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ON SIGN CHANGES FOR ALMOST PRIME COEFFICIENTS OF HALF-INTEGRAL WEIGHT MODULAR FORMS

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Abstract. For a half-integral weight modular form $f = \sum_{n=1}^{\infty} a_f(n) n^{(k-1)/2} q^n$ of weight $k = \ell + 1/2$ on $\Gamma_0(4)$ such that $a_f(n)$ $(n \in \mathbb{N})$ are real, we prove for a fixed suitable natural number r that $a_f(n)$ changes sign infinitely often as n varies over numbers having at most r prime factors, assuming the analog of the Ramanujan conjecture for Fourier coefficients of half-integral weight forms.

§1. *Introduction and statement of results.* Let $f = \sum_{n=1}^{\infty} a_f(n) n^{(k-1)/2} q^n$ be a half-integral weight modular form in the Kohnen + subspace of weight $k = \ell + 1/2$ on $\Gamma_0(4)$ with $\ell \geqslant 2$. Throughout the paper, we shall assume that $a_f(n)$ are real. In [**9**], Kohnen proved that for any half-integral weight modular form f, not necessarily an eigenform, for a square-free natural number t, the sequence $a_f(tn^2)$ $(n \in \mathbb{N})$ changes sign infinitely often, provided that there exists n_0 such that $a_f(tn_0^2) \neq 0$. In [**6**], Hulse *et al* proved that if f is an eigenform, then the sequence $a_f(t)$, where t runs over all square-free integers, changes sign infinitely often. In [**11**], Meher and Murty obtained some quantitative results on the number of sign changes in the sequence $a_f(n)$ $(n \in \mathbb{N})$.

Motivated by the Sato–Tate equidistribution theorem for integral weight Hecke eigenforms, it is natural to ask whether there is an equidistribution theorem in the half-integral weight setting also. To understand this question, the first step is to understand if the sequence $a_f(p)$, where p runs over prime numbers, changes sign infinitely often or not. Though we are not completely successful in answering this question, we try to answer the question to some extent in this paper. To this end, we consider the set $\mathcal{P}_r$ of numbers having at most r prime factors. Elements of $\mathcal{P}_r$ are called almost primes. We sometimes say that n is P_r or write $n = P_r$ if $n \in \mathcal{P}_r$. In this context, the paper will focus on the sequence $a_f(n)$, where n varies over $\mathcal{P}_r$ for a suitable fixed r (to be specified later). The purpose of the paper is to prove the following theorem.

THEOREM 1.1. *Let $\ell \geqslant 2$ be an integer and let f be an eigenform in the Kohnen + subspace of weight $k = \ell + 1/2$ on $\Gamma_0(4)$. Write*

$$f(z) = \sum_{n=1}^{\infty} a_f(n) n^{(k-1)/2} q^n, \quad q = e^{2\pi i z},$$

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and assume that:

(1) $a_f(n)$ *are real for all* $n \geqslant 1$*;*

(2) *the Ramanujan conjecture holds for* f*; that is, for all* $\epsilon > 0$ *and all* $n \geqslant 1$*, we have* $a_f(n) \ll_\epsilon n^\epsilon$.

Then there is an $r \geqslant 1$ *such that the sequence of numbers* $a_f(n)$ *changes sign infinitely often as* n *ranges over numbers in* $\mathcal{P}_r$*. More precisely, the number of sign changes with* $n = P_r$ *and* $n \leqslant x$ *is* $\gg \log x$.

In [13], the oscillations of Fourier coefficients of normalized Hecke eigenforms of integral weight were studied. However, we cannot apply those techniques to half-integral weight forms, since there is no multiplicative theory of eigenforms in this setting. In spite of this, we prove Theorem 1.1 by using sieve-theoretic techniques. Of course, we would like to prove the theorem with $r = 1$, but this goal seems out of reach with present knowledge and our sieve technique.

Here is our strategy. As outlined in [11], we need to have three ingredients to deduce sign-change results for any given sequence of numbers $a(n)$. First, we need an estimate of the form $a(n) = O(n^\alpha)$. Second, we need an estimate

$$\sum_{n\leqslant x} a(n) = O(x^\beta).$$

Third, we need an asymptotic formula

$$\sum_{n\leqslant x} a(n)^2 = cx + O(x^\gamma)$$

with α, β, γ non-negative constants and $c > 0$. Then, if $\alpha + \beta < \gamma < 1$, the sequence has infinitely many sign changes. If $\max(\alpha + \beta, \gamma) < b < 1$, then the last condition can be relaxed to

$$\sum_{x\leqslant n\leqslant x+y} a(n)^2 \gg y \quad \text{for any } y > x^b.$$

Such an inequality is vaguely referred to as the Hoheisel property in the literature (see, for example, [12]). In our situation, the assumption of the Ramanujan conjecture (see §2 below) gives the first estimate for our sequence $a_f(n)$ with n an almost prime. This can be relaxed a bit. We make some comments in this context at the end of the paper. For the second condition, we modify a result of Duke and Iwaniec [3], who showed the required estimate when the argument is restricted over primes. We derive the corresponding result for almost primes. We prove the following.

PROPOSITION 1.2. *If the* $a_f(n)$ *are as in Theorem 1.1, then for any natural number* r *and for sufficiently large* x*, we have for any* $\epsilon > 0$,

$$\sum_{1\leqslant n\leqslant x, n=P_r} a_f(n) = O(x^{155/156+\epsilon}). \tag{1}$$

For the third condition, we apply a lower bound sieve technique following a method of Hoffstein and Luo [5]. We show the following.

PROPOSITION 1.3. *Let f be as in Theorem 1.1. Then there exist a natural number r and $\delta > 0$ such that for any $Y > 3$, we have*

$$\sum_{n=P_r, Y^{\delta}<n<Y} a_f^2(n) \gg \frac{Y}{\log Y}.$$

With these results in place, we derive our main theorem following the axiomatic outline given above.

§2. *Notation and preliminaries.* For the sake of completeness, we review some rudimentary facts about half-integral weight modular forms as well as highlight why one expects Ramanujan's conjecture to be true. Let $f = \sum_{n\geqslant 1} a_f(n)n^{(k-1)/2}q^n$ be a cusp form of weight $k = \ell+1/2$ on $\Gamma_0(4)$. Consider the Dirichlet series

$$L(f, s) := \sum_{n\geqslant 1} \frac{a_f(n)}{n^s},$$

which converges for $\Re(s) > 3/2$ and represents a holomorphic function in this domain. Then, by means of the usual Mellin formula and using some standard arguments, one proves that the series admits an analytic continuation and functional equation for all complex values of s. The reader may consult [17, §5] or [8, p. 429].

From the results of Waldspurger [19] (see also [10]), we know that for all square-free m, $a_f^2(m) = O(L(1/2, g, \chi_m))$, where χ_m is the quadratic character $(\frac{m(-1)^{(l-1)/2}}{\cdot})$ and $g = \sum_{n\geqslant 1} c_g(n)n^{(2l-1)/2}q^n$ is the classical modular form of weight 2ℓ and level 2 corresponding to the half-integral modular form f via Shimura's correspondence [17]. In [7], Iwaniec showed that the $a_f(p)$ are bounded by the factor $p^{3/14}$. The exponent was later improved by Blomer and Harcos to $3/16 + \epsilon$ in [1]. One expects a stronger bound. Indeed, if we assume the analog of the Lindelöf hypothesis for $L(1/2, g, \chi_m)$ in the conductor aspect (that is, the m-aspect), then it is reasonable to expect the following.

CONJECTURE 2.1 (Ramanujan conjecture). *Let $f(z) = \sum_{n=1}^{\infty} a_f(n) n^{(k-1)/2}q^n$ be a half-integral weight modular form of weight $k = \ell + 1/2$ on $\Gamma_0(4N)$, where $k \in \mathbb{N}$, $k \geqslant 2$ and $q = e^{2\pi iz}$. Then, for any $\epsilon > 0$,*

$$a_f(n) = O(n^{\epsilon}). \tag{2}$$

We also need the following result of Duke–Iwaniec (see [3, §8]).

PROPOSITION 2.2. *If the $a_f(p)$ are as in Theorem 1.1, then, for sufficiently large x and for any $\epsilon > 0$, we have*

$$\sum_{1\leqslant p\leqslant x} a_f(p) = O(x^{155/156+\epsilon}), \tag{3}$$

where the sum runs over all prime numbers $p \leqslant x$.

We need a modified version of Proposition 2.2 in the context of almost primes. To this end, we introduce standard functions and apply Vaughan's identity to obtain this modified version. For any $r \in \mathbb{N}$, let Λ_r denote the generalized von Mangoldt function of order r. It is defined as

$$\Lambda_r(n) = \sum_{d|n} \mu(d)\left(\log\frac{n}{d}\right)^r.$$

By Möbius inversion,

$$(\log n)^r = \sum_{d|n} \Lambda_r(d).$$

For any $r \in \mathbb{N}$, we know that

$$\zeta^{(r)}(s) = (-1)^{r-1}\sum_{n=1}^{\infty}(\log n)^r n^{-s}.$$

This implies that

$$\frac{\zeta^{(r)}(s)}{\zeta(s)} = (-1)^r \sum_{n=1}^{\infty} \Lambda_r(n) n^{-s}.$$

The important feature of $\Lambda_r(n)$ is that it vanishes whenever n has more than r prime factors and so it is useful to detect when the condition $\omega(n) \leqslant r$ holds. We next recall a combinatorial partition of $\Lambda_1(n)$, which is a result of Vaughan [**18**] (cf. [**3**, equation (61)]).

PROPOSITION 2.3.

$$\Lambda_1(n) = \sum_{d|n, d\leqslant R} \mu(d)\log\frac{n}{d} - \sum_{lm|n, m\leqslant R, l\leqslant Q} \mu(m)\Lambda_1(l).$$

We need to generalize this identity to the case $\Lambda_r(n)$. We do this below.

§3. *Sums over almost primes and the proof of Proposition 1.2.* In Proposition 2.2, Duke–Iwaniec obtained the estimation for sums over primes. We modify their proof to show more generally that for any $r \geqslant 1$, we have Proposition 1.2.

The following combinatorial partition of $\Lambda_r(n)$ can also be deduced from [**16**, Lemma 1] with $c(n) = (-1)^r\Lambda_r(n)$, $\tilde{b}(n) = \mu(n)$, $b(n) = 1$ and $a(n) = (\log n)^r$.

PROPOSITION 3.1.

$$\Lambda_r(n) = \sum_{d|n, d\leqslant R} \mu(d)\left(\log\left(\frac{n}{d}\right)\right)^r - \sum_{lm|n, m\leqslant R, l\leqslant Q} \mu(m)\Lambda_r(l) + \sum_{lm|n, l\leqslant Q} \mu(m)\Lambda_r(l) + \sum_{lm|n, m>R, l>Q} \mu(m)\Lambda_r(l). \qquad (*)$$

Proof. We have, for any y,

$$\Lambda_r(n) = \sum_{d|n, d \leqslant y} \mu(d)\left(\log\left(\frac{n}{d}\right)\right)^r + \sum_{d|n, d > y} \mu(d)\left(\log\left(\frac{n}{d}\right)\right)^r.$$

Now, taking the second sum in the last line, we have, by Möbius inversion,

$$\begin{aligned}\sum_{d|n, d>y} \mu(d)\left(\log\left(\frac{n}{d}\right)\right)^r &= \sum_{d|n, d>y} \mu(d) \sum_{c|(n/d)} \Lambda_r(c) \\ &= \sum_{cd|n, d>y} \mu(d)\Lambda_r(c) \\ &= \sum_{cd|n, d>y, c>z} \mu(d)\Lambda_r(c) \\ &\quad + \sum_{cd|n, d>y, c\leqslant z} \mu(d)\Lambda_r(c) \quad \text{for any } z > 0.\end{aligned}$$

Here, z is a parameter chosen optimally in our later estimates. It should not be confused with a complex variable. Again, taking the second sum in the final line, we have

$$\sum_{cd|n, d>y, c\leqslant z} \mu(d)\Lambda_r(c) = \sum_{cd|n, c\leqslant z} \mu(d)\Lambda_r(c) - \sum_{cd|n, d\leqslant y, c\leqslant z} \mu(d)\Lambda_r(c).$$

Putting this together, with $y = R$, $z = Q$, we get the required partition of $\Lambda_r(n)$. $\square$

We obtain the following corollary, which is a generalized statement of Proposition 2.3 to any $r \in \mathbb{N}$.

COROLLARY 3.2. *Suppose that $n \in \mathbb{N}$ with $Q < n \leqslant QR = X$. Then*

$$\Lambda_r(n) = \sum_{d|n, d\leqslant R} \mu(d)\left(\log\left(\frac{n}{d}\right)\right)^r - \sum_{lm|n, m\leqslant R, l\leqslant Q} \mu(m)\Lambda_r(l).$$

Proof. If $n \leqslant QR$, note that the last sum in Proposition 3.1 is zero. If $n > Q$, then the third sum in $(*)$ $\sum_{lm|n, l\leqslant Q} \mu(m)\Lambda_r(l) = \sum_{l|n, l\leqslant Q} \Lambda_r(l) \sum_{m|(n/l)} \mu(m)$ is zero since $n/l > 1$ because $l \leqslant Q$ and $n = (n/l)l > Q$. $\square$

Let $\hat{f}_n = n^{(k-1)/2} a_n(f)$. Take $b_n = (1 - n/X)\hat{\psi}(n)$, where $\hat{\psi}$ is the Gauss sum of a Dirichlet character ψ to modulus $c \equiv 0 \pmod 4$ and $X \geqslant 2$. Now let us consider the sum

$$P(X) = \sum_{n\leqslant X} b_n \hat{f}_n \Lambda_r(n).$$

We follow closely the method of Duke and Iwaniec [3]. Similar to [3, proof of Proposition 2.2], we shall split the second sum in Corollary 3.2 over the dyadic intervals $L < l \leqslant 2L$, $M < m \leqslant 2M$ with $2L \leqslant Q$ and $2M \leqslant R$, and we write accordingly

$$\Lambda_r(n) = \Lambda_R^*(n) - \sum_L \sum_M \Lambda_{LM}^*(n),$$

where

$$\Lambda_R^*(n) = \sum_{d|n, d \leqslant R} \mu(d)\left(\log\left(\frac{n}{d}\right)\right)^r,$$
$$\Lambda_{LM}^*(n) = \sum_{lm|n, L<l\leqslant 2L, M<m\leqslant 2M} \mu(m)\Lambda_r(l).$$

LEMMA 3.3.

$$P(X) = P_R(X) - \sum_L \sum_M P_{LM}(X) + O(Q^{(k+1)/2} X^{r\epsilon}),$$

where

$$P_R(X) = \sum_{n \leqslant X} b_n \hat{f}_n \Lambda_R^*(n), \qquad P_{LM}(X) = \sum_{n\leqslant X} b_n \hat{f}_n \Lambda_{LM}^*(n).$$

Proof. The contribution to the sum from $n \leqslant Q$ is clearly $O(Q^{(k+1)/2} X^{\epsilon})$ by a simple application of Cauchy's inequality as in [3]. $\square$

§4. *Proof of Proposition 1.2.*

Proof. We follow [3], but give more details as the proof in [3] is terse. To treat $P_R(X)$, we apply partial summation and [3, equation (58)], which states that

$$\sum_{n\leqslant X, d|n} \hat{\psi}(n) \hat{f}_n \ll X^{k/2} \log X,$$

where the implied constant is independent of d. We deduce that

$$P_R(X) \ll R X^{k/2} (\log X)^{r+1}.$$

To treat $P_{LM}(X)$, we follow again [3] and split

$$P_{LM}(X) = P'_{LM}(X) + P''_{LM}(X),$$

where, in the first term, $n = lm$ is squarefree and, in the second term, $n = lm$ is not squarefree. Proceeding as in [3], we have (using [3, equation (7)])

$$P'_{LM}(X) \ll LM X^{k/2-1/4+\epsilon}.$$

For $P''_{LM}(X)$, we have (upon using of [3, equation (6)]),

$$P''_{LM}(X) \ll \sum_{l,m} |\mu(m)||\Lambda_r(l)|(lm)^{1/2} X^{(k-1)/2+\epsilon},$$

where the sum runs over l and m such that $L \leqslant l \leqslant 2L$, $M \leqslant m \leqslant 2M$ and $\mu(lm) = 0$. Since $\mu(lm) = 00$, and l is P_r, we see that $(l, m) \neq 1$ in the sum. We write

$$L < l = p_1 \dots p_r < 2L$$

and suppose that $p_1 | m$. Thus,

$$P''_{LM}(X) \ll X^{(k-1)/2+\epsilon} \sum_{L<p_1<2L} \sum_{L/p_1<p_2\dots p_r<2L/p_1} \frac{M^{3/2}}{p_1} \ll LM^{3/2}X^{(k-1)/2+\epsilon}.$$

Hence,

$$P''_{LM}(X) \ll LM^{3/2}X^{(k-1)/2+\epsilon}.$$

Thus, we get the first bound,

$$P_{LM}(X) \ll LMX^{k/2-1/4+\epsilon} + LM^{3/2}X^{(k-1)/2+\epsilon}.$$

For the second bound, one appeals to the estimate for the bilinear form:

$$\begin{aligned} &\left| \sum_{mn\leqslant X, M<m\leqslant 2M} a_m b_n \hat{f}_{mn} \right| \\ &\quad \ll \left(\sum_{mn \leqslant 2X} |a_m b_n|^2 \right)^{1/2} (X^{1/2}M^{-1/2} + X^{1/4}M^{3/4}) X^{(k-1)/2+\epsilon}, \end{aligned}$$

which is [3, equation (57)]. We put $a_m = \mu(m)$ to get

$$\left| \sum_{mn\leqslant X, M<m\leqslant 2M} \mu(m) b_n \hat{f}_{mn} \right| \ll (M^{-1/2} + M^{3/4}X^{-1/4}) X^{(k+1)/2+\epsilon}.$$

Combining this with our earlier discussion of $P_{LM}(X)$ finally leads to

$$P_{LM}(X) \ll (M^{-1/2} + M^{3/4}X^{-1/4}) X^{(k+1)/2+\epsilon},$$

which is valid for any M, Q, R satisfying $1 \leqslant M \leqslant R = X/Q$. Choosing $M = X^{1/26}$, $Q = X^{9/13}$, $R = X^{4/13}$ gives

$$P(X) \ll X^{(k+1)/2-1/52+\epsilon}.$$

The smoothing factor $(1 - n/X)$ is removed as in [3]. Therefore, we obtain that

$$\sum_{n\leqslant X} \hat{\psi}(n) \hat{f}_n \Lambda_r(n) \leqslant X^{(k+1)/2-1/156+\epsilon}.$$

Since $\Lambda_r(n)$ vanishes if n has more than r prime factors, we have by partial summation

$$\sum_{n\leqslant X, \omega(n)\leqslant r} \hat{\psi}(n) a_f(n) \leqslant X^{155/156+\epsilon}.$$

Hence,

$$\sum_{1\leqslant n\leqslant x, \omega(n)\leqslant r} \hat{\psi}(n) a_f(n) = O(x^{155/156+\epsilon}).$$

As in [3], we may take ψ to be the principal character (mod 4), from which the proposition follows. □

§5. *Proof of Proposition 1.3.*

Proof. Let r be as in the theorem of [5] and F a non-negative smooth function compactly supported in $(0, 1)$ with positive mean value. The argument on [5, p. 439] shows that there is a positive constant c such that

$$\sum_{n \leqslant Y, \omega(n) \leqslant r} a_f^2(n) F\left(\frac{n}{Y}\right) \geqslant \frac{cY}{\log Y} + O(Y^{14/15}).$$

Since F is bounded, this means that there is a positive constant c_1 such that

$$\sum_{n \leqslant Y, \omega(n) \leqslant r} a_f^2(n) \geqslant \frac{c_1 Y}{\log Y} + O(Y^{14/15}).$$

On the other hand, we have, assuming the Ramanujan conjecture,

$$\sum_{n \leqslant Y^\delta} a_f^2(n) \ll Y^{\delta+\epsilon}.$$

Actually, by the techniques of [14, 15], the ϵ in the exponent can be removed and one does not need to assume the Ramanujan conjecture here. In any case, for $\delta > 0$ and sufficiently small, we have

$$\sum_{Y^\delta < n < Y, n = P_r} a_f(n)^2 \gg \frac{Y}{\log Y}.$$

This completes the proof of Proposition 1.3. □

§6. *Proof of Theorem 1.1.* Theorem 1.1 follows from Conjecture 2.1 and Propositions 1.2 and 1.3 but for the convenience of the reader we include a proof.

Proof. We choose δ sufficiently small and show that $a_f(n)$ changes sign for $n = P_r$ and $x^\delta < n < x$. Suppose not. Without loss of generality, we can assume that $a_f(n)$ are positive for all n in the set $T = \{n : x^\delta < n \leqslant x, \omega(n) \leqslant r\}$. From Proposition 1.2, for sufficiently large x and sufficiently small $\delta > 0$, we have

$$\sum_{n \in T} a_f(n) = O(x^{155/156+\epsilon}). \tag{4}$$

Using Conjecture 2.1,

$$\sum_{n \in T} a_f^2(n) = O(x^{155/156+\epsilon_0+\epsilon}). \tag{5}$$

Replacing Y by x in Proposition 1.3, we get

$$\sum_{\substack{n = P_r \\ 1 \leqslant n \leqslant x}} a_f^2(n) \gg \frac{x}{\log x}. \tag{6}$$

Hence, for sufficiently large x, we have

$$\sum_{n\in T} a_f^2(n) \gg \frac{x}{\log x}. \tag{7}$$

We have a contradiction from (5) and (7). Thus, there is at least one sign change of $a_f(n)$ with $n = P_r$ in (x^δ, x). Thus, there is a sign change in each of the intervals of the form $(x^{\delta^t}, x^{\delta^{t-1}})$. The number of such disjoint intervals covering $(1, x)$ is clearly $\gg \log x$. This completes the proof of our theorem.

The value of r is determined by the results of [**5**]. In their paper, the authors suggest that by using metaplectic techniques, one can show that $r = 4$ is permissible. Again, based on comments of that paper, it is possible to sharpen this to $r = 3$ by using weighted sieve techniques (cf. [**5**, §3] for the details). This seems to be the limit of present-day knowledge. □

§7. *Concluding remarks.* In this section, by assuming a Siegel-type conjecture, we deduce that the sequence $a_f(p)$, where p varies over primes, changes sign infinitely often.

CONJECTURE 7.1. *If* $L(1/2, g, \chi_p) \neq 0$, *then* $|L(1/2, g, \chi_p)| \gg p^{-\epsilon}$ *for any* $\epsilon > 0$.

THEOREM 7.2. *Let f be as in Theorem 1.1. Assume that Conjecture 7.1 holds. Then the sequence $a_f(p)$, where p varies over primes, changes sign infinitely often.*

Proof. Without loss of generality, for sufficiently large x, we can assume that $a_f(p)$ are positive for all p in the set $T' = \{n : x_0 < p \leqslant x\}$ for some natural number x_0. From Proposition 2.2, we have

$$\sum_{p\in T'} a_f(p) = O(x^{155/156+\epsilon}). \tag{8}$$

Using Conjecture 7.1 and Waldspurger's theorem,

$$\sum_{p\in T'} a_f^2(p) \gg x^{1-\epsilon'}. \tag{9}$$

We have a contradiction from (8) and (9). Thus, there is at least one sign change of $a_f(p)$ for p prime in $(x_0, x]$. As before, this argument can be fine tuned to yield $\gg \log x$ sign changes for $p \leqslant x$. This completes the proof of our theorem. □

These results certainly extend to higher level, since both the Duke–Iwaniec theorem and the Hoffstein–Luo theorem do and our argument goes through. Finally, we remark that the assumption of Ramanujan's conjecture in our main theorem can be relaxed somewhat. A weaker assumption, namely $a_f(n) = O(n^\alpha)$ for any α such that $0 < \alpha < 1/156$, is sufficient to prove the results.

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References

1. V. Blomer and G. Harcos, Hybrid bounds for twisted L-functions. *J. Reine Angew. Math.* **621** (2008), 53–79.
2. A. C. Cojocaru and M. R. Murty, *An Introduction to Sieve Methods and Their Applications* (*London Mathematical Society Student Texts* **66**), Cambridge University Press (Cambridge, 2006).
3. W. Duke and H. Iwaniec, Bilinear forms in the Fourier coefficients of half-integral weight cusp forms and sums over primes. *Math. Ann.* **286**(4) (1990), 783–802.
4. H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press (London, 1974).
5. J. Hoffstein and W. Luo, Nonvanishing of L-series and the combinatorial sieve. *Math. Res. Lett.* **4** (1997), 435–444, with an appendix by David E. Rohrlich.
6. T. A. Hulse, E. M. Kiral, C. I. Kuan and L. Lim, The sign of Fourier coefficients of half-integral weight cusp forms. *Int. J. Number Theory* **8** (2012), 749–762.
7. H. Iwaniec, Fourier coefficients of modular forms of half-integral weight. *Invent. Math.* **87**(2) (1987), 385–401.
8. W. Kohnen, On Hecke eigenforms of half-integral weight. *Math. Ann.* **293**(3) (1992), 427–431.
9. W. Kohnen, A short note on Fourier coefficients of half-integral weight modular forms. *Int. J. Number Theory* **6** (2010), 1255–1259.
10. W. Kohnen and D. Zagier, Values of L-series of modular forms at the center of the critical strip. *Invent. Math.* **64** (1981), 175–198.
11. J. Meher and M. R. Murty, Sign changes of Fourier coefficients of half-integral weight cusp forms. *Int. J. Number Theory* **10**(4) (2014), 905–914.
12. C. J. Moreno, The Hoheisel phenomenon for generalized Dirichlet series. *Proc. Amer. Math. Soc.* **40** (1973), 47–51.
13. M. R. Murty, Oscillations of Fourier coefficients of modular forms. *Math. Ann.* **262** (1983), 431–446.
14. M. R. Murty and V. K. Murty, Mean values of derivatives of modular L-series. *Ann. of Math.* (2) **133** (1991), 447–475.
15. M. R. Murty and V. K. Murty, *Non-vanishing of L-functions and applications* (*Progress in Mathematics* **157**), Birkhäuser (Basel, 1997).
16. M. R. Murty and A. Sankaranarayanan, Averages of exponential twists of the Liouville function. *Forum Math.* **14** (2002), 273–291.
17. G. Shimura, On modular forms of half integral weight. *Ann. of Math.* (2) **97** (1973), 440–481.
18. R. C. Vaughan, Mean value theorems in prime number theory. *J. Lond. Math. Soc.* (2) **10** (1975), 153–162.
19. J.-L. Waldspurger, Sur les coefficients de Fourier des formes modulaires de poids demi-entier. *J. Math. Pures Appl.* (9) **60**(4) (1981), 375–484.

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