A Smooth Selberg Sieve and Applications

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Abstract We introduce a new technique for sieving over smooth moduli in the higher-rank Selberg sieve and obtain asymptotic formulas for the same.

Keywords The higher-rank Selberg sieve · Bounded gaps

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1 Introduction

The Bombieri–Vinogradov theorem establishes that the primes have a level of distribution θ for any $\theta < 1/2$. More precisely, letting $\pi(x)$ denote the number of primes upto x, we put for $(a, q) = 1$,

$$E_{\mathbb{P}}(x, q, a) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \chi_{\mathbb{P}}(n) - \frac{\pi(x)}{\phi(q)}, \tag{1.1}$$

where $\chi_{\mathbb{P}}$ is the characteristic function of the primes. Then, the primes are said to have a level of distribution θ if for any $A > 0$, we have

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$$\sum_{q \le x^{\theta}} \max_{(a,q)=1} \mid E_{\mathbb{P}}(x,q,a) \mid \ll \frac{x}{(\log x)^A}. \tag{1.2}$$

The chief innovation of Zhang [20] is the extension of the level of distribution of the primes to beyond $\theta = 1/2$, albeit in a weaker sense by restricting the moduli to be smooth or free of large prime factors. It is this breakthrough, combined with the classical GPY approach using the Selberg sieve that enabled him to obtain his spectacular result on bounded gaps between primes in May 2013. We encourage the reader to refer to [6, 16, 17], in addition to [20]. Collaborative efforts of a number of mathematicians [11] succeeded in improving the level of distribution in Zhang's equidistribution result from $\theta = \frac{1}{2} + \frac{1}{584}$ to $\theta = \frac{1}{2} + \frac{7}{300}$. More precisely, the following was proved. Let $P^+(q)$ denote the largest prime factor of q. Then, for any $\varpi, \xi \ge 0$ satisfying $600\varpi + 180\xi < 7$, and any $A > 0$, we have

$$\sum_{\substack{q \le x^{\Theta} \\ P^+(q) < x^{\xi}}} \max_{(a,q)=1} \mid E_{\mathbb{P}}(x,q,a) \mid \ll \frac{x}{(\log x)^A}, \tag{1.3}$$

where $\Theta = \frac{1}{2} + 2\varpi$. Applying this improved result to Zhang's work, along with sophisticated numerical techniques, the bound for gaps between primes was reduced from 70 million in [20] to 14950 in [11].

In October 2013, Maynard [9] and Tao [12] independently applied the higher-rank Selberg sieve to the problem of bounded gaps, thereby obtaining bounded gaps between primes for *any* positive level of distribution. They also obtained better numerical values. The natural next step in this sequence of ideas is to combine the new equidistribution estimate (1.3) with the higher-rank Selberg sieve. This has been done in [12], employing efficient numerical methods and extensive computations to reduce the bound still further to 246.

Recently, the authors ([14, 19] gave an axiomatic formulation of the higher-rank sieve as a general method, along with applications. This work allows one to see clearly the underlying structure of the sieve and motivates a more general way to incorporate smoothing into the higher-rank Selberg sieve. In [12], the moduli are constrained to be free of large prime factors by truncating the support of the function $\mathcal{F}$ appearing in (3.2). Our method imposes smoothing as an explicit condition and leads to expressions involving the Dickman and Buchstab functions (cf. Sect. 4) as would be expected. The general theory of the same forms the crux of this paper. In forthcoming work, we will discuss applications of this theory.

2 Notation

We will continue with the notation used in [19]. We include the same briefly here for the sake of completeness. We denote the k-tuple of integers $(d_1, \ldots, d_k)$ by $\underline{d}$.

A tuple is said to be square-free if the product of its components is square-free. For $R \in \mathbb{R}$, the inequality $\underline{d} \le R$ means that $\prod_i d_i \le R$. The notions of divisibility and congruence among tuples are defined component-wise. Divisibility relations between a tuple and a scalar are defined in terms of the product of the components of the tuple. For example,

$$q|\underline{d} \Longleftrightarrow q|\prod_i d_i.$$

We define the multiplicative vector function $f(\underline{d})$ as the product of its component (multiplicative) functions acting on the corresponding components of the tuple, that is,

$$f(\underline{d}) = \prod_{i=1}^{k} f_i(d_i).$$

We use $[\cdot, \cdot]$ and $(\cdot, \cdot)$ to denote LCM and GCD, respectively. In the case of tuples, this means the product of the LCMs (or GCDs) of the corresponding components. We employ the following multi-index notation to denote mixed derivatives of a function on k-tuples, $\mathcal{F}(\underline{t})$.

$$\mathcal{F}^{(\underline{\alpha})}(\underline{t}) := \frac{\partial^{\alpha}\mathcal{F}(t_1, \ldots, t_k)}{(\partial t_1)^{\alpha_1} \ldots (\partial t_k)^{\alpha_k}}, \tag{2.1}$$

for any k-tuple $\underline{\alpha}$ with $\alpha := \sum_{j=1}^{k} \alpha_j$.

Let $P^+(q)$ denote the largest prime factor of q. Then q is said to be m-smooth if $P^+(q) < m$. For a tuple $\underline{d}$, $P^+(\underline{d})$ denotes the largest prime factor dividing any of the components of $\underline{d}$. We use the convention $n \sim N$ to mean $N \le n < 2N$. In practice we have $N \to \infty$. We fix $D_0 = \log\log\log N$ and let $W = \prod_{p<D_0} p$. Then $W \sim \log\log N^{(1+o(1))}$ by an application of the prime number theorem. Let $\omega(n)$ denote the number of distinct prime factors of n. The greatest integer less than or equal to x is denoted as $\lfloor x \rfloor$. Throughout this paper, δ denotes a positive quantity which can be made as small as needed.

3 The Higher-Rank Selberg Sieve

In this section, we recall the salient features of the higher-rank Selberg sieve discussed in [19]. The exposition given here is concise for the sake of brevity, and the reader is encouraged to peruse Sect. 3.2 of the above-mentioned paper.

Given a set $\mathcal{S}$ of k-tuples (not necessarily finite),

$$\mathcal{S} = \{\underline{n} = (n_1, \ldots, n_k)\},$$

in [19], we undertook a systematic study of sums of the form

$$\sum_{\underline{n}\in\mathcal{S}} w_{\underline{n}}\Big(\sum_{\underline{d}|\underline{n}} \lambda_{\underline{d}}\Big)^2, \tag{3.1}$$

satisfying certain hypotheses. Here $w_{\underline{n}}$ is a 'weight' attached to the tuples $\underline{n}$ and $\lambda_{\underline{d}}$'s are sieve parameters chosen in terms of a fixed positive real number R and a smooth real valued test function $\mathcal{F}$ supported on the simplex

$$\Delta_k(1) := \{(t_1, \dots, t_k) \in [0, \infty)^k : t_1 + \cdots + t_k \le 1\}.$$

More precisely, we chose :

$$\lambda_{\underline{d}} = \mu(\underline{d})\mathcal{F}\left(\frac{\log d_1}{\log R}, \dots, \frac{\log d_k}{\log R}\right). \tag{3.2}$$

The sum (3.1) was assumed to satisfy the following hypotheses.

H1. If a prime p divides a tuple $\underline{n}$ such that p divides n_i and n_j, with $i \ne j$, then p must lie in some fixed finite set of primes $\mathcal{P}_0$.

This allows us to perform the 'W trick,' that is restrict $\underline{n}$ in the above sum to be congruent to a residue class $\underline{b} \pmod W$ such that $(b_i, W) = 1$ for all i.

H2. The function $w_{\underline{n}}$ satisfies

$$\sum_{\substack{\underline{d}|\underline{n} \\ \underline{n}\equiv\underline{b} \pmod W}} w_{\underline{n}} = \frac{X}{f(\underline{d})} + r_{\underline{d}},$$

for some multiplicative function f and some quantity X depending on the set $\mathcal{S}$.

H3. The components of f satisfy

$$f_j(p) = \frac{p}{\alpha_j} + O(p^t), \text{ with } t < 1$$

for some fixed $\alpha_j \in \mathbb{N}$.

We denote the tuple $(\alpha_1, \dots, \alpha_k)$ as $\underline{\alpha}$ and the sum of the components $\sum_{j=1}^k \alpha_j$ as α.

H4. There exists $\theta > 0$ and $Y \ll X$ such that

$$\sum_{[\underline{d},\underline{e}]<Y^\theta} |r_{[\underline{d},\underline{e}]}| \ll \frac{Y}{(\log Y)^A}$$

for any $A > 0$.

With all this in place, we state below the main results of the higher-rank sieve obtained in [19].

Lemma 3.1 *Set R to be some fixed power of X. Let f be a multiplicative function satisfying* H3 and

$$\mathcal{G}, \mathcal{H} : [0, \infty)^k \to \mathbb{R}$$

be smooth functions with compact support. We denote

$$\mathcal{G}\left(\frac{\log \underline{d}}{\log R}\right) := \mathcal{G}\left(\frac{\log d_1}{\log R}, \ldots, \frac{\log d_k}{\log R}\right)$$

and similarly for $\mathcal{H}$. Let the dash over the sum mean that we sum over k-tuples $\underline{d}$ and $\underline{e}$ with $[\underline{d}, \underline{e}]$ square-free and co-prime to W. Then,

$$\sum_{\underline{d},\underline{e}}{}' \frac{\mu(\underline{d})\mu(\underline{e})}{f([\underline{d}, \underline{e}])}\mathcal{G}\left(\frac{\log \underline{d}}{\log R}\right)\mathcal{H}\left(\frac{\log \underline{e}}{\log R}\right) = (1 + o(1))C(\mathcal{G}, \mathcal{H})^{(\underline{\alpha})} \frac{c(W)}{(\log R)^{\alpha}},$$

where

$$C(\mathcal{G}, \mathcal{H})^{(\underline{\alpha})} = \int_0^{\infty} \cdots \int_0^{\infty} \left(\prod_{j=1}^{k} \frac{t_j^{\alpha_j - 1}}{(\alpha_j - 1)!}\right) \mathcal{G}(\underline{t})^{(\underline{\alpha})}\mathcal{H}(\underline{t})^{(\underline{\alpha})} d\underline{t},$$

with $\mathcal{G}(\underline{t})^{(\underline{\alpha})}$ and $\mathcal{H}(\underline{t})^{(\underline{\alpha})}$ as in the notation of (2.1). Furthermore,

$$c(W) := \prod_{p|W} \frac{p^{\alpha}}{\phi(p)^{\alpha}}.$$

Theorem 3.2 *Let $\lambda_{\underline{d}}$'s be as chosen above. Suppose hypotheses H1 to H3 hold and H4 holds with $Y = X$. Set $R = X^{\theta/2-\delta}$ for small $\delta > 0$. Then,*

$$\sum_{\underline{n} \equiv \underline{b} \pmod{W}} w_n \left(\sum_{\underline{d}|\underline{n}} \lambda_{\underline{d}}\right)^2 = (1 + o(1))C(\mathcal{F}, \mathcal{F})^{(\underline{\alpha})} c(W) \frac{X}{(\log R)^{\alpha}},$$

with

$$c(W) := \frac{W^{\alpha}}{\phi(W)^{\alpha}}$$

and

$$C(\mathcal{F}, \mathcal{F})^{(\underline{\alpha})} = \int_0^{\infty} \cdots \int_0^{\infty} \left(\prod_{j=1}^{k} \frac{t_j^{\alpha_j - 1}}{(\alpha_j - 1)!}\right) \left(\mathcal{F}^{(\underline{\alpha})}(\underline{t})\right)^2 d\underline{t}.$$

4 A Refined Smoothing Procedure

In the axiomatization of the higher-rank Selberg sieve discussed in Sect. 3, it may be that the hypothesis H4 holds for θ in a range that is too restrictive to yield good asymptotic formulas. Motivated by estimates of the type (1.3), we would like to consider the following more relaxed condition on the error term instead of hypothesis H4:

H4* There exists $\Theta > 0$, $0 < \xi \leq 1$ and $Y \ll X$ such that

$$\sum_{\substack{[\underline{d},\underline{e}]<Y^{\Theta} \\ P^+([\underline{d},\underline{e}])<Y^{\xi\Theta/2}}} |r_{[\underline{d},\underline{e}]}| \ll \frac{Y}{(\log Y)^A}$$

for any $A > 0$.

Accordingly, we now consider the sum (3.1) with additional smoothing conditions imposed. Let $R_1 = X^{\Theta/2-\delta}$. We will replace R in (3.2) by R_1. The above setting motivates the analysis of smooth sums of the kind

$$\sum_{\underline{d},\underline{e}<R_1} \frac{\lambda_{\underline{d}}\lambda_{\underline{e}}}{f[\underline{d},\underline{e}]},$$

and hence a smooth version of the sum considered in Lemma 3.1 must be studied. We do so by emulating the Fourier analytic method adopted in [19], incorporating the smoothing conditions that arise by use of the partial zeta function as well as the Dickman and Buchstab functions.

The Dickman function ρ is defined recursively by the initial condition $\rho(u) = 1$, $(0 \leq u \leq 1)$ and the equation

$$\rho(u) = \rho(v) - \int_v^u \rho(t-1)\frac{dt}{t}, \quad (v \leq u \leq v+1).$$

The Buchstab function ω is defined similarly, by the initial condition $u\omega(u) := 1$, $(1 \leq u \leq 2)$ and the relation

$$u\omega(u) = 1 + \int_1^{u-1} \omega(v)dv, \quad (u > 2).$$

These functions have a long and venerable history. Though Dickman's paper [5] where he introduced the function was published in 1930, it seems that Ramanujan (unpublished) had studied it more than a decade earlier (see p. 337 of [15]). Indeed, Ramanujan writes down the following explicit formula for the Dickman function $\rho(u)$. Put $I_0 = 1$ and define (for $k \geq 1$) recursively

$$I_k(u) = \int \cdots \int_{\substack{t_1,\ldots,t_k \le 1 \\ t_1+\ldots+t_k \le u}} \frac{dt_1 \ldots dt_k}{t_1 \ldots t_k}.$$

Then,

$$\rho(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} I_k(u).$$

The study of $\rho(u)$ became dormant for almost two decades until 1947, when Chowla and Vijayaraghavan studied it unaware of any earlier work (see [3]). Two years later, Buchstab [2] studied the same function (again unaware of any earlier work). It was de Bruijn [4], in 1951, who began exhaustive research into the nature of this function and obtained an asymptotic expansion for it. In 1980, Hildebrand and Tenenbaum [7, 18] extended considerably the range of applicability of de Bruijn's formulas. We refer the reader to the excellent survey of Moree [10] for further details.

We state some results which will be useful in our analysis. These are from [18], after minor changes in notation.

Proposition 4.1 (p. 379 of [18]) *For the partial zeta function, defined as*

$$\zeta_y(s) := \prod_{p<y} \left(1 - \frac{1}{p^s}\right)^{-1},$$

we have,

$$\zeta_y(s) = \zeta(s) e^{-J((s-1)\log y)} \left(1 + O(L_\epsilon(y)^{-1})\right),$$

where

$$J(s) = \int_0^{\infty} \frac{e^{-s-t}}{s+t} dt$$

and

$$L_\epsilon(y) = \exp\{(\log y)^{3/5-\epsilon}\}.$$

Proposition 4.2 (Theorem 7, p. 372 of [18]) *Let ρ be the Dickman function and $\widehat{\rho}$ be the Laplace transform of ρ defined as*

$$\widehat{\rho}(s) = \int_0^{\infty} e^{-st} \rho(t) dt.$$

Then,

$$s\widehat{\rho}(s) = e^{-J(s)}.$$

We use the notation $\omega^+ := \boldsymbol{\delta} + \omega$, where $\boldsymbol{\delta}$ is the Dirac delta function. Then $\widehat{\omega^+}(s) = 1 + \widehat{\omega}(s)$, in the distributional sense.

Proposition 4.3 (Theorem 5, p. 404 of [18]) *Let ω denote the Buchstab function and $\widehat{\omega}$ be its Laplace transform given by*

$$\widehat{\omega}(s) = \int_0^\infty e^{-su}\omega(u)du.$$

Then,

$$s\widehat{\omega^+}(s) = \frac{1}{\widehat{\rho}(s)}.$$

Henceforth, ξ is a fixed number, $0 < \xi \le 1$. We also recall the following notation which will be widely used. If g is a vector function, that is, $g(\underline{t})$ is defined as $\prod_j g_j(t_j)$, we use the notation $g(\underline{t})^{\underline{\alpha}}$ to denote the product $\prod_j g_j(t_j)^{\alpha_j}$. It is clear that

$$\omega^+(\underline{t}) := \prod_j \omega^+(t_j) = \prod_j (\omega(t_j) + \boldsymbol{\delta}(t_j)).$$

We prove some results toward obtaining a smooth version of Lemma 3.1. These will play an important role in subsequent discussion.

Lemma 4.4 *Let f be a multiplicative function satisfying H3 with respect to the tuple $\underline{\alpha} = (\alpha_1, \dots, \alpha_k)$. Let $\mathcal{G}$, $\mathcal{H}$ be smooth functions with compact support as in Lemma 3.1. We retain all the notation used in Lemma 3.1. Then the R_1^ξ- smooth sum*

$$\sum_{\substack{\underline{d},\underline{e} \\ P^+([\underline{d},\underline{e}])<R_1^\xi}}{}' \frac{\mu(\underline{d})\mu(\underline{e})}{f([\underline{d},\underline{e}])}\mathcal{G}\left(\frac{\log \underline{d}}{\log R_1}\right)\mathcal{H}\left(\frac{\log \underline{e}}{\log R_1}\right).$$

is asymptotic (as $R_1 \to \infty$) to

$$(1 + o(1))\frac{c(W)}{(\log R_1)^\alpha}\xi^\alpha \mathscr{C}_{\mathcal{G},\mathcal{H}}(\xi)^{(\underline{\alpha})},$$

where $\mathscr{C}_{\mathcal{G},\mathcal{H}}(\xi)^{(\underline{\alpha})}$ is the integral

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v})(1 + i\underline{u})^{\underline{\alpha}}(1 + i\underline{v})^{\underline{\alpha}}\Upsilon(\underline{u}, \underline{v})^{(\underline{\alpha},\underline{\alpha},\underline{\alpha})}d\underline{u}d\underline{v}, \tag{4.1}$$

with

$$\Upsilon(\underline{u}, \underline{v})^{(\underline{\alpha},\underline{\alpha},\underline{\alpha})} = \widehat{\omega^+}((1 + i\underline{u})\xi)^{\underline{\alpha}}\,\widehat{\omega^+}((1 + i\underline{v})\xi)^{\underline{\alpha}}\,\widehat{\rho}((2 + i\underline{u} + i\underline{v})\xi)^{\underline{\alpha}},$$

$$\eta_{\mathcal{G}}(\underline{u}) = \int_{\mathbb{R}^k} G(\underline{t})\exp(\underline{t})\exp(i\underline{u}\cdot\underline{t})d\underline{t}, \quad \eta_{\mathcal{H}}(\underline{v}) = \int_{\mathbb{R}^k} G(\underline{t})\exp(\underline{t})\exp(i\underline{v}\cdot\underline{t})d\underline{t},$$

where $\exp(\underline{t}) = \prod_{j=1}^k e^{t_j}$ and the dot denotes the usual dot product of tuples.

Proof Let $y = R_1^{\xi}$. As the required sum is the same as the one considered in Lemma 3.1 with an additional smoothing condition imposed, we will follow the proof of the aforesaid lemma given in [19], with details to highlight any modifications. All references to [19] in this proof are understood to refer to the relevant steps in the proof of Lemma 3.1 in that paper.

Using Fourier inversion as in [19], this sum is given by the integral

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v}) Z_y(\underline{u}, \underline{v}) d\underline{u} d\underline{v}, \tag{4.2}$$

where $Z_y(\underline{u}, \underline{v})$ is now $Z(\underline{u}, \underline{v})$ of [19] along with a smoothing condition, that is,

$$Z_y(\underline{u}, \underline{v}) = \sideset{}{'}\sum_{\substack{\underline{d}, \underline{e} \\ P^+([\underline{d}, \underline{e}]) < y}} \frac{\mu(\underline{d})\mu(\underline{e})}{f([\underline{d}, \underline{e}])} \frac{1}{\underline{d}^{(1+i\underline{u})/\log R_1}} \frac{1}{\underline{e}^{(1+i\underline{v})/\log R_1}}$$

Again, we can write an Euler product for $Z_y(\underline{u}, \underline{v})$, as in [19], but it will run only over primes $D_0 < p < y$, as opposed to the Euler product that we had in [19] over primes greater than D_0. This is because, the dash over the sum constrains $[\underline{d}, \underline{e}]$ to be co-prime to W and the smoothing condition means that its prime factors must be below y. Hence, the Euler product for $Z_y(\underline{u}, \underline{v})$ is given by

$$\prod_{p \nmid W, p < y} \left(1 - \sum_{j=1}^{k} \frac{1}{f_j(p)} \left(\frac{1}{p^{\frac{1+iu_j}{\log R_1}}} + \frac{1}{p^{\frac{1+iv_j}{\log R_1}}} - \frac{1}{p^{\frac{1+iu_j}{\log R_1} + \frac{1+iv_j}{\log R_1}}} \right)\right).$$

After applying H3 to retrieve the behavior of $f_j(p)$ for each component $1 \leq j \leq k$, some algebraic manipulation along the lines in [19] gives us the following convenient approximation

$$Z_y(\underline{u}, \underline{v}) = (1 + o(1)) \prod_{j=1}^{k} \prod_{D_0 < p < y} \frac{\left(1 - \alpha_j p^{-1-\frac{1+iu_j}{\log R_1}}\right)\left(1 - \alpha_j p^{-1-\frac{1+iv_j}{\log R_1}}\right)}{1 - \alpha_j \, p^{-1-\frac{2+iu_j+iv_j}{\log R_1}}}. \tag{4.3}$$

This leads us to examine for each j, Euler products of the form

$$\prod_{D_0 < p < y} \left(1 - \frac{\alpha_j}{p^{1+s_j}}\right), \tag{4.4}$$

with $\mathrm{Re}(s_j) > 0$. We write the above Euler product as

$$\prod_{D_0<p<y}\left(1-\frac{\alpha_j}{p^{1+s_j}}\right)\left(1-\frac{1}{p^{1+s_j}}\right)^{-\alpha_j}\left(1-\frac{1}{p^{1+s_j}}\right)^{\alpha_j}$$
$$= D_j(1+s_j)\prod_{D_0<p<y}\left(1-\frac{1}{p^{1+s_j}}\right)^{\alpha_j},$$

where

$$D_j(s)=\prod_{D_0<p<y}\left(1-\frac{\alpha_j}{p^s}\right)\left(1-\frac{1}{p^s}\right)^{-\alpha_j}$$

is an Euler product supported on primes $D_0<p<y$ and analytic for $\mathrm{Re}(s)>0$. For $\mathrm{Re}(s)\geq 1$, we have

$$D_j(s)=1+O\big(\sum_{D_0<p<y}p^{-2}\big),$$

showing that $D_j(s)=1+o(1)$ as R_1 (and hence y as well as D_0) goes to ∞. Proceeding as in [19], we obtain

$$\prod_{D_0<p<y}\left(1-\frac{\alpha_j}{p^{1+s_j}}\right)=(1+o(1))\prod_{p|W}\left(1-\frac{1}{p^{1+s_j}}\right)^{-\alpha_j}\zeta_y\left(1+s_j\right)^{-\alpha_j}.$$

Fix some small $\epsilon>0$. As done in [19], it is possible to show that the main contribution to (4.2) comes from the region $|\underline{u}|, |\underline{v}|<(\log R)^{\epsilon}$. Hence, we would like to analyze $\zeta_y(1+s_j)$ as $s_j\to 0$. Combining Propositions 4.1 and 4.2, we obtain as $s_j\to 0^+$ and $y\to\infty$,

$$\begin{aligned}\zeta_y(1+s_j)&=(s_j\log y)\zeta(1+s_j)\widehat{\rho}\left(s_j\log y\right)(1+O(L_\epsilon(y)^{-1}))\\&=(1+o(1))(s_j\log y)\zeta(1+s_j)\widehat{\rho}\left(s_j\log y\right)\\&=(1+o(1))(\log y)\widehat{\rho}\left(s_j\log y\right),\end{aligned}$$

where we have used the asymptotic $\zeta(1+s)=(1+o(1))s^{-1}$ as $s\to 0^+$ for the last equality. Thus we obtain as $s_j\to 0^+$,

$$\prod_{D_0<p<y}\left(1-\frac{\alpha_j}{p^{1+s_j}}\right)=(1+o(1))\frac{W^{\alpha_j}}{\phi(W)^{\alpha_j}}(\log y)^{-\alpha_j}\widehat{\rho}\left(s_j\log y\right)^{-\alpha_j}$$

Applying this to each term appearing in (4.3), recalling that ξ is defined as $\log y/\log R_1$ and $\alpha:=\sum_{j=1}^k\alpha_j$, we have in the region $|\underline{u}|, |\underline{v}|<(\log R)^{\epsilon}$,

$$Z_y(\underline{u},\underline{v})=(1+o(1))\frac{W^{\alpha}}{\phi(W)^{\alpha}}(\log y)^{-\alpha}\prod_{j=1}^k\frac{\widehat{\rho}((1+iu_j)\xi)^{-\alpha_j}\,\widehat{\rho}((1+iv_j)\xi)^{-\alpha_j}}{\widehat{\rho}((2+u_j+v_j)\xi)^{-\alpha_j}},$$

as $R \to \infty$. To get rid of the denominator in the above expression, we use Proposition 4.3. This gives

$$Z_y(\underline{u}, \underline{v}) = (1 + o(1)) \frac{W^\alpha}{\phi(W)^\alpha} \frac{\xi^{2\alpha}}{(\log y)^\alpha} \prod_{j=1}^{k} (1 + iu_j)^{\alpha_j} (1 + iv_j)^{\alpha_j}$$

$$\prod_{j=1}^{k} \widehat{\omega^+}((1 + iu_j)\xi)^{\alpha_j}\, \widehat{\omega^+}((1 + iv_j)\xi)^{\alpha_j}\, \widehat{\rho}((2 + iu_j + iv_j)\xi)^{\alpha_j}$$

As $\xi = (\log y)/(\log R_1)$, we obtain

$$Z_y(\underline{u}, \underline{v}) = (1 + o(1))c(W) \frac{\xi^\alpha}{(\log R_1)^\alpha} (1 + i\underline{u})^{\underline{\alpha}} (1 + i\underline{v})^{\underline{\alpha}} \Upsilon(\underline{u}, \underline{v})^{(\underline{\alpha}, \underline{\alpha}, \underline{\alpha})}, \tag{4.5}$$

with notation as in the statement of this lemma. Plugging this into the integral expression (4.2) for the required sum yields the result. $\square$

In order to simplify the integral $\mathscr{C}_{\mathcal{G},\mathcal{H}}(\xi)^{(\underline{\alpha})}$ appearing in Lemma 4.4, we first consider the special case $\underline{\alpha} = \underline{1} = (1, \ldots, 1)$. We have the following result.

Lemma 4.5 *The integral*

$$\mathscr{C}_{\mathcal{G},\mathcal{H}}(\xi)^{(\underline{1})} = \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u}) \eta_{\mathcal{H}}(\underline{v}) (1 + i\underline{u})(1 + i\underline{v}) \Upsilon(\underline{u}, \underline{v})^{(\underline{1}, \underline{1}, \underline{1})} d\underline{u} d\underline{v}$$

is given by

$$\int_{\mathbb{R}^k} \rho(\underline{t}) \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \omega^+(\underline{s} - \underline{t}) \omega^+(\underline{r} - \underline{t}) \mathcal{G}^{(\underline{1})}(\xi \underline{r}) \mathcal{H}^{(\underline{1})}(\xi \underline{s}) d\underline{r} d\underline{s} d\underline{t},$$

which can be further simplified by writing each $\omega^+(\underline{x})$ as the product

$$\prod_j (\omega(x_j) + \boldsymbol{\delta}(x_j))$$

and expanding the resulting expression. (Here, we use the multi-index notation (2.1) *for $\mathcal{G}^{(\underline{1})}(\underline{u})$ and $\mathcal{H}^{(\underline{1})}(\underline{u})$.)*

Proof We have

$$\Upsilon(\underline{u}, \underline{v})^{(\underline{1}, \underline{1}, \underline{1})} = \widehat{\omega^+}((1 + i\underline{u})\xi)\, \widehat{\omega^+}((1 + i\underline{v})\xi)\, \widehat{\rho}((2 + i\underline{u} + i\underline{v})\xi).$$

Then, the interpretation of the vector notation and the definition of the Laplace transform give us

$$\widehat{\omega^+}((1+i\underline{u})\xi) = \prod_{j=1}^{k} \widehat{\omega^+}((1+iu_j)\xi) = \prod_{j=1}^{k} \int_{\mathbb{R}} \omega^+(r_j) e^{-\xi(1+iu_j)r_j} dr_j$$
$$= \int_{\mathbb{R}^k} \omega^+(\underline{r}) e^{-\xi(1+i\underline{u})\cdot\underline{r}} d\underline{r},$$

where the dot denotes dot product of the tuples $\xi(1+i\underline{u})$ and $\underline{r}$ and $\omega^+(\underline{r}) := \prod_{j=1}^{k} \omega^+(r_j)$. Similarly, we obtain

$$\widehat{\omega^+}((1+i\underline{v})\xi) = \int_{\mathbb{R}^k} \omega^+(\underline{s}) e^{-\xi(1+i\underline{v})\cdot\underline{s}} d\underline{s},$$
$$\widehat{\rho}((2+i\underline{u}+i\underline{v})\xi) = \int_{\mathbb{R}^k} \rho(\underline{t}) e^{-\xi(2+i\underline{u}+i\underline{v})\cdot\underline{t}} d\underline{t}.$$

Thus, $\Upsilon(\underline{u}, \underline{v})^{(\underline{1},\underline{1},\underline{1})}$ equals

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \rho(\underline{t})\omega^+(\underline{s})\omega^+(\underline{r}) e^{-\xi(1+i\underline{u})\cdot(\underline{t}+\underline{r})} e^{-\xi(1+i\underline{v})\cdot(\underline{t}+\underline{s})} d\underline{r}d\underline{s}d\underline{t}.$$

Plugging this into the required integral gives

$$\mathscr{C}_{\mathcal{G},\mathcal{H}}(\xi)^{(\underline{1})} = \int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \rho(\underline{t})\omega^+(\underline{s})\omega^+(\underline{r}) I_{\mathcal{G}} I_{\mathcal{H}} d\underline{r}d\underline{s}d\underline{t},$$

with

$$I_{\mathcal{G}}(\xi(\underline{t}+\underline{r})) = \int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u})(1+i\underline{u}) e^{-\xi(1+i\underline{u})\cdot(\underline{t}+\underline{r})} d\underline{u}$$
$$I_{\mathcal{H}}(\xi(\underline{t}+\underline{s})) = \int_{\mathbb{R}^k} \eta_{\mathcal{H}}(\underline{v})(1+i\underline{v}) e^{-\xi(1+i\underline{v})\cdot(\underline{t}+\underline{s})} d\underline{v}.$$

By Fourier inversion, we have the identities

$$\begin{aligned} \mathcal{G}(\underline{x}) &= \int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u}) \exp\left(-(1+i\underline{u})\cdot\underline{x}\right) d\underline{u} \\ \mathcal{H}(\underline{x}) &= \int_{\mathbb{R}^k} \eta_{\mathcal{H}}(\underline{v}) \exp\left(-(1+i\underline{v})\cdot\underline{x}\right) d\underline{v}. \end{aligned} \tag{4.6}$$

It is clear from this that $I_{\mathcal{G}}(\xi(\underline{t}+\underline{r}))$ is nothing but

$$(-1)^k \frac{\partial^k \mathcal{G}(\underline{x})}{\partial x_1 \ldots \partial x_k}\bigg|_{\underline{x}=\xi(\underline{t}+\underline{r})},$$

that is, $(-1)^k \mathcal{G}^{(\underline{1})}(\xi(\underline{t}+\underline{r}))$ in our notation. Repeating this argument for $\mathcal{H}$, we have $I_{\mathcal{H}}(\xi(\underline{t}+\underline{s})) = (-1)^k \mathcal{H}^{(\underline{1})}(\xi(\underline{t}+\underline{s}))$. Thus, the required integral $\mathscr{C}_{\mathcal{G},\mathcal{H}}(\xi)^{(\underline{1})}$ is given by

$$\begin{aligned}&\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \rho(\underline{t})\omega^+(\underline{s})\omega^+(\underline{r})\mathcal{G}^{(\underline{1})}(\xi(\underline{t}+\underline{r}))\mathcal{H}^{(\underline{1})}(\xi(\underline{t}+\underline{s}))d\underline{r}d\underline{s}d\underline{t} \\ &= \int_{\mathbb{R}^k} \rho(\underline{t}) \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \omega^+(\underline{s})\omega^+(\underline{r})\mathcal{G}^{(\underline{1})}(\xi(\underline{t}+\underline{r}))\mathcal{H}^{(\underline{1})}(\xi(\underline{t}+\underline{s}))d\underline{r}d\underline{s}d\underline{t} \\ &= \int_{\mathbb{R}^k} \rho(\underline{t}) \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \omega^+(\underline{s}-\underline{t})\omega^+(\underline{r}-\underline{t})\mathcal{G}^{(\underline{1})}(\xi\underline{r})\mathcal{H}^{(\underline{1})}(\xi\underline{s})d\underline{r}d\underline{s}d\underline{t},\end{aligned} \tag{4.7}$$

after suitable change of the variables $\underline{r}$ and $\underline{s}$. □

Let $\underline{\alpha}$, $\underline{\beta}$, $\underline{a}$, $\underline{b}$ and $\underline{c}$ be k-tuples. We now consider the general integral $\mathscr{C}_{\mathcal{G},\mathcal{H}}(\xi)^{(\underline{\alpha},\underline{\beta},\underline{a},\underline{b},\underline{c})}$, defined as

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v})(1+i\underline{u})^{\underline{\alpha}}(1+i\underline{v})^{\underline{\beta}}\Upsilon(\underline{u},\underline{v})^{(\underline{a},\underline{b},\underline{c})}d\underline{u}d\underline{v}, \tag{4.8}$$

with

$$\Upsilon(\underline{u},\underline{v})^{(\underline{a},\underline{b},\underline{c})} = \widehat{\omega^+}((1+i\underline{u})\xi)^{\underline{a}}\,\widehat{\omega^+}((1+i\underline{v})\xi)^{\underline{b}}\,\widehat{\rho}((2+i\underline{u}+i\underline{v})\xi)^{\underline{c}}.$$

Note that when all the tuples involved are the same, say $\underline{\alpha}$, then we will use the notation $\mathscr{C}_{\mathcal{G},\mathcal{H}}(\xi)^{(\underline{\alpha})}$ for convenience. We now emulate the proof of the above lemma for this general case.

Lemma 4.6 *The integral $\mathscr{C}_{\mathcal{G},\mathcal{H}}(\xi)^{(\underline{\alpha},\underline{\beta},\underline{a},\underline{b},\underline{c})}$ is given by*

$$(-1)^{\alpha+\beta}\int_{\mathbb{R}^k} \rho_{\underline{c}}(\underline{t}) \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \omega^+_{\underline{a}}(\underline{s}-\underline{t})\omega^+_{\underline{b}}(\underline{r}-\underline{t})\mathcal{G}^{(\underline{\alpha})}(\xi\underline{r})\mathcal{H}^{(\underline{\beta})}(\xi\underline{s})d\underline{r}d\underline{s}d\underline{t},$$

where $\rho_{\underline{c}}$, $\omega^+_{\underline{a}}$ and $\omega^+_{\underline{b}}$ are defined as follows. Let $*$ denote the convolution operator. Then,

$$\begin{aligned}\rho_{\underline{c}}(\underline{t}) := \rho^{*\underline{c}}(\underline{t}) &= \prod_{j=1}^k \rho(t_j)^{*c_j} \\ &= \prod_{j=1}^k \underbrace{\rho(t_j) * \ldots * \rho(t_j)}_{c_j \text{ times}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\omega^+_{\underline{a}}(\underline{r}) := \prod_{j=1}^k \omega^+(r_j)^{*a_j} &= \prod_{j=1}^k \underbrace{\omega^+(r_j) * \cdots * \omega^+(r_j)}_{a_j \text{ times}} \\ &= \prod_{j=1}^k \underbrace{(\delta+\omega(t_j)) * \cdots * (\delta+\omega(t_j))}_{a_j \text{ times}}.\end{aligned}$$

The definition of $\omega_{\underline{b}}^{+}$ *is exactly the same. As before, we use the multi-index notation* (2.1) *for* $\mathcal{G}^{(\underline{\alpha})}(\underline{u})$ *and* $\mathcal{H}^{(\underline{\beta})}(\underline{u})$.

Proof We have

$$\widehat{\omega^{+}}((1+i\underline{u})\xi)^{\underline{a}} = \prod_{j=1}^{k} \widehat{\omega^{+}}((1+iu_j)\xi)^{a_j}.$$

Then for each j, $\widehat{\omega^{+}}((1+iu_j)\xi)^{a_j}$ is the Laplace transform evaluated at $(1+iu_j)\xi$, of the convolution product

$$\omega^{+}(r_j)^{*a_j} = \omega^{+}(r_j) * \cdots * \omega^{+}(r_j),$$

where ω^{+} is convolved a_j times. Thus,

$$\widehat{\omega^{+}}((1+iu_j)\xi)^{a_j} = \int_{\mathbb{R}} \omega^{+}(r_j)^{*a_j} e^{-\xi(1+iu_j)r_j} dr_j$$

Reverting to the vector notation gives us

$$\widehat{\omega^{+}}((1+i\underline{u})\xi)^{\underline{a}} = \int_{\mathbb{R}^k} \omega_{\underline{a}}^{+}(\underline{r}) e^{-\xi(1+i\underline{u})\cdot\underline{r}} d\underline{r},$$

where $\omega_{\underline{a}}^{+}$ is as defined in the lemma. Proceeding similarly, we obtain

$$\widehat{\omega^{+}}((1+i\underline{v})\xi)^{\underline{b}} = \int_{\mathbb{R}^k} \omega_{\underline{b}}^{+}(\underline{s}) e^{-\xi(1+i\underline{v})\cdot\underline{s}} d\underline{s},$$
$$\widehat{\rho}((2+i\underline{u}+i\underline{v})\xi)^{\underline{c}} = \int_{\mathbb{R}^k} \rho_{\underline{c}}(\underline{t}) e^{-\xi(2+i\underline{u}+i\underline{v})\cdot\underline{t}} d\underline{t}.$$

We now proceed exactly as in the proof of the previous lemma with $\omega^{+}(\underline{r})$, $\omega^{+}(\underline{s})$, $\rho(\underline{t})$ replaced by $\omega_{\underline{a}}^{+}(\underline{r})$, $\omega_{\underline{b}}^{+}(\underline{s})$, $\rho_{\underline{c}}(\underline{t})$, respectively, to obtain that $\Upsilon(\underline{u},\underline{v})^{(\underline{a},\underline{b},\underline{c})}$ is given by

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \rho_{\underline{c}}(\underline{t})\omega_{\underline{b}}^{+}(\underline{s})\omega_{\underline{a}}^{+}(\underline{r}) e^{-\xi(1+i\underline{u})\cdot(\underline{t}+\underline{r})} e^{-\xi(1+i\underline{v})\cdot(\underline{t}+\underline{s})} d\underline{r}d\underline{s}d\underline{t}. \tag{4.9}$$

Thus, we obtain that the required integral is

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \rho_{\underline{c}}(\underline{t})\omega_{\underline{b}}^{+}(\underline{s})\omega_{\underline{a}}^{+}(\underline{r}) I_{\mathcal{G}}^{(\underline{\alpha})} I_{\mathcal{H}}^{(\underline{\beta})} d\underline{r}d\underline{s}d\underline{t},$$

with

$$I_{\mathcal{G}}^{(\alpha)} = \int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u})(1+i\underline{u})^{\underline{\alpha}} e^{-\xi(1+i\underline{u})\cdot(\underline{t}+\underline{r})} d\underline{u}$$
$$I_{\mathcal{H}}^{(\beta)} = \int_{\mathbb{R}^k} \eta_{\mathcal{H}}(\underline{v})(1+i\underline{v})^{\underline{\beta}} e^{-\xi(1+i\underline{v})\cdot(\underline{t}+\underline{s})} d\underline{v}.$$

As before, we use the identities (4.6) and change of variable to obtain the desired result. □

4.1 A Smooth Higher-Rank Sieve

We work with the setting of the sieve established in Sect. 3. Recall the hypotheses H1 to H3 in this section. Instead of H4 we will assume hypothesis H4* on the error terms. Our main result is then the following smooth version of Theorem 3.2 of [19], which can be thought of as the ξ-smooth higher-rank sieve.

Theorem 4.7 *With $\lambda_{\underline{d}}$'s chosen as in (3.2), hypotheses H1, H2, H3, H4* and $R_1 = X^{\Theta/2-\delta}$, we have*

$$\sum_{\substack{n\in\mathcal{S} \\ \underline{n}\equiv b \ (\mathrm{mod}\ W)}} w_n \Bigg(\sum_{\substack{\underline{d}|\underline{n} \\ P^+(\underline{d})<R_1^{\xi}}} \lambda_{\underline{d}} \Bigg)^2 = (1+o(1))c(W)\xi^{\alpha}\mathscr{C}_{\mathcal{F},\mathcal{F}}(\xi)^{(\underline{\alpha})} \frac{X}{(\log R_1)^{\alpha}},$$

with

$$c(W) = \frac{W^{\alpha}}{\phi(W)^{\alpha}}$$

and $\mathscr{C}_{\mathcal{F},\mathcal{F}}(\xi)^{(\underline{\alpha})}$ obtained from the expression in Lemma 4.6 by plugging in $\underline{\beta}, \underline{a}, \underline{b}, \underline{c} = \underline{\alpha}$.

Proof Expanding out the square, interchanging the order of summation gives us

$$\sum_{\underline{n}\equiv b \ (\mathrm{mod}\ W)} w_n \Bigg(\sum_{\substack{\underline{d}|\underline{n} \\ P^+(\underline{d})<R_1^{\xi}}} \lambda_{\underline{d}} \Bigg)^2 = \sum_{\substack{\underline{d},\underline{e}<R_1 \\ P^+([\underline{d},\underline{e}])<R_1^{\xi}}} \lambda_{\underline{d}}\lambda_{\underline{e}} \Bigg(\sum_{\substack{[\underline{d},\underline{e}]|\underline{n} \\ \underline{n}\equiv\underline{b} \ (\mathrm{mod}\ W)}} w_{\underline{n}} \Bigg)$$

Now one can argue exactly as in Theorem 3.2 of [19], using H1 and the W-trick to impose the same restrictions on the tuples $\underline{d}, \underline{e}$. Moreover, H2 along with the choice of $\lambda_{\underline{d}}$'s gives that the main term for the desired sum is

$$X \sum_{\substack{\underline{d},\underline{e} \\ P^+([\underline{d},\underline{e}])<R_1^{\xi}}}{}^{\prime} \frac{\mu(\underline{d})\mu(\underline{e})}{f([\underline{d},\underline{e}])} \mathcal{F}\left(\frac{\log \underline{d}}{\log R_1}\right) \mathcal{F}\left(\frac{\log \underline{e}}{\log R_1}\right).$$

As H3 holds, one can apply Lemma 4.4 to this sum, to obtain the asymptotic

$$(1+o(1))c(W)\xi^{\alpha}\mathscr{C}_{\mathcal{F},\mathcal{F}}(\xi)^{(\underline{\alpha},\underline{\alpha},\underline{\alpha})}\frac{X}{(\log R_1)^{\alpha}},$$

as X (and hence R_1) goes to infinity. As the choice of $\lambda_{\underline{d}}$'s (see (3.2)) in terms of the smooth compactly supported function $\mathcal{F}$ means that they are bounded absolutely, the error term is given by

$$O\Bigg(\sum_{\substack{\underline{d},\underline{e}<R_1\\ P^+([\underline{d},\underline{e}])<R_1^{\xi}}}|r_{[\underline{d},\underline{e}]}|\Bigg) \tag{4.10}$$

and can be neglected due to the choice of R_1, after applying H4*. □

5 Application to Bounded Gaps Between Primes

In this section, we apply the sieve with the smoothing procedure discussed above to the well-known prime k-tuples problem. A set $\mathscr{H}$ of distinct nonnegative integers is said to be admissible if for every prime p, there is a residue class $b_p \pmod{p}$ such that $b_p \notin \mathscr{H} \pmod{p}$. That is $|\mathscr{H} \pmod{p}| < p$, for every prime p. We will work with a fixed admissible k-tuple

$$\mathscr{H} = \{h_1, \ldots, h_k\}.$$

We use the 'W trick' to remove the effect of small primes, that is we restrict n to be in a fixed residue class b modulo W, where $W = \prod_{p<D_0} p$ and b is chosen so that $b + h_i$ is co-prime to W for each h_i. This choice of b is possible because of admissibility of the set $\mathscr{H}$. One can choose $D_0 = \log\log\log N$, so that $W \sim (\log\log N)^{1+o(1)}$ by an application of the prime number theorem, as noted earlier.

Recall that $\chi_{\mathbb{P}}$ denotes the characteristic function of the primes. Consider the expressions

$$S_1 = \sum_{\substack{n\sim N\\ n\equiv b \pmod{W}}} a_n$$

and

$$S_2 = \sum_{\substack{n\sim N\\ n\equiv b \pmod{W}}} \left(\sum_{m=1}^{k} \chi_{\mathbb{P}}(n+h_m)\right) a_n,$$

where a_n are nonnegative parameters.

For ρ positive, we denote by $S(N,\rho)$ the quantity

$$S_2 - \rho S_1 = \sum_{\substack{n \sim N \\ n \equiv b \ (\mathrm{mod}\ W)}} \left(\sum_{j=1}^{k} \chi_{\mathbb{P}}(n + h_j) - \rho \right) a_n. \tag{5.1}$$

The key idea then used is as follows. We state it as a proposition for convenient future reference.

Proposition 5.1 *Given a positive number ρ, if*

$$S(N, \rho) > 0$$

for all large N, then there are infinitely many integers n such that at least $\lfloor \rho \rfloor + 1$ of $n + h_1, \ldots, n + h_k$ are primes.

Proof The definition of $S(N, \rho)$ gives that the sum

$$\sum_{\substack{n \sim N \\ n \equiv b \ (\mathrm{mod}\ W)}} \left(\sum_{j=1}^{k} \chi_{\mathbb{P}}(n + h_j) - \rho \right) a_n > 0.$$

As a_n are nonnegative parameters, we must have

$$\sum_{j=1}^{k} \chi_{\mathbb{P}}(n + h_j) - \rho > 0,$$

for some $n \sim N$. As this happens for all large N,

$$\sum_{j=1}^{k} \chi_{\mathbb{P}}(n + h_j) > \rho$$

holds for infinitely many integers n. As each $\chi_{\mathbb{P}}(n + h_j)$ is an integer, this completes the proof. □

Fix some $0 < \xi \leq 1$. Writing $\underline{n}$ for the tuple $(n + h_1, \ldots, n + h_k)$, we make the following choice of sieve parameters a_n:

$$a_n = \Bigg(\sum_{\substack{\underline{d} | \underline{n} \\ P^+(\underline{d}) < R_1^{\xi}}} \lambda_{\underline{d}} \Bigg)^2,$$

with the sequence $(\lambda_{\underline{d}})$ chosen in terms of $\mathcal{F}$ as in (3.2). We will refer to the corresponding sums with this choice of sieve parameters as $S_1(\xi)$ and $S_2(\xi)$, respectively. We proceed to derive asymptotic formulas for $S_1(\xi)$ and $S_2(\xi)$ by applying our smooth higher-rank sieve.

5.1 Asymptotic Formula for $S_1(\xi)$

Recall that $S_1(\xi)$ denotes the ξ-smooth sum

$$S_1(\xi) := \sum_{\substack{n \sim N \\ n \equiv b \pmod W}} \Bigg(\sum_{\substack{d_j | n+h_j \forall j \\ P^+(\underline{d}) < R_1^{\xi}}} \lambda_{\underline{d}} \Bigg)^2$$

Theorem 5.2 *Choose $\Theta < 1$. With $\lambda_{\underline{d}}$'s chosen as in (3.2) and $R_1 = N^{\Theta/2-\delta}$, we have*

$$S_1(\xi) = (1+o(1)) \frac{W^{k-1}}{\phi(W)^k} \frac{N}{(\log R_1)^k} \xi^k \mathscr{C}_{\mathcal{F},\mathcal{F}}(\xi)^{(\underline{1})},$$

with $\mathscr{C}_{\mathcal{F},\mathcal{F}}(\xi)^{(\underline{1})}$ given by

$$\int_{(\mathbb{R}^+)^k} \rho(\underline{t}) \left(\int_{\Delta_k(1/\xi)} \omega^+(\underline{r}-\underline{t}) \mathcal{F}^{(\underline{1})}(\xi \underline{r}) d\underline{r} \right)^2 d\underline{t},$$

where $\Delta_k(1/\xi)$ is the simplex $\{\underline{t} \in [0,\infty]^k : \sum_{j=1}^k t_j \leq 1/\xi\}$.

Proof We wish to prove this as an application of Theorem 4.7. Note that the setting of the sieve and verification of conditions H1–H3 are the same as in the proof of Lemma 4.2 in [19]. As $r_{\underline{d}} = O(1)$ in this case, $|\lambda_{\underline{d}}|$'s are bounded, and $\Theta < 1$, H4* follows from the bound

$$\sum_{\substack{[\underline{d},\underline{e}] < N^{\Theta} \\ P^+([\underline{d},\underline{e}]) < N^{\xi\theta/2}}} 1 \ll \sum_{[\underline{d},\underline{e}] < N^{\Theta}} 1 \ll \frac{N}{(\log N)^A}$$

for any $A > 0$. The tuple $\underline{\alpha}$ is in this case just the tuple $\underline{1} = (1, \ldots, 1)$ and $\alpha = \sum_j \alpha_j = k$. We have

$$c(W) = \frac{W^k}{\phi(W)^k},$$

and $X = N/W$ exactly as before. It is clear that the result now follows directly from Theorem 4.7. The integral $\mathscr{C}_{\mathcal{F},\mathcal{F}}(\xi)^{(\underline{1})}$ is as in Lemma 4.5. It can be simplified to

$$\begin{aligned} &\int_{(\mathbb{R}^+)^k} \rho(\underline{t}) \int_{\Delta_k(1/\xi)} \int_{\Delta_k(1/\xi)} \omega^+(\underline{s}-\underline{t}) \omega^+(\underline{r}-\underline{t}) \mathcal{F}^{(\underline{1})}(\xi \underline{s}) \mathcal{F}^{(\underline{1})}(\xi \underline{r}) d\underline{r}\, d\underline{s}\, d\underline{t} \\ &= \int_{(\mathbb{R}^+)^k} \rho(\underline{t}) \left(\int_{\Delta_k(1/\xi)} \omega^+(\underline{r}-\underline{t}) \mathcal{F}^{(\underline{1})}(\xi \underline{r}) d\underline{r} \right)^2 d\underline{t}, \end{aligned}$$

where the limits of integration arise since the support of $\mathcal{F}(\underline{x})$ is the simplex $\Delta_k(1)$ and the support of the Dickman function $\rho(u)$ is $\mathbb{R}^+$. $\square$

Remark We remark that when $\xi = 1$, the above theorem gives back precisely Lemma 4.2 of [19] as a special case. Indeed, $S_1(1)$ is nothing but S_1, as the smoothing condition, is redundant when $\xi = 1$. Consider the final expression for $\mathscr{C}_{\mathcal{F},\mathcal{F}}(\xi)^{(1)}$ obtained from Lemma 4.5. If $\xi = 1$, then the support of the function $\mathcal{F}$ and hence the range of integration is the usual simplex

$$\Delta_k(1) := \{(t_1, \dots, t_k) \in [0, \infty)^k : t_1 + \cdots + t_k \le 1\}.$$

In particular while integrating over this simplex, for each t_j we have the bounds $0 \le t_j \le 1$. Recall that in this range, the Dickman function ρ is simply 1, while the Buchstab function ω is 0. Thus in the final expression of Lemma 4.5, putting $\xi = 1$ and $\mathcal{G}, \mathcal{H} = \mathcal{F}$, only the term involving the product $\boldsymbol{\delta}(\underline{s} - \underline{t})\boldsymbol{\delta}(\underline{r} - \underline{t})$ survives, giving us

$$\mathscr{C}_{\mathcal{F},\mathcal{F}}(1)^{(1)} = \int_{\Delta_k(1)} (\mathcal{F}^{(1)}(\underline{t}))^2 d\underline{t}.$$

This is nothing but the functional $\boldsymbol{\eta}(\mathcal{F})$ of Lemma 4.2 in [19].

5.2 Asymptotic Formula for $S_2(\xi)$

Let us recall the sum $S_2(\xi)$. We may write

$$S_2(\xi) = \sum_{m=1}^{k} S_2^{(m)}(\xi),$$

where

$$S_2^{(m)}(\xi) := \sum_{\substack{n \sim N \\ n \equiv b \pmod W}} \chi_{\mathbb{P}}(n + h_m) \Bigg(\sum_{\substack{d_j | n+h_j \forall j \\ P^+(\underline{d}) < R_1^{\xi}}} \lambda_{\underline{d}} \Bigg)^2$$

We proceed to derive an asymptotic formula for $S_2^{(m)}(\xi)$.

Theorem 5.3 *Choose $\Theta = 1/2 + 2\varpi - \delta$, with some small $\delta > 0$ and ϖ a positive number such that (1.3) holds (namely, ϖ satisfies $600\varpi + 180\eta < 7$, where $\eta = \Theta\xi$). With $\lambda_{\underline{d}}$'s chosen as in (3.2) and $R_1 = N^{\Theta/2-\delta}$, we have*

$$S_2^{(m)}(\xi) = (1 + o(1)) \frac{W^{k-1}}{\phi(W)^k} \frac{\pi(2N) - \pi(N)}{(\log R_1)^{k-1}} \xi^{k-1} \mathscr{C}_{\mathcal{F}_m,\mathcal{F}_m}(\xi)^{(1)},$$

where the function $\mathcal{F}_m$ acting on $(k-1)$-tuples is defined in terms of $\mathcal{F}$ by

$$\mathcal{F}_m(x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_k) = \mathcal{F}(x_1, \dots, x_{m-1}, 0, x_{m+1}, \dots, x_k)$$

and $\mathscr{C}_{\mathcal{F}_m,\mathcal{F}_m}(\xi)^{(1)}$ is given by

$$\int_{(\mathbb{R}^+)^{k-1}} \rho(\underline{t}) \left(\int_{\Delta_{k-1}(1/\xi)} \omega^+(\underline{r} - \underline{t}) \mathcal{F}_m^{(1)}(\xi \underline{r}) d\underline{r} \right)^2 d\underline{t}.$$

(Here, $\Delta_k(1/\xi)$ is as defined in Theorem 5.2.)

Proof Hypotheses H1, H2 and H3 hold as in the proof of Lemma 4.3 in [19] to give

$$X = \frac{\pi(2N) - \pi(N)}{\phi(W) \prod_{j \neq m} \phi(d_j)}$$

and $r_{\underline{d}} = E_{\mathbb{P}}(N, q, a)$, where a is some residue class co-prime to $q = W \prod_{j \neq m} d_j$ and $E_{\mathbb{P}}(x, q, a)$ is as defined by (1.1).

To check H4* with $Y = N$, it suffices to check that

$$\sum_{\substack{[\underline{d},\underline{e}] < N^{\Theta} \\ P^+([\underline{d},\underline{e}]) < N^{\Theta\xi}}} |E_{\mathbb{P}}(N, q, a)| \ll \frac{N}{(\log N)^A},$$

for any $A > 0$. As $W \ll \log\log N$, we see that there exists $\epsilon > 0$ small enough so that

$$\sum_{\substack{[\underline{d},\underline{e}] < N^{\Theta} \\ P^+([\underline{d},\underline{e}]) < N^{\Theta\xi}}} |E_{\mathbb{P}}(N, q, a)| \ll \sum_{\substack{q < N^{\Theta+\epsilon} \\ P^+(q) < N^{\Theta\xi}}} |E_{\mathbb{P}}(N, q, a)|$$

$$\ll \sum_{\substack{q < N^{\frac{1}{2}+2\varpi} \\ P^+(q) < N^{\eta}}} |E_{\mathbb{P}}(N, q, a)|$$

which is of the order of $N(\log N)^{-A}$, for any $A > 0$ by (1.3).

Keeping in mind the additional constraint $d_m = 1$ on tuples $\underline{d}$ as described in Lemma 4.3 of [19], which forces the mth component of the function $\mathcal{F}$ to be zero, the result follows as an application of Theorem 4.7. □

It can be observed as before that putting $\xi = 1$ yields Lemma 4.3 of [19]. Furthermore, in the expression obtained for $S_2^{(m)}(\xi)$ above, it is clear that the specific value of m has no role to play. Due to the symmetry of the integrals, we can write

$$\begin{aligned} S_2(\xi) &= \sum_{m=1}^{k} S_2^{(m)}(\xi) \qquad (5.2)\\ &= (1+o(1))\frac{W^{k-1}}{\phi(W)^k}\frac{\pi(2N)-\pi(N)}{(\log R_1)^{k-1}}\xi^{k-1}k\mathscr{C}_{\mathcal{F}_k,\mathcal{F}_k}(\xi)^{(\underline{1})}, \end{aligned}$$

with notation as in Theorem 5.3, setting $m = k$.

We remark that the densities depending on ξ in the expressions for $S_1(\xi)$ and $S_2(\xi)$ are strictly positive, as can be seen from the positive integrands and limits of integration. Thus, ξ- smoothing gives for the sums S_1 and S_2 the expected asymptotic formulas multiplied by some strictly positive density factor depending on ξ. This is indeed what one would expect, in the spirit of the classical Buchstab iteration procedure.

6 Some Material Toward Numerical Bounds

Choosing some Θ which is admissible in the derivation of the asymptotic formula for Q_1 as well as Q_2, one obtains the following after using the prime number theorem.

Theorem 6.1 *Choose $\Theta = 1/2 + 2\varpi$, with $\varpi > 0$ and $600\varpi + 180\xi < 7$. Then, with $\lambda_{\underline{d}}$'s chosen as in (3.2) in terms of $\mathcal{F}$, and $R = N^{\Theta/2-\delta}$, we have as $N \to \infty$,*

$$\begin{aligned} S(N,\nu) &:= S_2 - \nu S_1\\ &\sim \frac{W^{k-1}}{\phi(W)^k}\frac{N}{(\log R)^k}\xi^{k-1}\left(\left(\frac{\Theta}{2}-\delta\right)k\mathscr{C}_{\mathcal{F}_k,\mathcal{F}_k}(\xi)^{(\underline{1})} - \nu\xi\mathscr{C}_{\mathcal{F},\mathcal{F}}(\xi)^{(\underline{1})}\right). \end{aligned}$$

Combining Proposition 5.1 with the above result, we need

$$\nu < \left(\frac{\Theta}{2}-\delta\right)\frac{k}{\xi}\frac{\mathscr{C}_{\mathcal{F}_k,\mathcal{F}_k}(\xi)^{(\underline{1})}}{\mathscr{C}_{\mathcal{F},\mathcal{F}}(\xi)^{(\underline{1})}} \qquad (6.1)$$

This suggests that we should maximize the functional appearing above, prompting us to define

$$M_k(\xi) = \sup_{\mathcal{F}} k\frac{\mathscr{C}_{\mathcal{F}_k,\mathcal{F}_k}(\xi)^{(\underline{1})}}{\mathscr{C}_{\mathcal{F},\mathcal{F}}(\xi)^{(\underline{1})}},$$

where the supremum is taken over all symmetric smooth functions supported on $\Delta_k(1)$. This can be viewed as the 'ξ-smooth' analogue of the classical functional M_k encountered in [9, 12, 19]. We will express $M_k(\xi)$ in a more amenable form, which also makes it easier to check that $M_k(1)$ is indeed the functional M_k defined in (33) of [12].

We write $\mathcal{F}^{(\underline{1})}(\underline{x})$ as $G(\underline{x})$. Then G is a symmetric smooth function supported on the simplex $\Delta_k(1)$. Expressing G as

$$G(\underline{x}) = \frac{\partial}{\partial x_m}\left(\frac{\partial^{k-1}\mathcal{F}(\underline{t})}{\partial x_1 \ldots \partial x_{m-1}\partial x_m \ldots \partial x_k}\right),$$

we see from the fundamental theorem of calculus that the function $\mathcal{F}_m^{(1)}$ that appears in Theorem 5.3 is simply the anti-derivative of G with respect to the mth component, evaluated at $x_m = 0$. It is also clear that the anti-derivative of G with respect to the mth component has the same support as $\mathcal{F}$. Hence, we can write

$$\int_0^\infty G(\underline{x})dx_m = -\left(\frac{\partial^{k-1}\mathcal{F}(\underline{x})}{\partial x_1 \ldots \partial x_{m-1}\partial x_m \ldots \partial x_k}\right)\Bigg|_{x_m=0} = -\mathcal{F}_m^{(1)}(\underline{x}).$$

In particular, $\mathcal{F}_k^{(1)}$ *evaluated* at $\xi\underline{r}$, namely $\mathcal{F}_k^{(1)}(\xi\underline{r})$ equals

$$-\int_0^\infty G(\xi r_1, \ldots, \xi r_{k-1}, x_k)dx_k.$$

This allows us to recast M_k in terms of $G(\underline{x})$ as

$$M_k(\xi) = \sup_G \frac{kJ_k(G,\xi)}{I(G,\xi)}, \tag{6.2}$$

where the supremum is taken over all symmetric smooth functions supported on $\Delta_k(1)$ and the functionals $J_k(G,\xi)$ and $I(G,\xi)$ are defined as follows.

$$J_k(G,\xi) := \int_{(\mathbb{R}^+)^{k-1}} \rho(\underline{t}) \left(\int_{\Delta_{k-1}(1/\xi)} \omega^+(\underline{r}-\underline{t}) \left(\int_0^\infty G(\xi r_1, \ldots, \xi r_{k-1}, x)dx\right) d\underline{r}\right)^2 d\underline{t}, \tag{6.3}$$

and

$$I(G,\xi) := \int_{(\mathbb{R}^+)^k} \rho(\underline{t}) \left(\int_{\Delta_k(1/\xi)} \omega^+(\underline{r}-\underline{t}) G(\xi\underline{r})d\underline{r}\right)^2 d\underline{t}. \tag{6.4}$$

We would like estimates for the new integrals $J(G,\xi)$ and $I(G,\xi)$ in terms of the functionals $J(G,1)$ and $I(G,1)$ that appear for the higher-rank sieve without smoothing. Recall from [19] that

$$J(G,1) = \int_{\Delta_{k-1}(1)} \left(\int_0^\infty G(\underline{t})dt_k\right)^2 dt_1 \ldots dt_{k-1}, \tag{6.5}$$

and

$$I(G,1) = \int_{\Delta_k(1)} G(\underline{t})^2 d\underline{t}. \tag{6.6}$$

Let us define the functional

$$I_k(F, \xi) = \int_{(\mathbb{R}^+)^k} \rho(\underline{t}) \left(\int_{(\mathbb{R}^+)^k} \omega^+(\underline{r} - \underline{t}) F(\xi \underline{r}) d\underline{r} \right)^2 d\underline{t}$$

The problem thus reduces finding upper and lower bounds for this functional in terms of

$$I_k(F) = \int_{(\mathbb{R}^+)^k} F(\underline{t})^2 d\underline{t}.$$

Lower bound. Using the bound $\omega^+(u) = \omega(u) + \delta(u) \geq \delta(u)$ for any $u \in \mathbb{R}^+$, we can write

$$I_k(F, \xi) \geq \int_{(\mathbb{R}^+)^k} \rho(\underline{t}) F(\xi \underline{t})^2 d\underline{t} = \xi^{-k} \int_{(\mathbb{R}^+)^k} \rho(t_1/\xi) \ldots \rho(t_k/\xi) F(\underline{t})^2 d\underline{t}$$

Upper bound. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left(\int\limits_{(\mathbb{R}^+)^k} \omega^+(\underline{r} - \underline{t}) F(\xi \underline{r}) d\underline{r} \right)^2 &\leq \int\limits_{(\mathbb{R}^+)^k} \omega^+(\underline{r} - \underline{t})^2 d\underline{r} \int\limits_{(\mathbb{R}^+)^k} F(\xi \underline{r})^2 d\underline{r} \\ &= \xi^{-k} I(F) \int\limits_{(\mathbb{R}^+)^k} \omega^+(\underline{r} - \underline{t})^2 d\underline{r} \end{aligned}$$

This gives

$$I_k(F, \xi) \leq \xi^{-k} I(F) \left(\int_{(\mathbb{R}^+)^k} \rho(\underline{t}) \int_{(\mathbb{R}^+)^k} \omega^+(\underline{r} - \underline{t})^2 d\underline{r} d\underline{t} \right).$$

Bounding the integrals that arise in the lower and upper bounds above needs some work and we defer this to a future paper. We expect that effective bounds for these integrals should yield a general result involving a remainder term that would encompass contributions both with and without smoothing. More precisely, we should be able to capture contributions from moduli below Y^θ and also from moduli up to Y^Θ with prime factors below Y^ξ. Implementing these ideas and obtaining numerical improvements would entail the use of variational techniques as well as the following identities involving the Dickman and Buchstab functions.

Recall that the Dickman function is supported on $\mathbb{R}^+$. Broadhurst [1] gives a closed form for the Dickman function in terms of polylogarithms, in certain ranges. We state below the closed form in the range 0–2.

Proposition 6.2 *The Dickman function in the domain* $[0, 2]$ *is given by*

$$\rho(u) = \begin{cases} 1 & \textit{if } 0 \leq u \leq 1 \\ 1 - \log u & \textit{if } 1 \leq u \leq 2 \end{cases}$$

Proposition 6.3 *The convolution $\rho * \rho$ in the domain* $[0, 2]$ *is given by,*

$$\rho_2(u) = \begin{cases} u & \textit{if } 0 \leq u \leq 1 \\ 3u - 2u \log u - 2 & \textit{if } 1 \leq u \leq 2 \end{cases}$$

Proof The support of ρ gives

$$\rho_2(u) = \int_0^u \rho(t)\rho(u-t)dt.$$

If $0 \leq u \leq 1$, then the integrand is simply 1, giving the desired answer. For $1 \leq u \leq 2$, we write the above integral as

$$\int_0^{u-1} \rho(t)\rho(u-t)dt + \int_{u-1}^1 \rho(t)\rho(u-t)dt + \int_1^u \rho(t)\rho(u-t)dt.$$

Let us consider the first integral. The limits of integration imply that $0 \leq t \leq 1$ and $1 \leq u - t \leq 2$, giving that this integral is

$$\int_0^{u-1} (1 - \log(u-t))dt.$$

Similarly, the second integral is simply

$$\int_{u-1}^1 dt,$$

while the third is given by

$$\int_1^u (1 - \log t)dt.$$

Evaluating these integrals gives the desired expression for $\rho_2(u)$. $\square$

Recall that the Buchstab function $\omega(u)$ is supported on $u \geq 1$.

Proposition 6.4 *The Buchstab function in the domain* $[0, 2]$ *is given by*

$$\omega(u) = \begin{cases} 0 & \textit{if } 0 \leq u \leq 1 \\ 1/u & \textit{if } 1 \leq u \leq 2. \end{cases}$$

The actual implementation of these results we reserve for a future date.

7 Concluding Remarks

We believe that this implementation is just the beginning of a larger program. We endeavor to explore further applications of this theory to other classical questions of number theory.

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