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A VERY SIMPLE PROOF OF STIRLING'S FORMULA

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Abstract. We present a short proof of Stirling's formula that uses only a basic knowledge of calculus.

1. Introduction

Numerous proofs [1, 2, 3, 6, 7, 8] of Stirling's formula of varying levels of sophistication have appeared in the literature. For instance, the proof in [6] invokes Lebesgue's dominated convergence theorem while [7] uses Poisson distribution from probability theory; familiarity with these ideas cannot be expected of students in first year calculus courses. While the ideas in [8] are elementary, the work done in achieving a slight improvement to the barebones Stirling's formula does not clearly convey the basic idea behind this asymptotic formula. In this note, we present a short proof of Stirling's formula that uses only a basic knowledge of calculus.

We shall first prove that

$$n! \sim C\sqrt{n}\left(\frac{n}{e}\right)^n \tag{1.1}$$

for some constant C and subsequently prove that $C = \sqrt{2\pi}$. It is a curious fact that the discovery of (1.1) is due to de Moivre and Stirling's contribution to Stirling's formula is recognising that $C = \sqrt{2\pi}$. The first rigorous proof that the constant is $\sqrt{2\pi}$ is to be found in de Moivre's monograph "Miscellanea Analytica" [5] consisting of results about summation of series. This proof is also the widely known proof that uses Wallis's product formula. While Stirling offers no proof of his claim, it is likely that Stirling's own reasoning involves Wallis's formula. In his extensive analyses of Stirling's works, I. Tweddle [9] suggests that the digits of $\sqrt{\pi}$ may have been known to Stirling; Stirling computes the first nine places of $\sqrt{\pi}$ using Bessel's interpolation formula [9, p. 244] but "he certainly offers no proof here for the introduction of π".

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2. Proof of Stirling's Formula

We begin with the following easy observation about the logarithm function

2.1. **Observation.** For $|x| < 1$, we have the following convergent Taylor expansion centered at 0

$$\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

This tells us that, for $|x| < 1$, we have

$$|-\log(1-x) - x| \leqslant \sum_{k=2}^{\infty} |x|^k = \frac{|x|^2}{1-|x|}.$$

Changing x to $-x$ gives us

$$|\log(1+x) - x| \leqslant \frac{|x|^2}{1-|x|}. \tag{2.1}$$

2.2. **The proof.** We now study the function $\log n!$ which is

$$\sum_{k=1}^{n} \log k = \int_{\frac{1}{2}}^{n+\frac{1}{2}} \log t \,\mathrm{d}t + \sum_{k=1}^{n} \left(\log k - \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \log t \,\mathrm{d}t \right). \tag{2.2}$$

The first integral is evaluated easily

$$I_1 := \int_{\frac{1}{2}}^{n+\frac{1}{2}} \log t \,\mathrm{d}t = \left(n + \frac{1}{2}\right) \log\left(n + \frac{1}{2}\right) - n + \frac{\log 2}{2}. \tag{2.3}$$

Now, note that

$$\log\left(n + \frac{1}{2}\right) = \log n + \log\left(1 + \frac{1}{2n}\right)$$

and our observation (2.1) gives

$$\left|\log\left(1 + \frac{1}{2n}\right) - \frac{1}{2n}\right| \leqslant \frac{\left(\frac{1}{2n}\right)^2}{1 - \frac{1}{2n}} \leqslant \frac{1}{2n^2}$$

where the last inequality holds since $1 - (2n)^{-1} \geqslant \frac{1}{2}$ for $n \geqslant 1$. Hence we have

$$\log\left(n + \frac{1}{2}\right) = \log n + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

so that the integral (2.3) becomes

$$I_1 = \left(n + \frac{1}{2}\right) \log n - n + \frac{1 + \log 2}{2} + O\left(\frac{1}{n}\right). \tag{2.4}$$

Let us now consider the sum on the right hand side of (2.2). The kth summand is

$$
\begin{aligned}
&\log k - \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \log t \, \mathrm{d}t \\
&= \log k - \left\{ \left(k + \frac{1}{2}\right) \log\left(k + \frac{1}{2}\right) - \left(k - \frac{1}{2}\right) \log\left(k - \frac{1}{2}\right) - 1 \right\} \\
&= \log k - \left\{ k \log\left(\frac{k+1/2}{k-1/2}\right) + \frac{1}{2} \log\left(k^2 - \frac{1}{4}\right) - 1 \right\}. \qquad (2.5)
\end{aligned}
$$

Now, when $|x| < 1$, considering the Taylor expansions of the functions $\log(1+x)$ and $\log(1-x)$, we have

$$\log\left(\frac{1+x}{1-x}\right) = 2x + O(x^3) \qquad (2.6)$$

so that we get

$$\log\left(\frac{k+1/2}{k-1/2}\right) = \log\left(\frac{1+1/2k}{1-1/2k}\right) = \frac{1}{k} + O(k^{-3}). \qquad (2.7)$$

Finally, writing

$$\log\left(k^2 - \frac{1}{4}\right) = \log k^2 + \log\left(1 - \frac{1}{4k^2}\right), \qquad (2.8)$$

for $k \geqslant 1$, we have from our observation (2.1) that

$$\left|\log\left(1 - \frac{1}{4k^2}\right) + \frac{1}{4k^2}\right| \leqslant \frac{\frac{1}{16k^4}}{1 - \frac{1}{4k^2}} \leqslant \frac{1}{12k^4} \qquad (2.9)$$

so that $\log(1 - \frac{1}{4k^2}) = O(k^{-2})$. Thus, (2.8) becomes

$$\log\left(k^2 - \frac{1}{4}\right) = 2\log k + O(k^{-2}). \qquad (2.10)$$

Putting (2.7) and (2.10) together into (2.5), our summand now becomes

$$
\begin{aligned}
&\log k - \left\{ k\left(\frac{1}{k} + O(k^{-3})\right) + \frac{1}{2}\left(2\log k + O(k^{-2})\right) - 1 \right\} \\
&= O(k^{-2}).
\end{aligned}
$$

In particular, the sum converges as $n \to \infty$ by the comparison test. Thus, we estimate the sum on the right hand side of (2.2) by passing to the infinite sum: writing S_k for the kth summand, we get

$$
\begin{aligned}
\sum_{k=1}^{n} S_k &= \sum_{k>0} S_k - \sum_{k>n} S_k \\
&= A + O(n^{-1})
\end{aligned}
$$

for some constant A, since $\sum_{k>n} k^{-2} = O(n^{-1})$ by comparison with the integral. Putting all this together, we have proved that, for some constant A',

$$\log n! = \left(n + \frac{1}{2}\right)\log(n) - n + A' + r_n \qquad (2.11)$$

where $r_n = O(n^{-1})$. Exponentiating, we get

$$n! \sim C\sqrt{n}\left(\frac{n}{e}\right)^n \tag{2.12}$$

for some constant C.

2.3. **The constant.** We describe a method to arrive at the constant C in (1.1). For every non-negative integer n, consider the integral

$$I_n := \int_0^{\pi/2} \sin^n\theta \,\mathrm{d}\theta. \tag{2.13}$$

By integrating by parts, we have the following recurrence

$$I_n = \frac{n-1}{n} I_{n-2}. \tag{2.14}$$

In particular, as $n \to \infty$, we must have

$$\frac{I_n}{I_{n-2}} \to 1.$$

It is immediate from (2.13) that $I_0 = \frac{\pi}{2}$ and $I_1 = 1$. Thus, using (2.14), an induction on n shows that

$$I_{2n} = \binom{2n}{n}\frac{\pi}{2^{2n+1}} \text{ and } I_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

Now, since $0 \leqslant \sin\theta \leqslant 1$ when $\theta \in [0, \frac{\pi}{2}]$, it follows that $I_{n-2} \geqslant I_{n-1} \geqslant I_n$ and so

$$\lim_{n\to\infty} \frac{I_n}{I_{n-1}} = 1.$$

Computing along the even subsequence, we are immediately led to a limit due to de Moivre and Stirling [4, pp. 243–254]

$$1 = \lim_{n\to\infty} \frac{I_{2n}}{I_{2n-1}} = \lim_{n\to\infty} \frac{\pi n}{2^{4n}}\binom{2n}{n}^2. \tag{2.15}$$

Now, if the asymptotic formula (1.1) holds, then, we must have that $C = \sqrt{2\pi}$ in (1.1). To see this, we use the de Moivre's formula (2.15)

$$\lim_{n\to\infty} \frac{\sqrt{n}}{2^{2n}}\binom{2n}{n} = \frac{1}{\sqrt{\pi}}$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{2^{2n}} \frac{C\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}}{\left(C\sqrt{n}\left(\frac{n}{e}\right)^n\right)^2} = \frac{1}{\sqrt{\pi}}$$

$$\lim_{n\to\infty} \frac{\sqrt{2}}{C} = \frac{1}{\sqrt{\pi}}$$

thus proving our claim.

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