Functiones et Approximatio
*** (202*), 1–15*
doi: 10.7169/facm/2048

ON THE HARDY-LITTLEWOOD PRIME TUPLES CONJECTURE AND HIGHER CONVOLUTIONS OF RAMANUJAN SUMS

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Abstract: We introduce the study of triple convolution Ramanujan sums and apply it to give a heuristic derivation of the Hardy-Littlewood conjecture on prime 3-tuples without using the circle method. We also estimate the triple convolution of the Jordan totient function using Ramanujan sums.

Keywords: Hardy-Littlewood conjecture, twin primes, prime tuples, Ramanujan sums.

1. Introduction

In 1922, Hardy and Littlewood [7] generalized the celebrated twin prime conjecture and formulated what is now called the prime k-tuple conjecture, which is the following. Suppose that $d_1, ..., d_k$ are distinct integers, and let $b(p)$ be the number of distinct residue classes (mod p) represented by the d_i. If $b(p) < p$ for every prime p, the prime k-tuple conjecture asserts that the number of $n \leqslant x$ such that **all** the k numbers $n + d_i$ are prime for $1 \leqslant i \leqslant k$ is asymptotic to

$$\mathfrak{S}(d_1, ..., d_k) \frac{x}{(\log x)^k},$$

where

$$\mathfrak{S}(d_1, ..., d_k) = \prod_p \left(1 - \frac{b(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-k}, \tag{1}$$

and the product is over all primes p. Hardy and Littlewood formulated their conjecture using the intuition provided by the circle method and essentially ignoring the contribution from the so-called minor arcs emanating from the technique and focusing only on the major arcs. Though the idea is simple, the analysis of the

Research of the first author is supported by the Science and Engineering Research Board, Department of Science and Technology, Government of India under grant SB/S2/RJN-053/2018. Research of the third author is partially supported by an NSERC Discovery grant.

2020 Mathematics Subject Classification: primary: 11A05; secondary: 11L03, 11N37 major arcs could be more complex and delicate. In 1999, Gadiyar and Padma [6] discovered a simple heuristic to derive the case $k = 2$ (or the generalized twin prime conjecture) using a simple orthogonality principle for Ramanujan sums originally discovered by Carmichael [2]. Though their approach does not lead to a solution to the twin prime problem, it does open up other lines of productive investigations related to the theory of Ramanujan sums.

First, let us recall the Ramanujan sum:

$$c_q(n) = \sum_{(a,q)=1} e^{2\pi i a n/q},$$

originally defined by Ramanujan in [13] in 1918. We refer to [8, 11] and [13] for the general properties of Ramanujan sums. Ramanujan [13] obtained the trigonometric series representations of normalized arithmetical functions of n in the form of an infinite series

$$\sum_{q=1}^{\infty} a_q c_q(n). \tag{2}$$

In particular, the series expansion of some well-known functions is as follows:

$$\tau(n) = -\sum_{q\geqslant 1} \frac{\log q}{q} c_q(n),$$

$$\frac{\phi(n)}{n} = \frac{6}{\pi^2} \sum_{q\geqslant 1} \left(\frac{\mu(q)}{q^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{-1} \right) c_q(n),$$

where $\tau(n)$ and $\phi(n)$ denote the divisor and Euler totient function, respectively. These series are now called the Ramanujan-Fourier series. The existence and convergence properties of these series are subjects that generate significant interest. A comprehensive review paper by Lucht [9] discusses the Ramanujan expansion of arithmetical functions. Moreover, notable monographs in this direction include the works of [15] and [16].

Indeed, the theory of the Ramanujan-Fourier series for arithmetical functions that is nascent in their work can be applied successfully to other problems, and the reader can find a beginning of this theory in [5]. Murty and Saha [12] adopted the method in [5] to derive an asymptotic formula with explicit error terms for the shifted 2-convolution sums of arithmetical functions with absolutely convergent Ramanujan expansion. Subsequently, they and Coppola [3, 4, 14] extended the results with a weaker hypothesis.

In this note, we generalize the method of Gadiyar and Padma to derive a heuristic formulation of the general k-tuple conjecture. This leads to a cognate area of research for higher convolutions of Ramanujan sums. We also estimate the triple convolution of the Jordan totient function using the convolution of Ramanujan sums in Section 9.

2. Preliminaries

We first review the orthogonality principle for Ramanujan sums alluded to in the introduction and indicate how to generalize this to "triple convolutions". This will give us insight on how to generalize it even further.

Theorem 2.1 (Carmichael, 1932). *We have*

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x}c_r(n)c_s(n+h)=c_r(h),$$

if $r=s$ *and zero otherwise.*

Proof. We have

$$\sum_{n\leqslant x}c_r(n)c_s(n+h)=\sum_{(a,r)=1}\sum_{(b,s)=1}e^{2\pi ihb/s}\sum_{n\leqslant x}e^{2\pi in(a/r+b/s)}.$$

The innermost sum is bounded unless $a/r+b/s$ is an integer m (say). But then

$$as+br=mrs$$

forces $r=s$ because $(a,r)=(b,s)=1$. The result is now immediate. ∎

It is this argument we will now consider for "triple convolutions". More precisely, let us consider

$$\sum_{n\leqslant x}c_r(n)c_s(n+h)c_t(n+j)$$
$$=\sum_{(a,r)=1}\sum_{(b,s)=1}e^{2\pi ihb/s}\sum_{(c,t)=1}e^{2\pi ijc/t}\sum_{n\leqslant x}e^{2\pi in(a/r+b/s+c/t)}. \quad (3)$$

Let us first look at the case $r=s=t$. In that case, the innermost sum is bounded unless $a+b+c=0 \pmod r$ **and** $b+c$ is coprime to r. Therefore, in this case, we have

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x}c_r(n)c_r(n+h)c_r(n+j)=\sum_{\substack{(b,r)=(c,r)=1\\(b+c,r)=1}}e^{2\pi i(hb+jc)/r}.$$

This motivates the study of a two-variable variant of the Ramanujan sum:

$$\mathscr{K}_r(h,j):=\sum_{\substack{(b,r)=(c,r)=1\\(b+c,r)=1}}e^{2\pi i(hb+jc)/r}, \quad (4)$$

a function worthy of independent study. The case when $r=p$ is prime is easily evaluated as follows. The condition $b+c$ coprime to p is always satisfied unless $c=p-b$, so we have

$$\mathscr{K}_p(h,j)=c_p(h)c_p(j)-c_p(h-j).$$

Since $c_p(n) = c_p(-n)$, this is a symmetric function of h and j as it should be. It should be possible to derive similar formulas in the general case.

Let us consider the case when r, s, t are not all equal. The innermost sum in (3) is bounded unless $a/r + b/s + c/t$ is an integer. This means that

$$\frac{a}{r} + \frac{bt + cs}{st}$$

is an integer. We will first study the case that the three numbers r, s, t are mutually coprime. Then

$$\frac{a}{r} + \frac{bt + cs}{st}$$

being an integer implies that $r = st$ from the earlier discussion because then $bt+cs$ is coprime to st. Similarly, $s = rt$ and $t = rs$ from which we conclude $r = s = t = 1$. Thus, we have:

Theorem 2.2. *If r, s, t are mutually coprime, then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} c_r(n)c_s(n+h)c_t(n+j) = 0$$

unless $r = s = t = 1$, in which case the limit is 1.

This result holds in greater generality provided the p-adic valuations of r, s, t are distinct for some prime p. We examine this in the later sections.

3. Multiplicativity of $\mathscr{K}_r(h, j)$

In this section, we will prove that $\mathscr{K}_r(h, j)$ is a multiplicative function of r.

Lemma 3.1. *Let $(m, n) = 1$. Then $(b, m) = (c, n) = 1$, if and only if $(bn + cm, mn) = 1$.*

Proof. Let $n = \prod_p p^{v_p(n)}$ and $m = \prod_q q^{v_q(m)}$ be the prime factorization of n and m, where p's and q's are distinct since $(m, n) = 1$. Suppose that $(b, m) = (c, n) = 1$, and $(bn + cm, mn) = d$. Then, we have $d = \prod_{p,q} p^{v_p(d)} q^{v_q(d)}$ where $0 \leqslant v_p(d) \leqslant v_p(n)$ and $0 \leqslant v_q(d) \leqslant v_q(m)$. For some p, one can write $p^{v_p(d)}|bn + cm$ and so $p^{v_p(d)}|c$ which implies $p^{v_p(d)}|(c, n)$. It is possible only if $v_p(d) = 0$. Similarly, we can show that $v_q(d) = 0$, that is $d = 1$. Conversely, assume that $(bn+cm, mn) = 1$ and $(b, m) = d$, then $d|b$ and $d|m$. This implies $d|bn+cm$ and $d|mn$. Therefore, we have $(bn+cm, mn) = d$ which gives $d = 1$. Similarly, one can prove $(c, n) = 1$. ■

Remark 3.2. Let $(m, n) = 1$ and $(b, m) = (c, n) = 1$. If $1 \leqslant b \leqslant m$ and $1 \leqslant c \leqslant n$, then we have $\phi(mn)$ choices for $bn + cm$. The lemma shows that each of these numbers will be coprime to mn.

Remark 3.3. For $(m, n) = 1$, one can represent every coprime residue class of mn as $bn+cm$ where b and c belong to a coprime residue class of m and n, respectively.

Theorem 3.4. *If* $(m, n) = 1$, *then*

$$\mathscr{K}_{mn}(h,j) = \mathscr{K}_m(h,j)\mathscr{K}_n(h,j). \tag{5}$$

That is, $\mathscr{K}_r(h,j)$ *is a multiplicative function of* r.

Proof. From (4), we have

$$\mathscr{K}_{mn}(h,j) = \sum_{\substack{(b,mn)=(c,mn)=1\\(b+c,mn)=1}} e^{2\pi i(hb+jc)/mn},$$

Next, from Remark 3.2, Remark 3.3 and Lemma 3.1, we see that for every b and c, we can find b_1, b_2, c_1, c_2 such that $b = b_1n + b_2m$, $c = c_1n + c_2m$, and $(b_1, m) = (c_1, m) = (b_2, n) = (c_2, n) = 1$. This allows us to decompose the sum on the right-hand side as a product of two sums corresponding to the factors on the right-hand side of (5). This completes the proof. ■

4. Fractions revisited

In analogy with rational functions, it may be useful to think of a fraction as a rational number with "poles" at the prime divisors of the denominator. This motivates the following "partial fraction expansion" of a rational number.

Lemma 4.1. *Let*

$$q = \prod_p p^{v_p(q)}$$

be the unique factorization of q *as a product of prime powers. If* $1 \leqslant a < q$, *we can write*

$$\frac{a}{q} = \sum_{p|q} c_p p^{-v_p(q)} \pmod 1,$$

where $0 \leqslant c_p < p^{v_p(q)}$ *and this representation is unique.*

Proof. We write $Q_p = q/p^{v_p(q)}$. Since the greatest common divisor of all the Q_p's is 1, we have, by the Euclidean algorithm, integers x_p such that

$$a = \sum_{p|q} x_p Q_p$$

and dividing by q gives

$$\frac{a}{q} = \sum_{p|q} x_p p^{-v_p(q)}.$$

Now we choose c_p satisfying $0 \leqslant c_p < p^{v_p(q)}$ such that $c_p \equiv x_p \pmod{p^{v_p(q)}}$ which completes the proof. ■

This lemma also explains why we could prove Theorem 2.2 when r, s, t are mutually coprime relatively quickly. In that case, the "poles" of each fraction do not interfere with each other, and the only way the "poles" can all disappear is if $r = s = t = 1$. That is, there are no "poles" to start with.

Since we will be working (mod 1), we can drop the condition $p|q$ in the summation of the lemma. Thus, we can write any fraction as

$$\frac{a}{q} = \sum_p x_p(a) p^{-v_p(q)} \pmod 1$$

and speak of the "order of the pole" at p as $v_p(q)$. Now, the condition of the innermost sum of (3) is that for every prime p,

$$x_p(a)p^{-v_p(r)} + x_p(b)p^{-v_p(s)} + x_p(c)p^{-v_p(t)} = 0 \pmod{1}. \tag{6}$$

If $v_p(r), v_p(s), v_p(t)$ are all distinct for some prime p, this condition is never satisfied and so we get:

Theorem 4.2. *If for some prime p, $v_p(r), v_p(s), v_p(t)$ are all distinct, then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} c_r(n)c_s(n+h)c_t(n+j) = 0.$$

In other words, the only way we can have a possible non-zero limit is when, for every prime p, at least two of $v_p(r), v_p(s), v_p(t)$ are equal.

5. The squarefree case

When r, s, t are all squarefree, then the set of values of $v_p(r), v_p(s), v_p(t)$ can only be 0 or 1. Viewing this as a triple, $(v_p(r), v_p(s), v_p(t))$ for which (6) holds, the only possibilities are $(1, 1, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1)$. In any case, we can define

$$\mathscr{K}_{r,s,t}(k, h, j) := \sum_{\substack{(a,r)=1,(b,s)=1,(c,t)=1 \\ \frac{a}{r}+\frac{b}{s}+\frac{c}{t}\in\mathbb{Z}}} \exp\left(2\pi i \left(\frac{ak}{r} + \frac{bh}{s} + \frac{cj}{t}\right)\right), \tag{7}$$

and determine the conditions when this function is non-zero. This determination should be sufficient to extend the heuristic mentioned in the introduction to the Hardy-Littlewood 3-triple conjecture in view of Hardy's formula

$$\frac{\phi(n)\Lambda(n)}{n} = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n), \tag{8}$$

where the summation is over squarefree q. In this way, we seem to be getting multivariable generalizations of the classical Ramanujan sums that are new and worthy of independent study.

6. Carmichael's theorem revisited

It will be useful to re-derive Carmichael's theorem through this new optic. We consider

$$\sum_{n\leqslant x} c_r(n+h)c_s(n+j) = \sum_{(a,r)=1}\sum_{(b,s)=1} e^{2\pi iha/r}e^{2\pi ibj/s}\sum_{n\leqslant x} e^{2\pi in(a/r+b/s)}.$$

Therefore,

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x} c_r(n+h)c_s(n+j) = \sum_{\substack{(a,r)=1,(b,s)=1\\ \frac{a}{r}+\frac{b}{s}\in\mathbb{Z}}} e^{2\pi iha/r}e^{2\pi ibj/s}$$

Using our theory of fractions, it is now clear the condition

$$\frac{a}{r}+\frac{b}{s}\in\mathbb{Z}$$

can hold if and only if $r=s$ and $a=-b$ which gives the limit to be $c_r(h-j)$. We state this for future reference.

Lemma 6.1.

$$\sum_{\substack{(a,r)=1,(b,s)=1\\ \frac{a}{r}+\frac{b}{s}\in\mathbb{Z}}} e^{2\pi iha/r}e^{2\pi ibj/s} = c_r(h-j)\delta_{r,s}$$

where $\delta_{r,s}$ is the Kronecker delta function.

This calculation motivates a re-examination of (7).

7. The evaluation of $\mathscr{K}_{r,s,t}(k,h,j)$

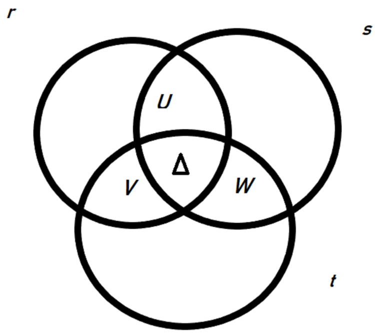
We will confine our attention to the case that r,s,t are all squarefree. As noted above, the integrality condition forces either all $v_p(r), v_p(s), v_p(t)$ are equal to 1 or exactly two are equal to 1 for every prime divisor p of rst. This suggests we write $\Delta=\gcd(r,s,t)$ and $U=\gcd(r,s)/\Delta$, $V=\gcd(r,t)/\Delta$ and $W=\gcd(s,t)/\Delta$. The Venn diagrams below may help the reader visualize the situation. The circles represent the set of prime divisors of r,s,t, respectively. Applying our theory of fractions, Figure 1 shows that r,s,t cannot have prime factors outside of Δ,U,V,W, and upon close analysis (Theorem 7.1), our diagram shrinks to Figure 2.

Indeed, using our theory of fractions (along with the Chinese remainder theorem), we can separate the sum (7) into parts corresponding to Δ, U, V and W. We then find:

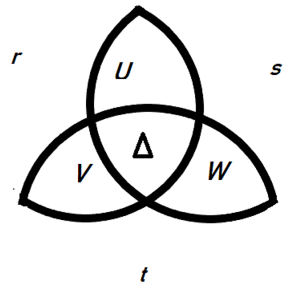
Theorem 7.1. *Let r,s,t be squarefree with $(a,r)=(b,s)=(c,t)=1$. Then*

$$\frac{a}{r}+\frac{b}{s}+\frac{c}{t}=0 \pmod 1 \tag{9}$$

implies $r=\Delta UV$, $s=\Delta UW$, and $t=\Delta VW$ with Δ,U,V,W all mutually coprime.



(a) Fig. 1



(b) Fig. 2

Proof. As noted earlier, using our theory of fractions, we see that if r has a prime divisor not dividing ΔUV, it gives rise to a "pole" on the left-hand side of (9). The same argument applies to s and t. ■

Theorem 7.2. *Let r, s, t be squarefree with $(a, r) = (b, s) = (c, t) = 1$. Suppose (9) holds. Then,*

$$\mathscr{K}_{r,s,t}(k, h, j) = \mathscr{K}_{\Delta}(h - k, j - k)c_U(h - j)c_V(j - k)c_W(h - k),$$

where the c_U, c_V, c_W are Ramanujan sums and $\mathscr{K}_r$ is given by (4).

Proof. We have already noted that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} c_r(n + k)c_s(n + h)c_t(n + j) = \mathscr{K}_{r,s,t}(k, h, j).$$

This limit is zero unless the conditions of Theorem 7.1 are met. Since r is squarefree and equal to ΔUV by the previous theorem, we see by the multiplicativity of the Ramanujan sum that

$$c_r(n) = c_{\Delta}(n)c_U(n)c_V(n).$$

Similarly, $c_s(n) = c_{\Delta}(n)c_U(n)c_W(n)$ and $c_t(n) = c_{\Delta}(n)c_V(n)c_W(n)$. The integrality condition, along with our earlier results, completes the proof. ■

We can now supply the heuristic argument for the Hardy-Littlewood prime 3-tuple conjecture, which we do in the next section.

8. A heuristic derivation of the Hardy-Littlewood 3-tuple conjecture

By partial summation, the Hardy-Littlewood 3-tuple conjecture is equivalent to

$$\sum_{n\leqslant x} \Lambda(n)\Lambda(n + h)\Lambda(n + j) \sim x \prod_{p} \left(1 - \frac{b(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-3}, \tag{10}$$

where $b(p)$ is the size of the image of $\{0, h, j\} \pmod p$. The product on the right-hand side of (10) can be rewritten as

$$\prod_p \frac{p^2(p - b(p))}{(p-1)^3}.$$

Our objective is to present a heuristic proof of (10) by employing the convolution of Ramanujan sums. First, we observe that

$$\Upsilon := \sum_{n \leqslant x} \Lambda(n)\Lambda(n+h)\Lambda(n+j) \sim \sum_{n \leqslant x} \frac{\phi(n)}{n}\Lambda(n)\frac{\phi(n+h)}{n+h}\Lambda(n+h)\frac{\phi(n+j)}{n+j}\Lambda(n+j).$$

Next, inserting the Ramanujan-Fourier series of Hardy from equation (8), we therefore have using Theorem 7.1

$$\frac{\Upsilon}{x} \sim \sum_{r,s,t} \frac{\mu(r)\mu(s)\mu(t)}{\phi(r)\phi(s)\phi(t)} \mathscr{K}_{r,s,t}(0, h, j).$$

Using Theorems 7.2 and 7.1, the right hand side can be written as

$$\sum_{\Delta, U, V, W} \mu^2(\Delta UVW) \frac{\mu(\Delta)^3}{\phi(\Delta)^3} \frac{\mu^2(U)\mu^2(V)\mu^2(W)}{\phi(U)^2\phi(V)^2\phi(W)^2} \mathscr{K}_\Delta(h, j) c_U(h-j) c_V(j) c_W(h),$$

where the term $\mu^2(\Delta UVW)$ ensures that Δ, U, V, W are all mutually coprime as required by Theorem 7.1. Writing

$$f_d(h, j); = \sum_{UVW = d} c_U(h-j)c_V(j)c_W(h), \tag{11}$$

we can rewrite our sum in a simpler way as

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{\phi(m)^2} \sum_{\Delta d = m} \frac{\mu(d)}{\phi(\Delta)} \mathscr{K}_\Delta(h, j) f_d(h, j).$$

The inner sum is a multiplicative function of m. For $m = p$ a prime, we have that the inner sum is

$$\frac{\mathscr{K}_p(h, j)}{p-1} - \mathscr{K}_1(h, j) f_p(h, j).$$

Thus the sum can be written as a product:

$$\prod_p \left(1 - \frac{1}{(p-1)^2}\left\{\frac{c_p(h)c_p(j) - c_p(h-j)}{p-1} - (c_p(h-j) + c_p(h) + c_p(j))\right\}\right).$$

Let us first consider the case $b(p) = 3$, that is when $0, h, j$ are all distinct mod p. In particular, h, j and $h - j$ are all coprime to p. Using our formulas for each of these terms, we find the Euler factor is

$$\frac{p^2(p-3)}{(p-1)^3}$$

as predicted. If $b(p) = 1$, then p divides both h and j. Again, the Euler factor turns out to be

$$\frac{p^2}{(p-1)^2}$$

as predicted by the conjecture. Finally, when $b(p) = 2$, then possible cases are p divides h and j is coprime to p or p divides j and h is coprime to p or both h and j are coprime to p and p divides $h - j$. In all cases, the Euler factor is

$$\frac{p^2(p-2)}{(p-1)^3}.$$

Thus, in all cases, the Euler factor is consistent with that predicted by the Hardy-Littlewood conjecture.

9. Triple convolution of the Jordan totient function

For a positive integer α, the Jordan totient function is defined as

$$\phi_\alpha(n) := n^\alpha \prod_{p|n} \left(1 - \frac{1}{p^\alpha}\right).$$

When $\alpha = 1$, it coincides with the Euler totient function. In [5], the authors obtained the shifted 2-convolution of the Euler totient function employing the Ramanujan-Fourier series and the orthogonality property of Ramanujan sums. Later, Balasubramanian and Giri [1] derived an asymptotic formula for the weighted shifted 2-convolution of a class of functions sufficiently close to the constant function $f(x) = 1$ for all x using the information about the average value of weight function in arithmetic progressions. Consequently, they also obtained the shifted 2-convolution of the Euler and Jordan totient functions. Alternatively, one can obtain the shifted 2-convolution of the Jordan totient function using the orthogonality property of Ramanujan sums. In this section, we estimate the triple convolution of the Jordan totient function.

Theorem 9.1. *For positive integers α, h, and j, we have as x tends to infinity,*

$$\sum_{n \leqslant x} \frac{\phi_\alpha(n)}{n^\alpha} \frac{\phi_\alpha(n+h)}{(n+h)^\alpha} \frac{\phi_\alpha(n+j)}{(n+j)^\alpha}$$
$$\sim x \prod_p \left(1 - \frac{3}{p^{\alpha+1}}\right) \prod_{\substack{p \\ b(p)=2}} \left(1 + \frac{1}{p^{2\alpha+1} - 3p^\alpha}\right) \prod_{\substack{p \\ b(p)=3}} \left(1 + \frac{3p^\alpha - 1}{p^{3\alpha+1} - 3p^{2\alpha}}\right),$$

where $b(p)$ denotes the number of distinct residue classes (mod p) represented by 0, h and j.

When $\alpha = 1$, Theorem 9.1 provides the convolution formula for the Euler function. Mirsky has also explored the convolution of the Euler function [10] through a distinct methodology. Mirsky examined a specific class of functions and employed their properties to express the k-convolution of a function as a simultaneous solution to k congruence equations. It is worth noting that Mirsky's method can also be employed to derive Theorem 9.1.

Proof. Ramanujan obtained the series expansion of Jordan totient function in terms of Ramanujan sums given by:

$$\frac{\phi_\alpha(n)}{n^\alpha} = \frac{1}{\zeta(\alpha+1)} \sum_{r=1}^{\infty} \frac{\mu(r)}{\phi_{\alpha+1}(r)} c_r(n). \tag{12}$$

The above Ramanujan-Fourier series is absolutely convergent. This implies

$$\begin{aligned}\sum_{n\leqslant x} \frac{\phi_\alpha(n)}{n^\alpha}\frac{\phi_\alpha(n+h)}{(n+h)^\alpha}\frac{\phi_\alpha(n+j)}{(n+j)^\alpha} \\ = \frac{1}{\zeta^3(\alpha+1)} \sum_{r,s,t=1}^{\infty} \frac{\mu(r)\mu(s)\mu(t)}{\phi_{\alpha+1}(r)\phi_{\alpha+1}(s)\phi_{\alpha+1}(t)} \sum_{n\leqslant x} c_r(n)c_s(n+h)c_t(n+j).\end{aligned}$$

We define

$$\Gamma := \sum_{n\leqslant x} \frac{\phi_\alpha(n)}{n^\alpha}\frac{\phi_\alpha(n+h)}{(n+h)^\alpha}\frac{\phi_\alpha(n+j)}{(n+j)^\alpha}.$$

Next, using Theorem 7.1, we have

$$\frac{\Gamma}{x} \sim \frac{1}{\zeta^3(\alpha+1)} \sum_{r,s,t=1}^{\infty} \frac{\mu(r)\mu(s)\mu(t)}{\phi_{\alpha+1}(r)\phi_{\alpha+1}(s)\phi_{\alpha+1}(t)} \mathscr{K}_{r,s,t}(0,h,j).$$

Therefore, using Theorems 7.2 and 7.1, the right hand side can be written as

$$\begin{aligned}\frac{1}{\zeta^3(\alpha+1)} \sum_{\Delta,U,V,W} \mu^2(\Delta UVW) \\ \times \frac{\mu^3(\Delta)\mu^2(U)\mu^2(V)\mu^2(W)}{\phi^3_{\alpha+1}(\Delta)\phi^2_{\alpha+1}(U)\phi^2_{\alpha+1}(V)\phi^2_{\alpha+1}(W)} \mathscr{K}_\Delta(h,j)c_U(h-j)c_V(j)c_W(h),\end{aligned}$$

where the term $\mu^2(\Delta UVW)$ ensures that Δ, U, V, W are all mutually coprime as required by Theorem 7.1. From (11), we can rewrite our sum in a simpler way as

$$\frac{1}{\zeta^3(\alpha+1)} \sum_{m=1}^{\infty} \frac{\mu(m)}{\phi^2_{\alpha+1}(m)} \sum_{\Delta d=m} \frac{\mu(d)}{\phi_{\alpha+1}(\Delta)} \mathscr{K}_\Delta(h,j) f_d(h,j).$$

The inner sum is a multiplicative function of m. For $m = p$ a prime, we have that the inner sum is

$$\frac{\mathscr{K}_p(h,j)}{p^{\alpha+1}-1} - \mathscr{K}_1(h,j) f_p(h,j).$$

Thus the sum can be written as a product:

$$\frac{1}{\zeta^3(\alpha+1)} \times \prod_p \left(1 - \frac{1}{(p^{\alpha+1}-1)^2}\left\{\frac{c_p(h)c_p(j)-c_p(h-j)}{p^{\alpha+1}-1} - (c_p(h-j)+c_p(h)+c_p(j))\right\}\right).$$

Let us first consider the case $b(p)=3$, that is, when 0, h, j are all distinct mod p. In particular, h, j and $h-j$ are all coprime to p. Using our formulas for each of these terms, we find the Euler factor is

$$\frac{1}{\zeta^3(\alpha+1)} \prod_p \left(1 - \frac{1}{(p^{\alpha+1}-1)^2}\left\{\frac{2}{p^{\alpha+1}-1} + 3\right\}\right).$$

If $b(p)=1$, then p divides both h and j. Again, the Euler factor turns out to be

$$\frac{1}{\zeta^3(\alpha+1)} \prod_p \left(1 - \frac{1}{(p^{\alpha+1}-1)^2}\left\{\frac{(p-1)(p-2)}{p^{\alpha+1}-1} - 3(p-1)\right\}\right).$$

Finally, when $b(p)=2$, then possible cases are p divides h and j is coprime to p or p divides j and h is coprime to p or both h and j are coprime to p and p divides $h-j$. In all cases, the Euler factor is

$$\frac{1}{\zeta^3(\alpha+1)} \prod_p \left(1 - \frac{1}{(p^{\alpha+1}-1)^2}\left\{\frac{(2-p)}{p^{\alpha+1}-1} - (p-3)\right\}\right).$$

By combining the above cases, we have

$$\Gamma \sim x \prod_p \left(1 - \frac{3}{p^{\alpha+1}}\right) \times \prod_{\substack{p \\ b(p)=2}} \left(1 + \frac{1}{p^{2\alpha+1}-3p^{\alpha}}\right) \prod_{\substack{p \\ b(p)=3}} \left(1 + \frac{3p^{\alpha}-1}{p^{3\alpha+1}-3p^{2\alpha}}\right). \qquad \blacksquare$$

Using Theorem 9.1, we conclude that

Corollary 9.2.

$$\sum_{n\leqslant x} \phi_\alpha(n)\phi_\alpha(n+h)\phi_\alpha(n+j) \sim \frac{x^{3\alpha+1}}{3\alpha+1} \prod_p \left(1 - \frac{3}{p^{\alpha+1}}\right) \prod_{\substack{p \\ b(p)=2}} \left(1 + \frac{1}{p^{2\alpha+1}-3p^{\alpha}}\right) \prod_{\substack{p \\ b(p)=3}} \left(1 + \frac{3p^{\alpha}-1}{p^{3\alpha+1}-3p^{2\alpha}}\right),$$

where $b(p)$ denotes the number of distinct residue classes (mod p) represented by 0, h and j.

Proof. We can write

$$\sum_{n\leqslant x}\phi_\alpha(n)\phi_\alpha(n+h)\phi_\alpha(n+j)$$
$$=\sum_{n\leqslant x}\frac{\phi_\alpha(n)}{n^\alpha}\frac{\phi_\alpha(n+h)}{(n+h)^\alpha}\frac{\phi_\alpha(n+j)}{(n+j)^\alpha}n^\alpha(n+h)^\alpha(n+j)^\alpha. \quad (13)$$

From Theorem 9.1, we have

$$A(x):=\sum_{n\leqslant x}\frac{\phi_\alpha(n)}{n^\alpha}\frac{\phi_\alpha(n+h)}{(n+h)^\alpha}\frac{\phi_\alpha(n+j)}{(n+j)^\alpha}\sim xC,$$

where

$$C=\prod_p\left(1-\frac{3}{p^{\alpha+1}}\right)\prod_{\substack{p\\ b(p)=2}}\left(1+\frac{1}{p^{2\alpha+1}-3p^\alpha}\right)\prod_{\substack{p\\ b(p)=3}}\left(1+\frac{3p^\alpha-1}{p^{3\alpha+1}-3p^{2\alpha}}\right).$$

Therefore, applying partial summation formula to the right side of (13) yields

$$\sum_{n\leqslant x}\phi_\alpha(n)\phi_\alpha(n+h)\phi_\alpha(n+j)$$
$$=x^\alpha(x+h)^\alpha(x+j)^\alpha A(x)-\int_1^x A(t)\frac{d}{dt}(t^\alpha(t+h)^\alpha(t+j)^\alpha)dt$$
$$\sim\left(1-\frac{3\alpha}{3\alpha+1}\right)x^{3\alpha+1}C=\frac{x^{3\alpha+1}}{3\alpha+1}C. \qquad \blacksquare$$

10. Concluding remarks

Our theory of the two variable variant of the Ramanujan sum (4) and triple convolutions of classical Ramanujan sums can be developed in various directions. Firstly, the theory of Ramanujan sums can be extended to the two-variable variant. In this context, it would be interesting to evaluate our sums in the non-squarefree case. Secondly, we can use this theory to study triple convolution sums of classical arithmetical functions:

$$\sum_{n\leqslant x}f(n)g(n+a)h(n+b)$$

using the theory of Ramanujan expansions and our results regarding triple convolution sums. In some cases, it may be possible to derive asymptotic formulas by following the methods used in [5]. In other cases, precise conjectures can be made. In either case, it is important to derive error terms in our limit theorems.

Acknowledgements. We are grateful to the referees for their valuable comments, which have significantly improved the current version.

References

[1] R. Balasubramanian and S. Giri, *The mean-value of a product of shifted multiplicative functions and the average number of points of elliptic curves*, J. Number Theory **157** (2015), 37–53.

[2] R.D. Carmichael, *Expansions of arithmetical functions in infinite series*, Proceedings of the London Math. Society **34** (1932), 1–26.

[3] G. Coppola, M. Ram Murty, and B. Saha, *On the error term in a Parseval type formula in the theory of Ramanujan expansions II*, J. Number Theory **160** (2016), 700–715.

[4] G. Coppola, M. Ram Murty, and B. Saha, *Finite Ramanujan expansions and shifted convolution sums of arithmetical functions*, J. Number Theory **174** (2017), 78–92.

[5] H.G. Gadiyar, M. Ram Murty, and R. Padma, *Ramanujan-Fourier series and a theorem of Ingham*, Indian Journal of Pure and Applied Mathematics **45(5)** (2014), 691–706.

[6] H.G. Gadiyar, R. Padma, *Ramanujan-Fourier series, the Wiener-Khintchine formula and the distribution of prime pairs*, Physica A. **269** (1999), 503–510.

[7] G.H. Hardy and J.E. Littlewood, *Some problems of "Partitio Numerorum": III. On the expression of a number as a sum of primes*, Acta Math. **44** (1922), 1–70.

[8] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, 6th edition, Oxford University Press, Oxford, 2008.

[9] L.G. Lucht, *A survey of Ramanujan expansions*, International Journal of Number Theory **6** (2010), 1785–1799.

[10] L. Mirsky, *Summation formulae involving arithmetic functions*, Duke Mathematical Journal **16** (1949), 261–272.

[11] M. Ram Murty, *Problems in analytic number theory*, 2nd edition, 2008.

[12] M. Ram Murty and B. Saha, *On the error term in a Parseval type formula in the theory of Ramanujan expansions*, J. Number Theory **156** (2015), 125–134.

[13] S. Ramanujan, *On certain trigonometric sums and their applications in the theory of numbers*, Transactions of the Cambridge Philosophical Society **22** (1918), 179–199.

[14] B. Saha, *A note on arithmetical functions with absolutely convergent Ramanujan expansions*, International Journal of Number Theory **12.06** (2016), 1595–1611.

[15] W. Schwarz and J. Spilker, *Arithmetical functions*, Lecture Notes Series, London Mathematical Socity, 1994.

[16] R. Sivaramakrishnan, *Classical theory of arithmetic functions*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York **126**, 1989.

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Received: 8 June 2022; **revised:** 23 June 2023