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Generalization of an identity of Ramanujan

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Abstract. In this article, we extend two identities proved by Ramanujan involving the Riemann zeta function and the Dirichlet L-function associated to the non-trivial Dirichlet character modulo 4. More precisely, given two power series

$$\sum_{n=0}^{\infty} a_n T^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n T^n$$

which are both rational functions with certain property, we then explicitly show that

$$\sum_{n=0}^{\infty} a_n b_n T^n$$

is again a rational function with the same property. We use this to explain Ramanujan's identities and also analyse Rankin-Selberg convolutions of automorphic L-functions.

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1. Introduction

For a complex number $s \in \mathbb{C}$ with $\Re(s) > 1$, define the Riemann zeta function by

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}$$

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and the Dirichlet L-function associated to the Dirichlet character χ modulo a natural number $q > 1$ by

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$

In 1916, Ramanujan (see page 83 of [8]) proved the following identities

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n \geq 1} \frac{\sigma_a(n)\sigma_b(n)}{n^s} \tag{1}$$

and $$\frac{\eta(s)\eta(s-a)\eta(s-b)\eta(s-a-b)}{(1-2^{-2s+a+b})\zeta(2s-a-b)} = \sum_{n \geq 1} \frac{\chi_4(n)\sigma_a(n)\sigma_b(n)}{n^s}.$$

Here

$$\sigma_k(n) := \sum_{\substack{d \geq 1, \\ d \mid n}} d^k$$

is the divisor function and

$$\eta(s) := L(s, \chi_4),$$

where χ_4 is the non-trivial Dirichlet character modulo 4. Since $\sigma_k(n)$ and $\chi_4(n)$ are multiplicative functions, the Dirichlet series, in each of the cases, can be expressed as an Euler product. For instance, in the first case, we have

$$\prod_p \sum_{m=0}^{\infty} \frac{\sigma_a(p^m)\sigma_b(p^m)}{p^{ms}}$$

and in the second case,

$$\prod_p \sum_{m=0}^{\infty} \frac{\sigma_a(p^m)\sigma_b(p^m)\chi_4(p^m)}{p^{ms}}.$$

Each of these cases suggests the following general question; if the two power series

$$\sum_{n=0}^{\infty} a_n T^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n T^n \tag{2}$$

are both rational functions, then is the power series

$$\sum_{n=0}^{\infty} a_n b_n T^n \tag{3}$$

also a rational function?

Indeed, in each of the two examples, the power series

$$\sum_{m=0}^{\infty} \sigma_a(p^m)T^m \quad \text{and} \quad \sum_{m=0}^{\infty} \chi_4(p)^m \sigma_a(p^m)T^m$$

are both rational functions. The Euler factors in the above product are then seen to be simple rational functions of p^{-s}, which then allows us to deduce Ramanujan's identities.

These questions have a colourful history. The expression (3) is often referred to as the Hadamard product of the two power series in (2).

The question of rationality of this function seems to have been answered in the positive by Emile Borel (see [3], Theorem 7).

The more general question of the algebraicity of (3) if the series in (2) are both algebraic also seems to have received considerable attention in automata theory (see [7]). Analogous questions can be asked for polynomials in several variables and we refer the reader to [9] for further details and history.

Here, our goal is more modest. If the two series in (2) are both rational with certain property, we show that (3) is also rational with the same property and can be given explicitly. Such an explicit description is often useful in analytic number theory especially in the study of the Rankin-Selberg convolution of two automorphic L-functions (see [1] for details) as well as section 4 below. In fact, it was the application to sign change questions about coefficients of Dirichlet series attached to automorphic representations that motivated this work (see [4]). To be precise, in [5], we investigate the sign changes of Dirichlet coefficients (a_n, say) of L-series attached to a self-dual automorphic representation π, in which case the a_n are real. In such a study, it is essential to know the precise properties of the series

$$\sum_{n=1}^{\infty} \frac{a_n^2}{n^s},$$

such as the location of the poles, their orders and the Laurent coefficients. In particular, we need to know its precise relationship to the Rankin-Selberg L-series $L(s, \pi \times \pi)$ (see section 4 below for elaboration). Ramanujan's identities (1) are then seen to be special cases of this more general result.

In this regard, what one finds in the literature are imprecise results. For instance, as the referee points out, Hadamard [2], using complex analysis, had identified the singularities of (3) in terms of the singularities of the functions in (2). Borel's work is discussed in [3]. But what is important for our applications is the shape of the numerator of the Hadamard product of the two series. We also refer the reader to the survey [6] in which there is a very special case of our result below here and this is discussed in the context of prime number theory and Ramanujan's work.

2. The case of simple roots

In this section, we prove the following generalisation of the Ramanujan identities (1).

Theorem 1. *For $i = 1, 2$, let $P_i(T)$, $Q_i(T)$ be non-zero polynomials over $\mathbb{C}$ such that degree of P_i is strictly less than the degree of Q_i. Also let*

$$Q_1(T) := \prod_{i=1}^{r}(1 - \alpha_i T) \quad \textit{and} \quad Q_2(T) := \prod_{j=1}^{t}(1 - \beta_j T),$$

where α_i's are distinct for $1 \leq i \leq r$ and β_j's are distinct for $1 \leq j \leq t$. Suppose that

$$\sum_{n \geq 0} a_n T^n = \frac{P_1(T)}{Q_1(T)} \quad \textit{and} \quad \sum_{n \geq 0} b_n T^n = \frac{P_2(T)}{Q_2(T)},$$

where $a_n, b_n \in \mathbb{C}$ for all n. Then

$$\sum_{n \geq 0} a_n b_n T^n = \frac{R(T)}{\prod_{i,j}(1 - \alpha_i \beta_j T)},$$

where $R(T) \in \mathbb{C}[T]$. Now if $a_0 = 1 = b_0$, then $R(0) = 1$. Further, if we have $P_1'(0) = 0 = P_2'(0)$, then $R'(0) = 0$. Here P' denotes the derivative of $P(T)$ with respect to T.

Proof. Using partial fractions, one can write

$$\sum_{n \geq 0} a_n T^n = \frac{P_1(T)}{Q_1(T)} = \sum_{i=1}^{r} \frac{c_i}{1 - \alpha_i T},$$

$$\sum_{n \geq 0} b_n T^n = \frac{P_2(T)}{Q_2(T)} = \sum_{j=1}^{t} \frac{e_j}{1 - \beta_j T},$$

where c_i, e_j's are suitable absolute constants. Then

$$a_n = \sum_{i=1}^{r} c_i \alpha_i^n \quad \text{and} \quad b_n = \sum_{j=1}^{t} e_j \beta_j^n. \tag{4}$$

Hence we have

$$\begin{aligned}
F(T) &:= \sum_{n\geq 0} a_n b_n T^n \\
&= \sum_{n\geq 0}\left(\sum_{i=1}^{r} c_i\alpha_i^n\right)\left(\sum_{j=1}^{t} e_j\beta_j^n\right)T^n, \text{ by equation (4)} \\
&= \sum_{i,j} c_i e_j \sum_{n\geq 0}\alpha_i^n\beta_j^n T^n \\
&= \sum_{i,j}\frac{c_i e_j}{1-\alpha_i\beta_j T} \\
&= \frac{R(T)}{\prod_{i,j}(1-\alpha_i\beta_j T)},
\end{aligned}$$

where $R(T) \in \mathbb{C}[T]$. Now if $a_0 = 1 = b_0$, then $a_0 b_0 = 1 = R(0)$. Further $a_0 = 1 = b_0$ implies that

$$\sum_{i=1}^{r} c_i = 1 = \sum_{j=1}^{t} e_j. \tag{5}$$

Using (5) and the assumption that the coefficient of T in P_1 and P_2 are zero, we get

$$S_1 := \sum_{i=1}^{r}\alpha_i = \sum_{i=1}^{r} c_i\alpha_i \quad \text{and} \quad S_2 := \sum_{j=1}^{t}\beta_j = \sum_{j=1}^{t} e_j\beta_j. \tag{6}$$

Write

$$D(T) := \prod_{i,j}(1-\alpha_i\beta_j T).$$

Then $R'(0) = (D(T)F(T))'|_{T=0} = D'(0)F(0) + F'(0)D(0)$. Now by equation (5), we have

$$D(0) = 1 = \sum_{i,j} c_i e_j = F(0).$$

Also by equation (6), we get

$$D'(0) = -\sum_{i,j}\alpha_i\beta_j = -S_1 S_2, \quad F'(0) = \sum_{i,j} c_i\alpha_i e_j\beta_j = S_1 S_2.$$

Hence the theorem. $\square$

3. The case of multiple roots

Related results in the case that the denominators $Q_1(T)$ and $Q_2(T)$ of our rational functions have multiple roots can be derived. These will take various forms. We begin with:

Theorem 2. *For $i = 1, 2$, let $P_i(T), Q_i(T) \in \mathbb{C}[T]$ be non-zero polynomials such that degree of P_i is strictly less than the degree of Q_i. Also let*

$$Q_1(T) := \prod_{i=1}^{r}(1-\alpha_i T)^{\ell_i} \quad \textit{and} \quad Q_2(T) := \prod_{j=1}^{t}(1-\beta_j T)^{m_j},$$

where α_i's are distinct for $1 \le i \le r$ and β_j's are distinct for $1 \le j \le t$. Suppose that

$$\sum_{n\ge 0} a_n T^n = \frac{P_1(T)}{Q_1(T)} \quad \textit{and} \quad \sum_{n\ge 0} b_n T^n = \frac{P_2(T)}{Q_2(T)},$$

where $a_n, b_n \in \mathbb{C}$ for all n. Then

$$\sum_{n\ge 0} a_n b_n T^n = \frac{R^*(T)}{\prod_{i,j}(1-\alpha_i\beta_j T)^{\ell_i+m_j-1}},$$

where $R^*(T) \in \mathbb{C}[T]$. If $a_0 = 1 = b_0$, then $R^*(0) = 1$. We also have

$$\sum_{n=0}^{\infty} a_n b_n T^n = \frac{R(T)}{\prod_{i,j}(1-\alpha_i\beta_j T)^{\ell_i m_j}}.$$

If $a_0 = b_0 = 1$, then $R(0) = 1$. Further suppose that $P_1'(0) = 0 = P_2'(0)$. Then $R'(0) = 0$. Here P' denotes the derivative of $P(T)$ with respect to T.

Remark 3. It is the second formulation of the power series that should be viewed as the analogue of our Theorem 1 in the case of multiple roots. As will be evident from the application to the Rankin-Selberg convolution discussed in section 4, it is this version that is applicable in a wide context.

Proof. Using partial fractions, we can write

$$\sum_{n\ge 0} a_n T^n = \frac{P_1(T)}{Q_1(T)} = \sum_{i=1}^{r}\sum_{u=0}^{\ell_i-1}\frac{c_{i,u}}{(1-\alpha_i T)^{\ell_i-u}}$$

$$\text{and} \quad \sum_{n\ge 0} b_n T^n = \frac{P_2(T)}{Q_2(T)} = \sum_{j=1}^{t}\sum_{v=0}^{m_j-1}\frac{e_{j,v}}{(1-\beta_j T)^{m_j-v}},$$

where $c_{i,u}$'s and $e_{j,v}$'s are suitable absolute constants. If $D = d/dT$ is the derivative operator, we have

$$\frac{1}{(1-\gamma T)^k} = \frac{\gamma^{-(k-1)}}{(k-1)!} D^{(k-1)} \left(\frac{1}{1-\gamma T} \right).$$

Then it is evident from the above that

$$a_n = \sum_{i=1}^{r} U_i(n) \alpha_i^n \tag{7}$$

$$\text{and} \quad b_n = \sum_{j=1}^{t} V_j(n) \beta_j^n,$$

where U_i and V_j are polynomials of degree $\ell_i - 1$ and $m_j - 1$ respectively. Hence we can write by equation (7),

$$\begin{aligned} F(T) &:= \sum_{n \geq 0} a_n b_n T^n \\ &= \sum_{n \geq 0} \sum_{i,j} U_i(n) V_j(n) \alpha_i^n \beta_j^n T^n. \end{aligned}$$

It is convenient to introduce the Pochammer symbol $(n)_k$ to designate

$$n(n-1)\cdots(n-k+1)$$

for $k \geq 1$ and 1 if $k = 0$. If $D = d/dT$ is the derivative operator, then

$$D^k \left(\frac{1}{1-\gamma T} \right) = \sum_{n=0}^{\infty} (n)_k \gamma^n T^{n-k}.$$

With this notation, we may write the polynomial $R_{ij}(n) := U_i(n) V_j(n)$ which is a polynomial of degree $\ell_i + m_j - 2$ as

$$\sum_k c_{ijk} (n)_k$$

so that our expression above becomes

$$\begin{aligned}
F(T) &:= \sum_{n\geq 0} a_n b_n T^n \\
&= \sum_{n\geq 0} \sum_{i,j} \sum_k c_{ijk}(n)_k \alpha_i^n \beta_j^n T^n. \\
&= \sum_{i,j} \sum_{k=0}^{\ell_i+m_j-2} c_{ijk} T^k D^k \left(\frac{1}{1-\alpha_i\beta_j T} \right) \\
&= \sum_{i,j} \sum_{k=0}^{\ell_i+m_j-2} c_{ijk} T^k \frac{k!(\alpha_i\beta_j)^k}{(1-\alpha_i\beta_j T)^{k+1}}
\end{aligned}$$

Taking common denominators, we see that we can write

$$F(T) = \frac{R^*(T)}{\prod_{i,j}(1-\alpha_i\beta_j T)^{\ell_i+m_j-1}}.$$

If $a_0 = b_0 = 1$, we see immediately that $R^*(0) = 1$.

To deduce the final assertion of the theorem, we write

$$\sum_{n=0}^{\infty} a_n b_n T^n = \frac{R(T)}{\prod_{i,j}(1-\alpha_i\beta_j T)^{\ell_i m_j}}$$

so that

$$R(T) = R^*(T) \prod_{i,j} (1-\alpha_i\beta_j T)^{(\ell_i-1)(m_j-1)}.$$

As before, $R(0) = 1$ if $a_0 = b_0 = 1$. Moreover, from the fact that

$$R(T) = \left(\sum_{n=0}^{\infty} a_n b_n T^n \right) \prod_{i,j} (1-\alpha_i\beta_j T)^{\ell_i m_j},$$

we see that $R'(0) = a_1 b_1 - (\sum_i \ell_i \alpha_i)(\sum_j m_j \beta_j) = 0$, since

$$a_1 = \left(\frac{P_1(T)}{Q_1(T)} \right)' |_{T=0} = -\, Q_1'(0) = \sum_i \ell_i \alpha_i$$

and

$$b_1 = \left(\frac{P_2(T)}{Q_2(T)} \right)' |_{T=0} = -\, Q_2'(0) = \sum_j m_j \beta_j.$$

This completes the proof. $\square$

Now that we have dealt with the multiple root case, we can safely derive the following result.

Corollary 4. *For $a_n, b_n \in \mathbb{C}$, consider the power series*

$$\sum_{n\geq 0} a_n T^n = \frac{P_1(T)}{Q_1(T)} \quad \textit{and} \quad \sum_{n\geq 0} b_n T^n = \frac{P_2(T)}{Q_2(T)},$$

where $P_i, Q_i \in \mathbb{C}[T]$ for $i = 1, 2$ are non-zero polynomials satisfying the assumptions of Theorem 2. Then for any natural number $k, \ell \geq 1$, we have

$$\sum_{n\geq 0} a_n^k b_n^\ell T^n = \frac{R(T)}{\prod_{i_1,\ldots,i_k; j_1,\ldots,j_\ell}(1 - \alpha_{i_1}\cdots\alpha_{i_k}\beta_{j_1}\cdots\beta_{j_\ell}T)},$$

where $R(T) \in \mathbb{C}[T]$. Further if $a_0 = 1 = b_0$ and the coefficient of T in P_1 and P_2 are zero, then $R(0) = 1$ and $R'(0) = 0$.

Proof. The result follows by applying Theorem 2 and an easy induction argument. □

Remark 5. Corollary 4 is true if we replace $\mathbb{C}$ by any field K of characteristic zero.

4. Applications to automorphic L-functions

We indicate here briefly how the results of the previous section can be applied to the theory of automorphic L-functions. For the background and technical preparation, we refer the reader to [1].

Theorem 6. *Let $L(s, \pi_1)$ and $L(s, \pi_2)$ be two automorphic L-functions whose associated Dirichlet series can be written as*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

respectively. Suppose that π_1 and π_2 satisfy Ramanujan's conjecture. Then the series

$$\sum_{n=1}^{\infty} \frac{a_n b_n}{n^s} = L(s, \pi_1 \times \pi_2) g(s),$$

where $g(s)$ is a Dirichlet series absolutely convergent for $\Re(s) > 1/2$ and $L(s, \pi_1 \times \pi_2)$ denotes the Rankin-Selberg L-series attached to π_1 and π_2.

Proof. By the standard theory of automorphic L-functions, we know that

$$\sum_{n=0}^{\infty} a_{p^n} T^n = \prod_{j=1}^{r_1} (1 - \alpha_{p,j} T)^{-1}$$

and

$$\sum_{n=0}^{\infty} b_{p^n} T^n = \prod_{j=1}^{r_2} (1 - \beta_{p,j} T)^{-1},$$

for certain natural numbers r_1 and r_2. Notice that the numerator for each of the rational functions here is 1 and thus satisfies the condition in our theorems. Thus, by our theorem, the associated $R(T)$ is a polynomial such that $R(0) = 1$ and $R'(0) = 0$. A routine calculation shows that $R(T)$ is a polynomial of degree less than $r_1 r_2$ whose coefficients are bounded (the bound depending only on r_1 and r_2). In this way, we easily identify the local factors of the Rankin-Selberg series attached to $\pi_1 \times \pi_2$. The function $g(s)$ can be shown to converge for $Re(s) > 1/2$ since

$$g(s) = \prod_p R(p^{-s}).$$

□

The applications of our theorem of course are not limited to the automorphic context. They are applicable whenever we have Dirichlet series, associated to a multiplicative function, whose associated p-Euler factor is a rational function in p^{-s}, for all but finitely many primes p. For instance, in Ramanujan's case, the functions $\sigma_a(n)$ are not (in general) coefficients of automorphic L-series, except in cases where a is a positive odd integer, in which case the coefficient is coming from the classical Eisenstein series or the quasi-modular form E_2. In this way, we can deduce analytic properties of series of the form

$$\sum_{n=1}^{\infty} \frac{a_n \sigma_a(n)}{n^s}.$$

For instance, it is easy to check that

$$\sum_{n=0}^{\infty} \sigma_a(p^n) T^n = \frac{1 + p^{a+1} T^2}{(1 - T)(1 - p^a T)},$$

so that this power series is in the form required of our main theorem.

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