}{\log\left(\dfrac{k + \sqrt{k^2 - \lambda(X)^2}}{\lambda(X)}\right)},$$

if X is not bipartite and by

$$2 + \frac{\log n}{\log\left(\dfrac{k + \sqrt{k^2 - \lambda(X)^2}}{\lambda(X)}\right)},$$

if X is bipartite.

Proof. Let A be the adjacency matrix of X. We consider the inner product space $L^2(V)$ and let ϕ_j, $1 \leq j \leq n$ be an orthonormal basis of eigenfunctions of A with corresponding eigenvalues λ_j. As X is k-regular, we have $\lambda_1 = k$ and so $\phi_1(v) = 1/\sqrt{n}$ for all vertices v. By the spectral theorem (for real symmetric matrices), we have for any polynomial P,

$$P(A)_{v,w} = \sum_{j=1}^{n} P(\lambda_j)\phi_j(v)\phi_j(w).$$

If for v, w, we have the distance $d(v,w) > N$, and the degree of the polynomial P is at most N then clearly $P(A)_{v,w} = 0$. Thus,

$$\sum_{j=1}^{n} P(\lambda_j)\phi_j(v)\phi_j(w) = 0.$$

Let us first suppose X is not bipartite. Then, $-k$ is not an eigenvalue of X. Separating the $j = 1$ term from the others, we have

$$\left|\frac{P(k)}{n}\right| = \left|\sum_{j=2}^{n} P(\lambda_j)\phi_j(v)\phi_j(w)\right| \leq \sup_{|\lambda| \leq \lambda(X)} |P(\lambda)| \sum_{j=2}^{n} |\phi_j(v)\phi_j(w)|.$$

Applying the Cauchy-Schwarz inequality on the last sum yields,

$$|P(k)| \leq n \sup_{|\lambda| \leq \lambda(X)} |P(\lambda)|,$$

for any polynomial P. We now choose our polynomial optimally. Indeed, letting

$$P_N(x) := T_N(x/\lambda(X)),$$

where

$$T_N(x) = \cos(N \arccos x) = \frac{1}{2}\left\{\left(x + i\sqrt{1-x^2}\right)^N + \left(x - i\sqrt{1-x^2}\right)^N\right\},$$

is the N-th Chebycheff polynomial (of the first kind). Since $|P_N(x)| \leq 1$ for $|x| \leq \lambda(X)$, we deduce $|P_N(k)| \leq n$. This implies

$$\frac{1}{2}\left(\frac{k + \sqrt{k^2 - \lambda(X)^2}}{\lambda(X)}\right)^N \leq n,$$

for every N strictly less than the diameter of X. The assertion of the theorem is now immediate upon taking logarithms. If X is bipartite, we have

$$\left|\frac{P(k)}{n} \pm \frac{P(-k)}{n}\right| = \left|\sum_{j=2}^{n-1} P(\lambda_j)\phi_j(v)\phi_j(w)\right| \leq \sup_{|\lambda| \leq \lambda(X)} |P(\lambda)| \sum_{j=2}^{n} |\phi_j(v)\phi_j(w)|.$$

As $T_N(x)$ is even or odd according as N is even or odd, we have

$$\left(\frac{k+\sqrt{k^2-\lambda(X)^2}}{\lambda(X)}\right)^N \leq n,$$

for N either one less or two less than the diameter of X from which the theorem is now immediate. □

Theorem 2 shows that to minimize the diameter, we need to minimize $\lambda(X)$, and this explains our interest in Ramanujan graphs. One way of constructing families of k-regular graphs is using group theory which we discuss in the next section.

5. Cayley Graphs

Let G be a finite group of order n and S a k element subset of G which is stable under conjugation. We construct the *Cayley graph* $X(G,S)$ as follows. The vertices of this graph are the elements of the group G listed as $g_1, g_2, \ldots, g_n$. We join g_i to g_j if $g_i g_j^{-1} \in S$. This is a directed graph unless S is stable under the map $s \mapsto s^{-1}$ for all $s \in S$. It is remarkable that one can determine all the eigenvalues and eigenvectors of the adjacency matrix A of this graph (even in the directed case) using group theory. In the case that G is abelian, this is relatively easy to see. Indeed, for each irreducible character χ of G, we define the column vector

$$v_\chi = \begin{pmatrix} \chi(g_1) \\ \chi(g_2) \\ \vdots \\ \chi(g_n) \end{pmatrix}.$$

Then

$$(Av_\chi)_i = \sum_{j=1}^{n} a_{ij}\chi(g_j) = \sum_{j:g_i g_j^{-1}\in S} \chi(g_j) = \sum_{s\in S} \chi(g_i s^{-1}) = \left(\sum_{s\in S}\chi(s^{-1})\right)(v_\chi)_i.$$

In other words,

$$\sum_{s\in S}\chi(s^{-1})$$

is an eigenvalue of A with eigenvector v_χ. For distinct characters, the v_χ are distinct and in fact, linearly independent by the linear independence of characters of a finite group. This gives a complete description of all the eigenvalues and eigenvectors of the Cayley graph $X(G,S)$ in the case G is abelian.

Nascent in the above calculation is the following determinant of Dedekind. If G is a finite abelian group of order n as above, and if $x_{g_1}, \dots, x_{g_n}$ are indeterminates, then

$$\det[x_{g_i g_j^{-1}}] = \prod_\chi \left(\sum_{i=1}^n \chi(g_i^{-1}) x_{g_i} \right),$$

where the product is over all characters of G. In particular, if $f : G \to \mathbb{C}$ is a complex valued function, the eigenvalues of the matrix $[f(gh^{-1})]$ are precisely the sums

$$\sum_{g \in G} f(g)\chi(g^{-1}).$$

Thus, the eigenvalues of the Cayley graph $X(G,S)$ are obtained by taking f to be the characteristic function of S. For further discussions of this determinant, along with combinatorial variations, we refer the reader to [47]. Both Dedekind and Frobenius viewed this result with wonder and awe since the left hand side could easily be a messy determinant and yet, it miraculously factors as a product of linear forms! The problem of generalizing this result to non-abelian finite groups marks the beginning of the development of representation theory of finite groups.

This calculation allows us to give an important family of Ramanujan graphs. If $\mathscr{K}_n$ is the complete graph on n vertices, it is clear that its adjacency matrix is $J - I$ where J is $n \times n$ matrix with every entry equal to 1 and I the $n \times n$ identity matrix. If we consider the *additive group* $\mathbb{Z}/n\mathbb{Z}$, and define $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ by $f(0) = 0$ and $f(i) = 1$ for $i \neq 0$, we immediately deduce that all the eigenvalues of the adjacency matrix of $\mathscr{K}_n$ are given by the character sums

$$\sum_{j=1}^{n-1} e^{2\pi i a j/n} = \begin{cases} n-1 & \text{if } a = 0 \\ -1 & \text{if } 1 \le a \le n-1. \end{cases}$$

This means that the regular graph $\mathscr{K}_n$ is Ramanujan.

The determination of the eigenvalues and eigenvectors of Cayley graphs in the non-abelian case is not so simple and requires some representation theory (which we review below). The complete description of the eigenvalues was first given in 1975 by Lovász [37], then re-discovered independently in 1979 by Babai [3], in 1981 by Diaconis and Shahshahani [17] and then later in 1988 by Zieschang [60]! We will present a simplified and self-contained treatment after a brief interlude into the origins of the representation theory of finite groups.

6. A Short History of Character Theory

The rapid development of algebra can be traced back to the end of the nineteenth century and the beginning of the twentieth. Motivated by a desire to solve Fermat's last theorem, Kummer began an ardent study of algebraic number fields. The first step was to extend the familiar unique factorization theorem of natural numbers into higher realms. He was partly successful in that he created the theory of "ideal numbers". But it was Dedekind who realized that the theory of modules over rings needs to be developed in order to fully understand the concepts of divisibility and factorization in algebraic number fields.

With the metaphor of the unique factorization theorem as a guiding principle, the concept of a *module* and an *algebra* began to emerge as the synthesizing ideas for a foundational theory. One also had to move away from commutative rings.

An *algebra* over a field K is a ring $\mathcal{A}$ with identity which is at the same time a vector space over K, and which satisfies the rule:

$$\lambda(ab) = (\lambda a)b = a(\lambda b), \qquad \lambda \in K, \quad a, b, \in \mathcal{A},$$

with respect to scalar multiplication.

There are many examples of algebras that the reader is already familiar with. The set of $n \times n$ matrices over the field K, denoted $M_n(K)$ is an algebra over K. If V is a vector space over K, then $\mathrm{End}(V)$, the set of linear transformations of V into itself is an algebra over K with multiplication being composition.

If G is a finite group and K is a field, we can construct the *group algebra*, denoted $K[G]$ as being the set of all formal sums

$$\sum_{g \in G} a_g g, \qquad a_g \in K.$$

Then, $K[G]$ is a K-vector space in the obvious way. It becomes an algebra if we define

$$\left(\sum_{g \in G} a_g g\right)\left(\sum_{h \in G} b_h h\right) := \sum_{g,h \in G} a_g b_h gh.$$

It is the structure of this group algebra (with $K = \mathbb{C}$) that lies at the heart of the problem of determining the eigenvalues and eigenvectors of Cayley graphs.

The center of this algebra is easily determined. It is not difficult to see that any element in the center is a linear combination of conjugacy classes. Indeed, if

$$\sum_{g \in G} a_g g$$

lies in the center, we must have

$$\sum_{g \in G} a_g h^{-1} g h = \sum_{g \in G} a_g g, \qquad \forall g \in G.$$

This means that a_g is constant on the conjugacy classes of G. In other words, the center is a vector space over K of dimension equal to the number of conjugacy classes of G.

An algebra can also be viewed as a module over itself, and thus, the structure theory of algebras is subsumed by the structure theory of modules over rings.

Let R be a ring with unity and M a (left) module over R. The *length* of the module M is the upper bound of the length r of chains of submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_r = 0,$$

where the inclusions are strict. If such an upper bound does not exist, we set $r = \infty$.

A module is called *simple* if it has no proper non-zero submodules. If M and N are simple modules over a ring R, then any morphism $f : M \to N$ is either the zero map or an isomorphism. This is called *Schur's lemma* and the proof is self-evident since ker f is either zero or M. In the latter case, f is the zero map, and in the former case, f is injective. But then Im f is a submodule of N isomorphic to M and this must be N because N is simple.

An alternate way of stating Schur's lemma is using the Hom operator. If M and N are simple R-modules, we write Hom (M, N) to designate the group of R-homomorphisms from M into N. Then, Schur's lemma says that Hom (M, N) equals 0 if M and N are not isomorphic. Otherwise, it is a division ring since any non-zero homomorphism $f : M \to M$ is invertible.

When Schur's lemma is applied to finite dimensional algebras over an algebraically closed field K, we deduce that if M is a simple algebra, then $\mathrm{End}(M) \simeq K$ because if $f \in \mathrm{Hom}(M, M)$, then $f - \lambda$ is the zero map for some $\lambda \in K$ (which is an eigenvalue of f which exists as K is algebraically closed).

We say M is *semisimple* if it is a direct sum of simple modules. It is easy to check that every submodule and quotient module of a semisimple module is again semisimple. Since a ring R can be viewed as a module over itself, we say R is *semisimple* if R (viewed as a module over itself) is semisimple. Since any R-module is a quotient module of a free module, and a free module is semisimple, we see that if R is semisimple, then any R-module is semisimple. Thus, the concept of semisimplicity allows us to "factorize" the module as a sum of simple modules.

An elegant structure theorem for semisimple modules of finite length was discovered by Wedderburn in 1908 and we describe it below and give a simple proof in the appendix. In essence, it says that every semisimple module is isomorphic to a direct sum of matrix rings over division rings. We do not need its full strength for our discussion, but include it here for the elucidation of ideas.

Let M be a semisimple algebra over an algebraically closed field. Schur's lemma shows immediately that any decomposition of M as a direct sum of simple modules is unique. Indeed, if

$$M = \oplus_i n_i M_i = \oplus_i m_i M_i,$$

are two different decompositions into non-isomorphic simple modules M_i (where we allow n_i and m_i to be zero), then for each k,

$$\mathrm{Hom}(M_k, M) = \oplus_i n_j \mathrm{Hom}(M_k, M_i) = \oplus_j m_j \mathrm{Hom}(M_k, M_j).$$

Hence, $n_k \mathrm{Hom}(M_k, M_k) = m_k \mathrm{Hom}(M_k, M_k)$. As K is algebraically closed, we have $\mathrm{Hom}(M_k, M_k) \simeq K$. We deduce $n_k = m_k$ by taking dimensions of both sides.

A module M is said to be *completely reducible* if for any submodule U there is a complementary submodule V such that $M = U \oplus V$. When the order of G is coprime to the characteristic of K, the complete reducibility of the group algebra $K[G]$ is a theorem due to Maschke. If $K = \mathbb{C}$, this is easy to see. Indeed, $M = \mathbb{C}[G]$ is isomorphic (as a vector space) to $\mathbb{C}^n$ where $n = |G|$. This has the usual inner product (see for example, p. 6 of Serre [54]), which we can make G-invariant by performing an "averaging trick." Given a submodule W of M we can take the orthogonal complement of W with respect to this G-invariant inner product as the complementary submodule of W in M. Thus, $\mathbb{C}[G]$ is completely reducible and therefore semisimple. In essence, this is Maschke's original proof.

Recall that a representation of a group G is a vector space V on which the group G acts. If V is finite dimensional, we say the representation is finite dimensional of *degree* equal to the dimension of V. Any representation V of G turns V into a $\mathbb{C}[G]$-module. As $\mathbb{C}[G]$ is semisimple, so is V. Thus, every representation of G can be decomposed into a direct sum of irreducible representations. The action of G on $M = \mathbb{C}[G]$ given by left multiplication is called the (left) *regular representation.* For $g \in G$, the trace of the linear transformation given by $v \mapsto gv$ for $v \in M$ is called the regular character and denoted χ_{reg}. Checking how this transformation acts on the basis elements and then taking its trace leads to:

$$\chi_{\mathrm{reg}}(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

χ_{reg} extends to $\mathbb{C}[G]$ by linearity.

We can decompose our group algebra $M = \mathbb{C}[G]$ as a direct sum of simple algebras:

$$M = \oplus_i M_i. \tag{2}$$

The map $v \mapsto gv$ restricted to M_i must be scalar multiplication, according to Schur's lemma. If we denote this scalar as $\chi_i(g)$, we see on noting $\chi_i(1)$ is the dimension of M_i, that

$$\chi_{\mathrm{reg}}(g) = \sum_i \chi_i(1)\chi_i(g). \tag{3}$$

In particular, we have from the decomposition (2) that the unit element of M can be written as:

$$1 = \sum_i e_i,$$

and $M_i = e_iM$, because each M_i is simple. Clearly, $e_ie_j = 0$ for $i \neq j$, and multiplying the above equation by e_j, we see that the e_j's are idempotents. That is, $e_j^2 = e_j$. We want to determine the e_j's explicitly.

If we write

$$e_i = \sum_{g \in G} a_g g,$$

then using (1), we have $\chi_{\mathrm{reg}}(e_ig^{-1}) = a_g|G|$ on the one hand. On the other hand, from (3),

$$|G|a_g = \sum_j \chi_j(1)\chi_j(e_ig^{-1}).$$

If we let ρ_j be the representation afforded by χ_j (that is, the representation given by the action of G on M_j), then

$$\rho_j(e_ig^{-1}) = \rho_j(e_i)\rho_j(g^{-1}),$$

which is zero unless $i = j$ in which case it is $\rho_j(g^{-1})$ because $\rho_i(e_i)$ is the identity matrix (since e_i is idempotent and $e_i(e_iv) = e_iv$). Therefore $\chi_j(e_ig^{-1}) = 0$ unless $i = j$, in which case it is $\chi_j(g^{-1})$. We swiftly deduce that

$$e_i = \frac{1}{|G|}\sum_{g \in G} \chi_i(1)\chi_i(g^{-1})g. \tag{4}$$

The familiar orthogonality relations for the irreducible characters are now immediate. But we actually have a slightly general result.

Theorem 3 (Generalized orthogonality relation). *For every $h \in G$, we have*

$$\frac{1}{|G|}\sum_{g \in G} \chi_i(gh)\chi_j(g^{-1}) = \delta_{ij}\frac{\chi_i(h)}{\chi_i(1)},$$

where δ_{ij} equals 1 if $i = j$ and zero otherwise.

Proof. This follows from comparing the left hand side and the right hand side of the equation $e_ie_j = \delta_{ij}e_i$. Indeed from (4), the coefficient of a fixed $h \in G$ on the right hand side is equal to

$$\frac{\delta_{ij}}{|G|}\chi_i(1)\chi_i(h^{-1}).$$

The coefficient of h on the left hand side is

$$\frac{\chi_i(1)\chi_j(1)}{|G|^2}\sum_{g_1g_2\in G; g_1g_2=h}\chi_i(g_1^{-1})\chi_j(g_2^{-1}) = \frac{\chi_i(1)\chi_j(1)}{|G|^2}\sum_{g\in G}\chi_i(gh^{-1})\chi_j(g),$$

from which the result is clear. □

Putting $h=1$ gives the familiar orthogonality relation:

Corollary 4 (The first orthogonality relation).

$$\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\chi_j(g^{-1}) = \delta_{ij}.$$

The orthogonal idempotents e_i also allow us to deduce that the number of irreducible characters χ_i is equal to the number of conjugacy classes. Indeed, as we noted above $\chi_j(e_i)=0$ if $i\neq j$ and $\chi_j(1)$ otherwise. Thus, the χ_j are distinct functions on $\mathbb{C}[G]$ and consequently distinct functions on G since the χ_j are extended to $\mathbb{C}[G]$ by linearity. As the χ_j are class functions (that is, they are constant on the conjugacy classes of G), we deduce that the number of them is exactly equal to the number of conjugacy classes.

We can deduce the familiar second orthogonality relation from the first as follows. Since the characters are class functions, we let $x_1,\ldots,x_t$ be the representatives of the conjugacy classes $C_1,\ldots,C_t$ (say). The first orthogonality relation can be re-written as

$$\sum_{k=1}^{t}|C_k|\chi_i(x_k)\overline{\chi_j(x_k)} = \delta_{ij}|G|. \tag{5}$$

This can be written as a matrix equation as follows. Let D be the diagonal matrix $\operatorname{diag}(|C_1|,\ldots,|C_t|)$. Let $\mathcal{X}$ be the $t\times t$ matrix whose (i,k) entry is $\chi_i(x_k)$. In other words, $\mathcal{X}$ is the character table viewed as a matrix. Then, equation (5) can be re-written as

$$\mathcal{X}D\overline{\mathcal{X}}^T = |G|I.$$

That is, $|G|^{-1}D\overline{\mathcal{X}}^T$ is a right inverse of $\mathcal{X}$. By linear algebra, it is also a left inverse. Thus,

$$D\overline{\mathcal{X}}^T\mathcal{X} = |G|I,$$

or in other words,

$$\sum_{k,\ell}D_{ik}(\overline{\mathcal{X}}^T)_{k\ell}\mathcal{X}_{\ell j} = \delta_{ij}|G|.$$

As D is a diagonal matrix, this gives:

Corollary 5 (The second orthogonality relation).

$$\sum_{\ell=1}^{t} \overline{\chi_\ell(x_i)}\chi_\ell(x_j) = \delta_{ij}\frac{|G|}{|C_i|}.$$

Let us note for future reference that putting $x_i = x_j = 1$ in the corollary gives

$$\sum_{\ell=1}^{t} \chi_\ell(1)^2 = |G|.$$

Thus, from (3), we deduce that each irreducible representation occurs in the regular representation with multiplicity equal to its degree.

This is a snapshot of character theory presented here in this way using the theory of semisimple algebras because most textbook treatments (such as Lang [33]) give the impression that one needs to develop a substantial amount of algebra before the main results are derived.

7. Eigenvalues of Cayley Graphs

We will consider a slightly more general situation from which will emerge the theorem concerning eigenvalues of Cayley graphs. Let G be a finite group and $f : G \to \mathbb{C}$ be a class function. Our goal is to determine the eigenvalues of the matrix $[f(gh^{-1})]$ and evaluate its determinant. This should be seen as a generalization of the Dedekind determinant.

Let V be the left regular representation of G. As we noted in the previous section, this can be realized as the module $\mathbb{C}[G]$ with the action of G being given by left multiplication. Then, as remarked earlier, V decomposes as a direct sum of irreducible representations V_i:

$$V = \oplus_i V_i,$$

corresponding to the inequivalent irreducible representations V_i. Consider the linear map

$$T := \sum_{h \in G} f(h)h.$$

Since f is a class function, with value f_i (say) on class C_i, we can re-write this as

$$T = \sum_i f_i K_i, \qquad K_i = \sum_{h \in C_i} h.$$

Being a linear combination of conjugacy classes, this is an element in the center of $\mathbb{C}[G]$. Thus $Tg = gT$ for all $g \in G$. In other words, T is a $\mathbb{C}[G]$-morphism. As each

V_i is irreducible, T restricted to each V_i is an algebra endomorphism of V_i which by Schur's lemma must be multiplication by a scalar λ_i. Thus,

$$\sum_{h\in G} f(h)h\Big|_{V_i} = \lambda_i I,$$

and by taking traces of both sides, we get

$$\sum_{h\in G} f(h)\chi_i(h^{-1}) = \lambda_i \chi_i(1).$$

As each V_i has dimension $\chi_i(1)$, and V_i occurs with multiplicity $\chi_i(1)$ in the regular representation, the multiplicity of λ_i is $\chi_i(1)^2$. We have therefore proved:

Theorem 6. *If G is a finite group and f a class function on G, the eigenvalues of the matrix $[f(gh^{-1})]$ are given by*

$$\frac{1}{\chi(1)} \sum_{g\in G} f(g)\chi(g)$$

which has multiplicity $\chi(1)^2$ as χ ranges over the irreducible characters of G. In particular, the determinant of the matrix is

$$\prod_{\chi} \left(\frac{1}{\chi(1)} \sum_{g\in G} f(g)\chi(g) \right)^{\chi(1)}.$$

Corollary 7. The eigenvalues of the Cayley graph $X(G,S)$ are given by

$$\frac{1}{\chi(1)} \sum_{s\in S} \chi(s),$$

as χ ranges over the irreducible characters of G. Moreover, the multiplicity of each of these eigenvalues is $\chi(1)^2$.

This can be seen as a generalization of the Dedekind determinant theorem. It should be noted that in the determination of the eigenvalues of Cayley graphs, we only used Schur's lemma and Maschke's theorem, namely that $\mathbb{C}[G]$ is semisimple.

8. The Alon-Bopanna Theorem

If X is a k-regular loopless graph with vertices labelled $1, 2, \ldots, n$, and A is its adjacency matrix, then, as remarked earlier, the (i,j)-th entry of A^r is the number of walks of length r from i to j. In particular, the i-th entry on the diagonal of A^2

represents the number of paths of length two from i to i. Since such a path can only be a visit to the neighbor of i and back to i, the number of such paths is k. Therefore, $\operatorname{tr}(A^2) = kn$. Consequently,

$$kn \leq k^2 + \lambda(X)^2(n-1),$$

from which we deduce

$$\lambda(X) \geq \left(\frac{n-k}{n-1}\right)^{1/2} \sqrt{k},$$

whence

$$\liminf_{n\to\infty} \lambda(X) \geq \sqrt{k}.$$

The Alon-Bopanna theorem [1] states that

$$\liminf_{n\to\infty} \lambda(X) \geq 2\sqrt{k-1}.$$

In other words, Ramanujan graphs are extremal in the sense that they attain this bound.

We proceed to give a simple proof of this theorem. As we noted, the (i, j)-th entry of A^r is the number of walks of length r from i to j. If we let A_r be the matrix whose (i, j)-the entry is the number of "proper walks" (that is, there is no backtracking) of length r from i to j, then we have the following simple recursion:

$$AA_r = A_{r+1} = (k-1)A_{r-1}. \tag{6}$$

This is easily proved by induction. For $r = 1$, we need to show $A^2 = A_2 + kI$. But this is clear since the only paths of A^2 that backtrack are the ones encoded by the diagonal and there are k such paths for each vertex. The induction is completed by noting that

$$(A_{r+1})_{ij} = \sum_{t=1}^{n} (A_r)_{it} A_{tj} - (k-1)(A_{r-1})_{ij},$$

because the left hand side represents the number of paths of length $r + 1$ from i to j without backtracking. These can be counted first by extending a proper walk of length r from i to t and then moving to the vertex j which can be done in A_{tj} ways. But now, we need to remove from this count, any "improper" walks, which can happen only at the last step where we retraced our step. This was really a proper walk of length $r - 1$ from i to j which had been extended "improperly" in $(k - 1)$ ways. This is the right hand side of the equation.

We now follow Pizer [49] to give a simple proof of the Alon-Bopanna theorem. To this end, it is convenient to introduce the matrices B_r defined recursively as follows: $B_0 = I$, $B_1 = A$, and

$$B_1 B_r = B_{r+1} + (k-1)B_{r-1}, \qquad r \geq 1. \tag{7}$$

An easy induction argument shows that if r is odd,

$$B_r = A_r + A_{r-1} + \cdots + A_1,$$

and if r is even,

$$B_r = A_r + A_{r-1} + \cdots + A_2 + I.$$

In either case, $\mathrm{tr}\,(B_r) \geq 0$. We renormalize and set $C_r = B_r(k-1)^{-r/2}$. Then, the recurrence (7) becomes

$$C_1 C_r = C_{r+1} + C_{r-1}, \tag{8}$$

which is reminiscent of the familiar recurrence for the cosine function:

$$(2\cos\theta)(2\cos r\theta) = 2\cos(r+1)\theta + 2\cos(r-1)\theta.$$

This suggests we define C_{-r} as C_r for negative subscripts also and verify that the recurrence (8) holds for all integers. Motivated by this analogy with the cosine function, we are led to conjecture that

$$C_1^r = \sum_{j\geq 0} \binom{r}{j} C_{r-2j}. \tag{9}$$

Indeed, this is actually true and can be proved directly by induction (which we leave as an exercise to the reader). We are now ready to deduce the Alon-Bopanna theorem. Taking traces of both sides of (9) and noting that the traces are non-negative, we get for $r = 2s$:

$$(k-1)^{-s}\mathrm{tr}\,(A^{2s}) \geq \binom{2s}{s} n.$$

In other words,

$$k^{2s} + \lambda(X)^{2s}(n-1) \geq \binom{2s}{s}(k-1)^s n,$$

if X is not bipartite and

$$2k^{2s} + \lambda(X)^{2s}(n-2) \geq \binom{2s}{s}(k-1)^s n,$$

if X is bipartite. In either case, dividing through by $n-1$ and letting n tend to infinity, gives

$$\lim_{|X|\to\infty} \lambda(X)^{2s} \geq \binom{2s}{s}(k-1)^s. \tag{10}$$

Now,

$$(2s+1)\binom{2s}{s} \geq (1+1)^{2s} \geq \binom{2s}{s},$$

which yields that

$$\lim_{s\to\infty} \binom{2s}{s}^{1/2s} = 2,$$

from which we deduce the result by taking $(2s)$-th roots of both sides of (10).

Refinements of the Alon-Bopanna theorem have been obtained by Serre [55] who shows the following. For any $\epsilon > 0$, there is a constant $c = c(\epsilon, k)$ such that for every k-regular graph X on n vertices, the number of eigenvalues λ_i of X such that $\lambda_i > (2-\epsilon)\sqrt{k-1}$ is at least cn. A simpler proof of Serre's theorem was given by Cioabă [11].

Serre's paper [55] opens the vista on three parallel worlds that exhibit an underlying "Ramanujan" theme. The first is the world of modular forms and the eigenvalues of the Hecke operator T_p (with p prime) acting on the space of cusp forms. The second is the world of curves over the finite field $\mathbb{F}_p$ of p elements. The third is the world of $(p+1)$-regular graphs in which the adjacency matrix seems analogous to the Hecke operator. The discovery of the analogy between the first two worlds was a pinnacle of achievement of 20th century mathematics, fusing number theory and algebraic geometry in the form of the celebrated Weil conjectures and culminating in the proof of the Ramanujan conjecture in the theory of modular forms. The discovery of the connection to the world of regular graphs is an exciting and emerging chapter of the 21st century that will surely connect number theory and graph theory more intimately than before. This is evident in the discussion of the next section. The study of these analogies may open our minds to other parallel worlds.

9. The Ihara Zeta Function of a Regular Graph

In 1966, Ihara [29] introduced the zeta function of a regular graph. In this, he was motivated by the construction of the Selberg zeta function of a Riemannian manifold. The celebrated Selberg trace formula [53] arises in this context. In many ways, true to the Artinian aphorism of viewing all primes equally (including the prime at infinity), Ihara considered his construction to be the p-adic version of the Selberg zeta function.

Let X be a k-regular graph and write $q = k-1$. Recall that a walk of length $r-1$ is an ordered sequence of vertices $(x_1, \ldots, x_r)$ such that x_i is adjacent to x_{i+1} for $1 \le i \le r-1$. A proper walk is such that $x_{i+1} \neq x_{i-1}$ for some i (where we interpret the subscripts modulo r). That is, there is no backtracking. A proper walk whose endpoints are equal is called a *closed geodesic*. If γ is a closed geodesic, we use the notation γ^s to denote the closed geodesic obtained by repeating s times the path γ.

A closed geodesic which is not the power (with $s \geq 2$) of another closed geodesic is called a *prime geodesic*. We define an equivalence relation on the closed geodesics by declaring $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_t)$ are *equivalent* if and only if $r = t$ and there is a d such that $y_i = x_{i+d}$ for all i (and the subscripts are interpreted modulo t. In other words, two closed geodesics are equivalent if and only if they have the same length and they are essentially the same closed path, the starting point of the path being irrelevant. An equivalence class of a prime geodesic P is called a *prime geodesic cycle* and denoted $[P]$ and the length of any element in this class is denoted $\ell(P)$.

These notions may seem odd at first but they are better understood from the standpoint of algebraic topology. Indeed, any graph can be viewed as a topological space. To be precise, it is a 1-complex. Any segment of a path that backtracks is contractible to a shorter path and the two paths are homotopically equivalent. Each equivalence class of a prime geodesic cycle corresponds to a unique homotopy class of the fundamental group of X. The reader can find a friendly introduction to the notion of a graph as a topological space in Chapter 11 of [50].

Ihara [29] defined the zeta function of a k-regular graph X as:

$$Z_X(s) := \prod_{[P]} \left(1 - q^{-s\ell(P)}\right)^{-1},$$

where the product is over prime geodesic cycles. He then showed the following result (though the terminology of "Ramanujan" came much later):

Theorem 8. *Let X be a k-regular graph and put $q = k - 1$ and $g = (q-1)|X|/2$. Then*

$$Z_X(s) = (1 - u^2)^{-g} \det(I - Au + qu^2)^{-1}, \qquad u = q^{-s}.$$

Moreover, $Z_X(s)$ satisfies the "Riemann hypothesis" (that is, all the singularities lie on $\mathrm{Re}(s) = 1/2$) if and only if X is Ramanujan.

Proof. The fact that the zeta function is a rational function is not hard to prove and after some thought, essentially follows from (6). We refer the reader to various papers where this is done in a simple way (for example, pages 417–420 of Terras [57]). So we assume that the Ihara zeta function has the form given and show that it satisfies the analog of the Riemann hypothesis if and only if X is Ramanujan. We follow [12]. Let $\phi(z) = \det(zI - A)$ be the characteristic polynomial of the adjacency matrix A. If we set $z = (1 + qu^2)/u$, then the singularities of $Z_X(s)$ come from the zeros of $\phi(z)$. Since

$$u = \frac{z \pm \sqrt{z^2 - 4q}}{2q},$$

and any zero z_0 of ϕ is real (because A is symmetric and all its eigenvalues are real), we deduce that for some u,

$$z_0 = \frac{(1+qu^2)\overline{u}}{u\overline{u}} = \frac{\overline{u} + q|u|^2 u}{|u|^2},$$

is also real. Thus, the numerator is real and so, we must have $q|u|^2 = 1$ which is equivalent to the analog of the Riemann hypothesis. $\square$

Ihara [29] was the first person to (implicitly) discover Ramanujan graphs in 1968 and connect some of them to algebraic geometry. Some open questions arising from Ihara's approach are highlighted in the book by Lubotzky [38]. The notion of a zeta function for general graphs (and not just regular graphs) was extended and studied by Bass [4]. Further results were obtained by Hashimoto [27], Stark and Terras [56] who initiated a theory of Artin L-series for general graphs. The analogy between zeta functions of graphs and zeta functions in number theory is further amplified in Winnie Li's recent book [36].

10. Expander Graphs and Ramanujan Graphs

Ramanujan graphs form an important subset of a larger collection of k-regular graphs called *expander graphs*. In words, an expander graph is a highly connected, sparse graph. It is this strange combination of contradictory properties that makes them important. A friendly introduction to expander graphs can be found in [52]. Here is a precise definition. If X is a k-regular graph on n vertices, we define for each subset S of vertices of V, the *edge boundary* of S, denoted ∂S, by

$$\partial S := \{(u,v) \in E : u \in S, v \notin S\}.$$

In other words, the edge boundary of S counts the number of edges connecting S to its complement in V. Since, S or the complement of S, has size at most $n/2$, we define the *edge expansion* of X, denoted $h(X)$, by

$$h(X) := \min_{S \subset V : |S| \leq n/2} \frac{|\partial S|}{|S|}.$$

For a fixed $\delta > 0$, we say X is a (n, k, δ)-*expander* if $h(X) \geq \delta$. A famous theorem of Dodziuk [18] (see also [2]) states:

Theorem 9. *If X is a k-regular graph, then*

$$\frac{k - \lambda(X)}{2} \leq h(X) \leq \sqrt{2k(k - \lambda(X))}.$$

We will not prove this theorem, but refer the reader to [16] where a simple proof is given.

Expander graphs seem to have first appeared in the 1960's in a fundamental paper of Kolmogorov and Barzdin [31] who were studying the network of nerve cells in the human brain! In 1973, Pinsker [48] gave the first formal definition. Since then, they have found important applications in the design of optimal communication networks and error correcting codes. Excellent surveys can be found in [28] and [39]. For a detailed discussion of how one can use Cayley graphs to explicitly construct expanders, we refer the reader to [39]. It has even been suggested that expander graphs can be applied to study human thought and other problems in neurobiology [59]. So from the above theorem and brief discussion, we see that Ramanujan graphs have optimal expansion properties and the notion of expansion seems to be woven into the fabric of the cosmos.

11. Construction of Ramanujan Graphs

The first explicit constructions of Ramanujan graphs were given (independently) by Margulis [42] and Lubotzky, Phillips and Sarnak [40] in 1988, almost twenty years after Ihara's work. They constructed infinite families of Ramanujan graphs of degree $p+1$ with p prime. The proof that their graphs were Ramanujan relied on the analogue of Ramanujan's conjecture for cusp forms of weight 2 (proved by Eichler and Shimura) and this explains the name, "Ramanujan graphs", to some extent. Later, in 1994, Morgenstern [43] constructed explicitly infinite families of Ramanujan graphs of degree $q+1$ when q is any prime power. His work used the analogue of Ramanujan's conjecture for Drinfeld modules proved by Drinfeld [19]. No explicit construction of infinite families of Ramanujan graphs for other degrees is known.

Here is a short description of the explicit construction of Ramanujan graphs of degree $p+1$ for every prime p given in [40]. Let p and q be unequal primes with both being $\equiv 1 \pmod 4$. By elementary number theory, the congruence $u^2 \equiv -1 \pmod q$ has an integer solution. Fix any such solution. Now, by classical theorems of Lagrange and Jacobi, the equation $p = a^2 + b^2 + c^2 + d^2$ has precisely $8(p+1)$ integer solutions. Among these, there are exactly $p+1$ solutions with $a > 0$ and b, c, d, even. To each such solution, we associate the matrix

$$\begin{pmatrix} a+ub & c+ud \\ -c+ud & a-ub \end{pmatrix},$$

which gives $p+1$ matrices in the group $PGL_2(\mathbb{F}_q)$. Let S be the set of these matrices and define the Cayley graphs

$$X^{p,q} = \begin{cases} X(PGL_2(\mathbb{F}_q), S) & \text{if } \left(\frac{p}{q}\right) = -1 \\ X(PSL_2(\mathbb{F}_q), S) & \text{if } \left(\frac{p}{q}\right) = 1, \end{cases}$$

where $\left(\frac{p}{q}\right)$ is the Legendre symbol which equals 1 if p is a square modulo q and -1 otherwise. Then, the graphs $X^{p,q}$ form an infinite family of $p+1$ regular Ramanujan graphs when we vary q as shown in [40]. The proof of this is not easy. We will sketch the main ideas.

To show that the graphs $X^{p,q}$ are Ramanujan, it suffices to show that the Ihara zeta function of these graphs satisfies the analogue of the Riemann hypothesis. Indeed, if we let f_r to be the trace of the matrix A_r in (6), we have for every m (see p. 122 of [16]):

$$2 \sum_{0 \le r \le m/2} f_{m-2r} = \frac{2p^m}{n} \sum_{j=1}^{n} U_m\left(\frac{\lambda_j}{2\sqrt{p}}\right), \tag{11}$$

where $n = |X^{p,q}|$ and $U_m(x)$ is the Chebycheff polynomial of the second kind:

$$U_m(\cos\theta) = \frac{\sin(m+1)\theta}{\sin\theta}$$

with $x = \cos\theta$. Using the theory of quaternions, the left hand side of (11) is shown to be the number of ways of writing p^m as the quadratic form:

$$a^2 + 4q^2(b^2 + c^2 + d^2).$$

Denoting this number by $s(p^m)$, one can use the theory of modular forms of weight 2 to show that

$$s(p^m) = \delta(p^m) + a(p^m),$$

where $\delta(p^m)$ is the coefficient of an Eisenstein series of weight 2 and $a(p^m)$ is the coefficient of a cusp form of weight 2 on $\Gamma(16q^2)$. (For a quick introduction to modular forms, we refer the reader to [46].) Now, one can show

$$\delta(p^m) = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \frac{4}{n} \sum_{d | p^m} d & \text{if } m \text{ is even.} \end{cases}$$

This term corresponds to the "trivial" eigenvalue $\lambda_1 = p + 1$ appearing on the right hand side of (11), and so we see that bounding the non-trivial eigenvalues of the $(p+1)$-regular graphs $X^{p,q}$ is equivalent to estimating the cusp form coefficient $a(p^m)$.

Keeping in mind that q is fixed, the dimension of the space of cusp forms of weight 2 on $\Gamma(16q^2)$ is independent of p, and we see on invoking the Ramanujan conjecture (proved in this case by Eichler and Shimura) that the Ihara zeta function satisfies the analog of the Riemann hypothesis. This is a cursory sketch of the argument. The reader is referred to [40, 16] and [51] for the details.

12. Infinite Ramanujan Graph Families for Every Degree

In 2015, Marcus, Spielman and Srivastava [41] showed that infinite familes of bipartite Ramanujan graphs exist for every degree k. They did this using an averaging, probabilistic technique which we describe below. Sadly, the proof is not constructive and it would be desirable, for network applications, to have a constructive proof. At the same time, it must be acknowledged that the proof is quite elegant and resolves a major problem in graph theory that had been around for some time.

At the heart of their proof is the notion of a *2-lift* of a graph $X = (V, E)$. This is a graph $\widehat{X} = (\widehat{V}, \widehat{E})$ that has two vertices $\{v_0, v_1\} \in \widehat{V}$ for each vertex $v \in V$. This pair of vertices is called the *fibre* of the original vertex. Every edge in X corresponds to two edges in $\widehat{X}$. If $(u, v) \in E$ and $\{u_0, u_1\}$ and $\{v_0, v_1\}$ are the fibres of u and v respectively, then $\widehat{E}$ can either contain the pair of edges $\{(u_0, v_0), (u_1, v_1)\}$ (type (1)) or $\{(u_0, v_1), (u_1, v_0)\}$ (type (2)) (see Figure 2).

This concept was first introduced by Bilu and Linial [6] in 2006. To study the eigenvalues of a 2-lift of a given graph $X = (V, E)$, they studied the *signings* $s : E \to \{\pm 1\}$ of the edges where $s(u, v) = 1$ if edges of type (1) appear in the 2-lift and $s(u, v) = -1$ if edges of type (2) appear. Then they define the *signed adjacency matrix* A_s of X to be the adjacency matrix A of X except that the entry corresponding to the edge (u, v) is $s(u, v)$. Thus, one can reconstruct the lift from A_s. They showed that the set of eigenvalues of the 2-lift $\widehat{X}$ is the union of the eigenvalues of A and A_s (taken with multiplicity). Eigenvalues of A are called the *old eigenvalues*

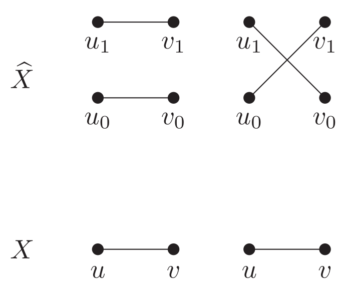

Figure 2. Lifting of edges in a 2-lift

and eigenvalues of A_s are called the *new eigenvalues* of $\widehat{X}$. They conjectured that every k-regular graph X has a 2-lift $\widehat{X}$ in which all the new eigenvalues have absolute value at most $2\sqrt{k-1}$. This conjecture is still open, but if true, immediately implies the existence of infinite families of k-regular Ramanujan graphs for every value of k because, as we noted at the outset, the complete graph $\mathscr{K}_{k+1}$ on $k+1$-vertices is a k-regular graph which is Ramanujan and one need only take the appropriate 2-lift iteratively, to get the desired infinite family.

Marcus, Spielman and Srivastava [41] prove a variant of this conjecture. They show that every k-regular graph has a signing in which all the new eigenvalues are at most $2\sqrt{k-1}$, the point being that they are unable to control the size of the smallest eigenvalue. This is why they consider bipartite graphs because such graphs have eigenvalues symmetric about zero (see for example, Theorem 4.3.2 of [12]) and so, controlling the largest eigenvalue allows you to control the smallest eigenvalue also. The strategy now is to take the complete bipartite graph $\mathscr{K}_{k,k}$ which we observed earlier is Ramanujan and take appropriate successive 2-lifts. It is not difficult to see that the 2-lift of a bipartite graph is again bipartite. This is a key idea used later.

The strategy of their method is not complicated. Recall that a *matching* in a graph X is a subset of its edges in which no two share a common vertex. If m_i denotes the number of matchings of X consisting of i edges, then the *matching polynomial* is defined to be

$$M_X(\lambda) := \sum_{i\geq 0}(-1)^i m_i \lambda^{n-2i},$$

where n is the number of vertices of X.

As is often the case with many graph theoretic polynomials, the matching polynomial satisfies simple recurrence relations. For example, if $e=(a,b)$ is an edge, we have clearly

$$M_X(\lambda) = M_{X-e}(\lambda) + M_{X-a-b}(\lambda), \tag{12}$$

where the first term on the right hand side counts matchings not involving e and the second term counts those involving e. Another useful recurrence is given by the following.

Lemma 10.

$$M_X(\lambda) = \lambda M_{X-a}(\lambda) - \sum_{b\sim a} M_{X-a-b}(\lambda).$$

Proof. Let us fix a vertex a and consider the matchings of size i that involve a and that do not involve a respectively. The latter is clearly $m_i(X-a)$. Those that involve a, connect a to a neighbor. It is then evident that

$$m_i(X) = m_i(X-a) + \sum_{b\sim a} m_{i-1}(X-a-b).$$

Inserting this recurrence into the polynomial defining $M_X(\lambda)$ gives the result. $\square$

The key theorem in [41] relies on the study of roots of the matching polynomial. The following simple lemma is helpful.

Lemma 11. *If T is a forest, its matching polynomial coincides with its characteristic polynomial.*

Proof. As every forest is a collection of disjoint trees, it suffices to prove the theorem for trees. As every tree T has a leaf, we label this vertex as 1, without any loss of generality, and its neighbor as 2. Then the adjacency matrix A has the form

$$\left(\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & * & * & * \\ \hline 0 & * & * & * & * \\ \vdots & * & * & * & * \\ 0 & * & * & * & * \end{array}\right).$$

Thus, expanding $p_T(\lambda) := \det(\lambda I - A)$ by the first row, gives easily the recursion

$$p_T(\lambda) = \lambda p_{T-1}(\lambda) - p_{T-1-2}(\lambda).$$

We can now proceed by induction using (12) and noting that λ is the characteristic polynomial of the graph consisting of one vertex. $\square$

A remarkable 1972 theorem of Heilmann and Lieb [26], states that $M_X(\lambda)$ has only real roots and if k is the maximum degree in X, then all roots of $M_X(\lambda)$ have absolute value at most $2\sqrt{k-1}$. (We provide a simple proof due to Godsil in Appendix 2.) This is then related to a theorem of Godsil and Gutman [23] that says that the expected value of uniformly random signings of the adjacency matrix is $M_X(\lambda)$. That is, if X has m edges, to each random assignment of ± 1 to the edges, we have a *signing* $s \in \{\pm 1\}^m$ and the associated adjacency matrix A_s. Letting $f_s(\lambda) = \det(\lambda I - A_s)$, we have the expected value given by the following theorem:

Theorem 12.

$$\mathbb{E}_{s \in \{\pm 1\}^m}[f_s(\lambda)] = M_X(\lambda).$$

Proof. The short and elegant proof of this given in [41] is as follows. Let n be the number of vertices of X and S_n denote the usual symmetric group on n letters. For $\sigma \in S_n$, we denote by $|\sigma|$ the number of inversions in σ. This is an even number if σ

is an even permutation and an odd number otherwise. For a subset R of $[n]$, we write $\mathrm{sym}(R)$ to be the permutation group on the elements of R. With this preamble, the left hand side of the equation of the theorem is $\mathbb{E}(\det(\lambda I - A_s))$ and so using the formal definition of the determinant, this is equal to

$$\mathbb{E}\left(\sum_{\sigma\in S_n}(-1)^{|\sigma|}\prod_{i=1}^{n}(\lambda I - A_s)_{i,\sigma(i)}\right)$$

which is (using the notation $[n]$ to designate the set $\{1,2,\ldots,n\}$),

$$\mathbb{E}\left(\sum_{\sigma\in S_n}(-1)^{|\sigma|}\sum_{\substack{R\subset[n]\\|R|=j}}\prod_{i\notin R}(\lambda I)_{i,\sigma(i)}(-1)^j\prod_{i\in R}(A_s)_{i,\sigma(i)}\right).$$

The only way the product over $i \notin R$ can survive is if σ fixes pointwise every element not in R because λI is diagonal. Thus, the left hand side is

$$\mathbb{E}\left(\sum_{j=0}^{n}\lambda^{n-j}(-1)^j\sum_{R\subset[n],|R|=j}\;\sum_{\tau\in\mathrm{sym}(R)}(-1)^{|\tau|}(A_s)_{i,\tau(i)}\right).$$

As the expectation is additive, we finally get that this is

$$\sum_{j=0}^{n}\lambda^{n-j}(-1)^j\sum_{R\subset[n],|R|=j}\;\sum_{\tau\in\mathrm{sym}(R)}(-1)^{|\tau|}\mathbb{E}(s_{i,\tau(i)}).$$

Since the s_{ij} are independent assignments of ±1 signs, we have $\mathbb{E}(s_{ij}) = 0$ and only those products that contain even powers (0 or 2) of the s_{ij} survive. So we can restrict our attention to those permutations τ the contain only orbits of size two (that is, disjoint transpositions). But these correspond to perfect matchings of R. As there are no perfect matchings when R is of odd size, we conclude that this is

$$\sum_{j=0}^{n}\lambda^{n-j}(-1)^j m_j$$

which is the matching polynomial. □

Since the matching polynomial is the expected characteristic polynomial of real symmetric matrices A_s, it is not surprising that all its roots turn out to be real, though this is not a proof. The following is a theorem of Heilmann and Lieb [26].

Theorem 13. *Let X be a graph with maximal degree k. Then, all the roots of the matching polynomial $M_X(\lambda)$ are real and have absolute value at most $2\sqrt{k-1}$.*

Godsil noted that one can give a simple proof of this by introducing the *path tree* of a graph: we fix a vertex a of the graph X. The *path tree* $T_a(X)$ of X rooted at a has vertices corresponding to paths that start at a and do not contain any vertex twice. Two paths are adjacent if one extends the other path by one vertex.

Godsil [23] proves that $M_X(\lambda)$ divides $M_{T_a(X)}(\lambda)$ which by Lemma 11 is the same as the characterstic polynomial of the tree $T_a(X)$. Consequently, $M_X(\lambda)$ has real roots. If X has maximal degree k, the path tree $T_a(X)$ also has degree at most k and so by Theorem 1, all the eigenvalues of the characteristic polynomial have absolute value at most $2\sqrt{k-1}$. The proof of this is given in Appendix 2.

We conclude that all the roots of the matching polynomial of a k-regular graph are real and lie in the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$. Since this is the expected value of the signed adjacency matrices A_s, we need "only" prove the existence of a signing s for which all the eigenvalues of A_s have absolute value bounded by $2\sqrt{k-1}$.

The authors of [41] are thus led to study interlacing families of polynomials which we explain below.

A polynomial

$$g(\lambda) = \prod_{i=1}^{n-1}(\lambda - \alpha_i),$$

is said to *interlace*

$$f(\lambda) = \prod_{i=1}^{n}(\lambda - \beta_i),$$

if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n.$$

Observe that the polynomial that interlaces f has degree one less than that of f. For example, a polynomial of degree n with n real roots is interlaced by its derivative. A collection of polynomials $F_1, \ldots, F_t$ is said to have a *commong interlacing* if there is a single polynomial g that interlaces each of the F_i. Here now is a result that belongs in a first-year calculus course and surprisingly plays a key role in the paper of Marcus, Spielman and Srivastava [41].

Lemma 14. *Let $F_1, \ldots, F_r$ be polynomials of degree n each having real roots, and positive leading coefficients. Let*

$$F = \sum_{i=1}^{r} F_i.$$

If $F_1, \ldots, F_r$ have a common interlacing, then there exists an i for which the largest root of F_i is at most the largest root of F.

Proof. Let G be a polynomial that interlaces all of the F_i. Let α_{n-1} be the largest root of G. As each $F_i(x)$ has a positive lead coefficient, it is positive for x sufficiently large. Each F_i has exactly one root that is at least α_{n-1} and $F_i(\alpha_{n-1}) \leq 0$. Therefore, $F(\alpha_{n-1}) \leq 0$. As F eventually becomes strictly positive, F has a root $\geq \alpha_{n-1}$. This must be the largest root of F. If we let β_n denote this root, then $F(\beta_n) = 0$ implies that $F_i(\beta_n) \geq 0$ for some i. As F_i has at most one root that is at least α_{n-1}, and $F_i(\alpha_{n-1}) \leq 0$, the largest root of F_i lies in $[\alpha_{n-1}, \beta_n]$ and we are done. □

Here is how the proof is now completed. Assume that we can show that the polynomials $\{f_s\}$ appearing in Theorem 12 form an interlacing family. We apply the previous lemma to this family to deduce that there is a signing s for which the largest root of $f_s(\lambda)$ is bounded by the largest root of

$$\sum_s f_s(\lambda)$$

which is a constant multiple of $M_X(\lambda)$ by Theorem 12. By Theorem 13, all the roots of $f_s(\lambda)$ are bounded by $2\sqrt{k-1}$ as desired.

13. An Equidistribution Result

One interesting consequence of the result of [40] is the following equidistribution theorem.

Theorem 15. *For each prime $p \equiv 1 \pmod 4$, write the $p+1$ solutions $\alpha_i = (a_i, b_i, c_i, d_i)$ for $1 \leq i \leq p+1$ such that*

$$p = a_i^2 + b_i^2 + c_i^2 + d_i^2, \qquad a_i > 0, \text{ and odd} \quad b_i, c_i, d_i \text{ even.}$$

For $q \equiv 1 \pmod 4$, let u be a solution to $u^2 \equiv -1 \pmod q$, and associate the matrix

$$\widehat{\alpha}_i = \begin{pmatrix} a_i + ub_i & c_i + ud_i \\ -c_i + ud_i & a_i - ub_i \end{pmatrix} \in PGL_2(\mathbb{F}_q).$$

Then for each conjugacy class C of $G = PGL_2(\mathbb{F}_q)$, the number N of $\widehat{\alpha}_i \in C$ satisfies

$$\left| N - \frac{|C|}{|G|}(p+1) \right| \leq 2|C|^{1/2}\sqrt{p}.$$

Proof. Since the Ramanujan graphs constructed explicitly in [40] are Cayley graphs $X(G, S)$ with the group $G = PGL_2(\mathbb{F}_q)$ and $S = \{\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{p+1}\}$ we can count the number of elements in a fixed conjugacy class C of G as follows. Using character theory, we see that this number is

$$\frac{|C|}{|G|}(p+1) + \frac{|C|}{|G|}\sum_{\chi\neq 1}\overline{\chi}(g_C)\sum_{i=1}^{p+1}\chi(\widehat{\alpha}_i), \tag{13}$$

where for each conjugacy class C, the element g_C is any representative and the outer sum is over irreducible characters of G. On the other hand, by Corollary 7 the numbers

$$\frac{1}{\chi(1)}\sum_{i=1}^{p+1}\chi(\widehat{\alpha}_i)$$

are the eigenvalues of the (Cayley) Ramanujan graphs $X^{p,q}$, as χ ranges over the irreducible characters of G. Thus,

$$\left|\sum_{i=1}^{p+1}\chi(\widehat{\alpha}_i)\right| \leq 2\chi(1)\sqrt{p}.$$

Applying the Cauchy-Schwarz inequality to estimate the double sum in (13), we get

$$\left|\frac{|C|}{|G|}\sum_{\chi\neq 1}\overline{\chi}(g_C)\sum_{i=1}^{p+1}\chi(\widehat{\alpha}_i)\right| \leq \frac{|C|}{|G|}\left(\sum_{\chi}|\chi(g_C)|^2\right)^{1/2}\left(\sum_{\chi\neq 1}4\chi(1)^2 p\right)^{1/2}.$$

Since

$$\sum_{\chi}|\chi(g_C)|^2 = \frac{|G|}{|C|} \quad \text{and} \quad \sum_{\chi}\chi(1)^2 = |G|,$$

we deduce that the number N of $\widehat{\alpha}_i$ in a fixed conjugacy class C satisfies

$$\left|N - \frac{|C|}{|G|}(p+1)\right| \leq 2|C|^{1/2}\sqrt{p}.$$

Since the order of $PGL_2(\mathbb{F}_q)$ is $q(q^2-1)$ and the size of the largest conjugacy class is $\asymp q^2$, we see that if $p \gg q^4$, then every conjugacy class of $PGL_2(\mathbb{F}_q)$ is represented by the map $\alpha_i \mapsto \widehat{\alpha}_i$ for $1 \leq i \leq p+1$. □

14. Concluding Remarks

We have presented an introductory survey highlighting some recent developments in the theory of Ramanujan graphs, aimed primarily at the graduate student. We have not given a full encyclopedic account of the topic and for this, the reader may consult other surveys such as [14, 28, 34, 58] as well as the book by Lubotzky [38] and the recent book by Kowalski [32]. The connection between the theory of automorphic representations and Ramanujan graphs is best explained in Chapter 9 of [35]. In [40], the authors give explicit construction of $(p+1)$-regular Ramanujan graphs when $p \equiv 1 \pmod 4$. In [8], the author remarks that minor modification of this construction also allows explicit construction in the case $p \equiv 3 \pmod 4$. The case $p = 2$ is treated in [8] and the case of prime powers is treated in [43]. Now that we know infinite families of Ramanujan graphs exist for every degree, it would be highly desirable to construct these explicitly.

Ramanujan graphs have other extremal properties that we have not discussed and these are described at length in [40]. For example, they are the best known explicit expander graphs with large girth.

It would of interest to show the existence of infinite families of non-bipartite Ramanujan graphs of every degree. In this context, there are the famous conjectures of Bilu and Linial [6] alluded to earlier: every k-regular Ramanujan graph has a 2-lift such that all new eigenvalues lie in the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$. A stronger conjecture is that every k-regular graph has such a 2-lift (see p. 492 of [28]).

The reader may be wondering if abelian groups can be used to construct Ramanujan graphs. That this is not possible was first shown by Klawe [30] in 1984. Further refinements of her results were obtained in [22] and [45] where (among other results) it is shown that there are only finitely many abelian Cayley graphs which are Ramanujan. On the other hand, Friedman [20, 21] has shown that a random k-regular graph is almost Ramanujan in the sense that the second largest eigenvalue λ_1 satisfies

$$\lambda_1 \leq 2\sqrt{k-1} + 2\log k + O(1).$$

Nearly Ramanujan graphs can be constructed explicitly using results about gaps between consecutive prime numbers. This is discussed in [13].

The theory of Ramanujan graphs is a new world to explore that is fascinating from many perspectives. The zeta functions of Ramanujan graphs are one more family (in addition to the zeta functions of curves and varieties over finite fields) for which the analog of the Riemann hypothesis is true. Yet, they are not arithmetically defined. Rather, they seem to be group theoretically defined. Indeed, if G is a group acting on the left on a set $\mathcal{X}$ and S is a subset of G, the *Schreier graph* $\mathscr{S}(G, \mathcal{X}, S)$ has vertex set X and edge set (x, sx) as s ranges over elements of S and x ranges over elements of $\mathcal{X}$. Cayley graphs are special cases of Schreier graphs as is seen by taking $G = \mathcal{X}$ with G acting on $\mathcal{X}$ by right multiplication. A theorem of Gross [25] says that every finite regular graph of even degree is a Schreier graph with G and $\mathcal{X}$ both finite. Thus, it may be enlightening to explore the world of Schreier graphs in our search for Ramanujan graphs.

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Appendix 1: A Simple Proof of Wedderburn's Theorem

A ring is called *simple* if it has no proper two-sided ideals. Wedderburn's theorem says that any simple ring is isomorphic to $M_n(D)$ for some division ring D. Here, $M_n(D)$ is the ring of $n \times n$ matrices with entries in a division ring D.

We first show that if D is a division ring, then $R = M_n(D)$ is simple. In other words, R has no non-trivial ideals. Indeed, if I is a non-zero two-sided ideal, let $A = (a_{ij})$ be a non-zero element of I. Suppose that $a_{km} \neq 0$ for some k, m. For $1 \leq i, j \leq n$, let E_{ij} be the $n \times n$ matrix with 1 in the ij-th place and zero everywhere else. Thus,

$$A = \sum_{i,j} a_{ij} E_{ij}.$$

Then,

$$E_{km} = E_{kk} A (a_{km}^{-1} E_{mm}) \in I.$$

But then, for any i, j and any $b \in D$,

$$bE_{ij} = bE_{ik}(E_{km})E_{mj} \in I,$$

since I is a two-sided ideal. By taking sums of these elements, we deduce that every element of D lies in I.

Schur's lemma tells us that if M is a simple R-module, then $\mathrm{End}(M)$ is a division ring. Now, R can be viewed as a module M over itself. We already noted that if R is semisimple, then any module over R is semisimple. Thus, R can be decomposed as a direct sum of simple R-modules, S_i say:

$$R = \oplus_i n_i S_i,$$

where the S_i are pairwise non-isomorphic. Taking endomorphisms of both sides, we see by Schur's lemma that

$$\mathrm{End}(R) \simeq \oplus_i \mathrm{End}(n_i S_i), \tag{14}$$

so that each component is isomorphic to $M_{n_i}(D_i)$ with $D_i = \mathrm{End}(S_i)$ which is a division ring by Schur's lemma. The proof will be complete if we can show that the left hand side of (14) is isomorphic to R which by our assumption is simple.

To this end, we introduce the notion of the *opposite ring* R^{op} for any ring R. This is essentially R with the order of the multiplication reversed. That is, we define $R^{op} = (R, +, *)$ where $+$ is as in R and

$$a * b := ba,$$

where the multiplication on the right hand side is in R. R and R^{op} are said to be *anti-isomorphic*. For any ring R, we have $\mathrm{End}(R) \simeq R^{op}$. To see this, we define two maps $\phi : \mathrm{End}(R) \to R$ given by $\phi(f) := f(1)$ and $\rho : R \to \mathrm{End}(R)$ given by $\rho(a)r := ra$ which is multiplication on the right. It is easy to verify that these maps are inverses of each other, giving an anti-isomorphism between the two rings. Indeed,

$$\rho(ab)r = r(ab) = [\rho(b)\rho(a)]r.$$

Thus, $\mathrm{End}(R) \simeq R^{op}$ and we have proved:

Theorem 16 (Wedderburn, 1908). *Any simple algebra is isomorphic to a matrix ring over a division ring.*

Appendix 2: Godsil's Theorem

Lemma 17. *For any graph X and vertex a, we have*

$$\frac{M_X(\lambda)}{M_{X-a}(\lambda)} = \frac{M_{T_a(X)}(\lambda)}{M_{T_a(X)-a}(\lambda)},$$

where $T_a(X)$ is the path tree of X rooted at a.

Proof. The proof is by induction on the number of vertices of X. The essential idea is to apply the recursion from Lemma 10. We leave the (easy) details to the reader. □

Theorem 18 (Godsil, 1981). *For any vertex a of X, $M_X(\lambda)$ divides $M_{T_a(X)}(\lambda)$.*

Proof. As usual, we induct on the number of vertices. For graphs with one or two vertices, the assertion is clear. If (a, b) is an edge, then by induction $M_{X-a}(\lambda)$ divides $M_{T_b(X-a)}(\lambda)$. But

$$T_a(X) - a = \bigcup_{b \sim a} T_b(X - a)$$

and so the matching polynomial of $T_a(X) - a$ is the product of the matching polynomials of the graphs on the right. In particular, $M_{T_b(X-a)}(\lambda)$ divides $M_{T_a(X)-a}(\lambda)$. Since $M_{X-a}(\lambda)$ divides $M_{T_b(X-a)}(\lambda)$, we deduce that $\frac{M_{T_a(X)-a}(\lambda)}{M_{X-a}(\lambda)}$ is a polynomial. By Lemma 17,

$$M_{T_a(x)}(\lambda) = M_{T_a(X)-a}(\lambda)\frac{M_X(\lambda)}{M_{X-a}(\lambda)} = M_X(\lambda)\frac{M_{T_a(X)-a}(\lambda)}{M_{X-a}(\lambda)},$$

and the proof is complete. □

References

[1] N. Alon, Eigenvalues and expanders, *Combinatorica*, **6** (1986) 83–96.

[2] N. Alon and V. Milman, λ_1 isoperimetric inequalities for graphs and superconcentrators, *J. Comb. Theory*, Series B, **38** (1985) 73–88.

[3] L. Babai, Spectra of Cayley graphs, *Journal of Combinatorial Theory*, Series B, **27** (1979) 180–189.

[4] H. Bass, The Ihara-Selberg zeta function of a tree lattice, *International Journal of Math.*, **3** (1992) 717–797.

[5] F. Bien, Construction of telephone networks by group representations, *Notices of the American Math. Society*, **36** (1989) 5–22.

[6] Y. Bilu and N. Linial, Lifts, discrepancy and nearly optimal spectral gap, *Combinatorica*, **26** (2006) 495–519.

[7] A. E. Brouwer and W. H. Haemers, Spectra of Graphs, Springer, (2012).

[8] P. Chiu, Cubic Ramanujan graphs, *Combinatorica*, **12(3)** (1992) 275–285.

[9] F. Chung, Diameters and eigenvalues, *Journal of the American Math. Society*, **2** (1989) 187–200.

[10] F. R. K. Chung, Spectral Graph Theory, Conference Board of the Mathematical Sciences, *American Mathematical Society*, Providence, Rhode Island, **92** (1997).

[11] S. M. Cioabă, On the extreme eigenvalues of regular graphs, *J. Combinatorial Theory*, Series B, **96** (2006) 367–373.

[12] S. M. Cioabă and M. R. Murty, A First Course in Graph Theory and Combinatorics, *Texts and Readings in Mathematics*, Hindustan Book Agency, **55** (2009).

[13] S. M. Cioabă and M. R. Murty, Expander graphs and gaps between primes, *Forum Math.*, **20** (2008) 745–756.

[14] S. M. Cioabă and M. R. Murty, Ramanujan and expander graphs, in Handbook of Finite Fields, edited by G. L. Mullen and D. Panario, CRC Press, (2013) 642–658.

[15] E. R. van Dam and W. H. Haemers, Eigenvalues and the diameters of graphs, *Linear and Multilinear Algebra*, **39** (1995) 33–44.

[16] G. Davidoff, P. Sarnak and A. Valette, Elementary Number Theory, Group Theory and Ramanujan Graphs, *London Mathematical Society Student Texts*, Cambridge University Press, **55** (2003).

[17] P. Diaconis and M. Shahshahani, Generating a random permutation with random transpositions, *Zeitschrift für Wahrscheinlichkeitstheorie*, **57** (1981) 159–179.

[18] J. Dodziuk, Difference equations, isoperimetric inequality and transcience of certain random walks, *Trans. Amer. Math. Society*, **284** (1984) 787–794.

[19] V. G. Drinfeld, The proof of Peterson's conjecture for $GL(2)$ over a global field of characteristic p, *Functional Analysis and its Applications*, **22** (1988) 28–43.

[20] J. Friedman, On the second eigenvalue and random walks in random d-regular graphs, *Combinatorica*, **11** (1991) 331–362.

[21] J. Friedman, A proof of Alon's second eigenvalue conjecture and related problems, *Mem. Amer. Math. Soc.*, **195** (2008).

[22] J. Friedman, M. R. Murty and J.-P.Tillich, Spectral estimates for Abelian Cayley graphs, *Journal of Combinatorial Theory*, Series B, **96** (2006) 111–121.

[23] C. Godsil and I. Gutman, On the matching polynomial of a graph, in Algebraic Methods in Graph Theory, Vol. I, II (Szeged, 1978) *Colloq. Math. Society Janos Bolyai*, North-Holland, New York, **25** (1981) 241–249

[24] C. Godsil and G. Royle, Algebraic Graph Theory, *Graduate Texts in Mathematics*, Springer-Verlag, New York, **207** (2001).

[25] J. L. Gross, Every connected regular graph of even degree is a Schreier coset graph, *J. Comb. Theory, Ser. B.*, **22(3)** (1977) 227–232.

[26] O. J. Heilmann and E. H. Lieb, Theory of monomer-dimer systems, *Comm. Math. Physics*, **25** (1972) 190–232.

[27] K. Hashimoto, Zeta functions of finite graphs and representations of p-adic groups, *Advanced Studies in Pure Mathematics*, **15** (1989) 211–280.

[28] S. Hoory, N. Linial and A. Wigderson, Expander graphs and their applications, *Bulletin of the American Math. Society*, **43** (2006) 439–562.

[29] Y. Ihara, On discrete subgroups of the two by two projective linear group over p-adic fields, *Journal of the Math. Society of Japan*, **18** (1966) no. 3, 219–235.

[30] M. Klawe, Limitations on explicit constructions of expanding graphs, *SIAM J. Comput.*, **13(1)** (1984) 156–166.

[31] A. N. Kolmogorv and Y. M. Barzdin, On the realization of networks in three-dimensional space in Selected Works of Kolmogorov, Kluwer, Dordrecht, **3** (1993) 194–202.

[32] E. Kowalski, An introduction to expander graphs, Cours Spécialisés, Société Mathématique de France, **26** (2019).

[33] S. Lang, Algebra, Third Edition, Addison Wesley, (1993).

[34] W.-C. W. Li, A survey of Ramanujan graphs, in Arithmetic, Geometry, and Number Theory, (Luminy, 1993) de Gruyter, Berline, (1996) 127–143.

[35] W.-C. W. Li, Number Theory with Applications, *Series on University Mathematics*, World Scientific, **7** (1996).

[36] W.-C. W. Li, Zeta and L-functions in Number Theory and Combinatorics, Conference Board of Mathematical Sciences, *American Mathematical Society*, Providence, Rhode Island, **92** (2019).

[37] L. Lovász, Spectra of graphs with transitive groups, *Periodica Mathematica Hungarica*, **6(2)** (1975) 191–195.

[38] A. Lubotzky, Discrete Groups, Expanding Graphs and Invariant Measures, *Progress in Mathematics*, Birkhäuser, **125** (1994).

[39] A. Lubotzky, Expander graphs in pure and applied mathematics, *Bulletin of the American Math. Society*, **49(1)** (2012) 113–162.

[40] A. Lubotzky, R. Phillips and P. Sarnak, Ramanujan graphs, *Combinatorica*, **8** (1988) 261–277.

[41] A. W. Marcus, D. A. Spielman and N. Srivastava, Interlacing families I: Bipartite Ramanujan graphs of all degrees, *Annals of Math.*, **182** (2015) 307–325.

[42] G. Margulis, Explicit group theoretic constructions of combinatorial schemes and their applications to the design of expanders and concentrators, *J. Prob. of Info. Trans.*, (1988) 39–46.

[43] Existence and explicit construction of $q+1$ regular Ramanujan graphs for every prime power q, *Journal of Combinatorial Theory (Series B)*, **62** (1994) 44–62.

[44] M. Ram Murty, Ramanujan graphs, *Journal of the Ramanujan Mathematical Society*, **18** (2003) no. 1, 1–20.

[45] M. Ram Murty, Ramanujan graphs and zeta functions, in Algebra and Number Theory, *Proceedings of the Silver Jubilee Conference*, University of Hyderabad, edited by R. Tandon, Hindustan Book Agency, (2005) 269–280.

[46] M. Ram Murty, M. Dewar and H. Graves, Problems in the Theory of Modular Forms, *IMSC Lecture Notes*, Hindustan Book Agency, **1** (2015).

[47] M. Ram Murty and Kaneenika Sinha, The generalized Dedekind determinant, in SCHOLAR - a Scientific Celebration Highlighting Open Lines of Arithmetic Research, edited by A. C. Cojocaru, C. David, F. Pappalardi, Contemporary Mathematics, *American Mathematical Society*, Providence, Rhode Island, **655** (2015) 153–164.

[48] M. S. Pinsker, On the complexity of a concentrator, Seventh *International Teletraffic Conference*, Stockholm, (June 1973) 318/1–318/4.

[49] A. Pizer, Ramanujan graphs, *AMS/IP Studies in Advanced Mathematics*, **7** (1998) 159–178.

[50] J. Rotman, The Theory of Groups, An introduction, Second Edition, Allyn and Bacon Inc., Boston, (1973).

[51] P. Sarnak, Some Applications of Modular Forms, *Cambridge Tracts in Mathematics*, Cambridge University Press, **99** (1990).

[52] P. Sarnak, What is ... an expander? *Notices of the American Math. Society*, **51** (2004) no. 7, 762–763.

[53] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces and applications to Dirichlet series, *Journal of the Indian Math. Society*, **20** (1956) 47–87.

[54] J.-P. Serre, Linear Representations of Finite Groups, Springer-Verlag, New York, (1977).

[55] J.-P. Serre, Répartition asymptotique des valeurs propres de l'operateur de Hecke T_p, *Journal of the American Mathematical Society*, **10** (1997) 75–102.

[56] H. M. Stark and A. Terras, Zeta functions of finite graphs and coverings, *Advances in Math.*, **121** (1996) 124–165.

[57] A. Terras, Fourier Analysis on Finite Groups and Applications, *London Math. Society Student Texts*, Cambridge University Press, **43** (1999).

[58] A. Valette, Graphes de Ramanujan, *Astérisque*, **245** (1997) 247–276.

[59] L. Valiant, What must a global theory of cortex explain? *Current Opinion in Neurobiology*, **25** (2014) 15–19.

[60] P.-H. Zieschang, Cayley graphs of finite groups, *Journal of Algebra*, **118** (1988) 447–454.