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A simple proof of the Wiener–Ikehara Tauberian Theorem

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Abstract

The Wiener–Ikehara Tauberian theorem is an important theorem giving an asymptotic formula for the sum of coefficients of a Dirichlet series $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$. We provide a simple and elegant proof of the Wiener–Ikehara Tauberian theorem which relies only on basic Fourier analysis and known estimates for the given Dirichlet series. This method also allows us to derive a version of the Wiener–Ikehara theorem with an error term.

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1. Introduction

Generally speaking, a Tauberian theorem is one in which the asymptotic behavior of a sequence of numbers or a function is deduced from the knowledge of their averages. Tauberian theorems derive their name from a theorem of A. Tauber [16] published in 1897, which states that if

$$\lim_{x\to 1^-} \sum_{n=0}^{\infty} a(n)x^n = A \tag{1.1}$$

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and $a(n) = o(1/n)$, then

$$\sum_{n=0}^{\infty} a(n) = A. \tag{1.2}$$

This means that if the left-hand limit of the power series exists at the boundary of the radius of convergence and certain restrictions on the coefficients are met, then the power series converges to this limit at the boundary. This is a conditional converse of Abel's theorem, which states that if (1.2) holds, then (1.1) is true. But to go from (1.1) to (1.2), one often needs an extra condition, usually called a Tauberian condition, like the one above that $na(n) \to 0$ as $n \to \infty$. To see why some extra conditions are needed, consider the case when $a(n) = (-1)^n$ where we have

$$\lim_{x\to 1^-} \sum_{n=0}^{\infty} (-1)^n x^n = \lim_{x\to 1^-} \frac{1}{1+x} = \frac{1}{2},$$

whereas the partial sums of the $a(n)$'s do not converge.

Some of the most interesting applications of Tauberian theorems pertain to analytic number theory in the context of Dirichlet series, as opposed to power series. In this context, Tauberian results can be thought of as estimates for the partial sums of the coefficients of a Dirichlet series, after imposing certain conditions on the series. An important result of this type is the Wiener–Ikehara theorem. Introduced by Ikehara [7] in 1931, it generalizes a theorem of Landau [10], by applying a Tauberian result obtained by Wiener. A well known application of the Wiener–Ikehara theorem is to the logarithmic derivative of the Riemann zeta function, using the fact that $\zeta(s) \neq 0$ on $\mathrm{Re}(s) = 1$, thereby deriving the prime number theorem. Similarly, one can derive instantly the prime number theorem for arithmetic progressions once analytic continuations to $\mathrm{Re}(s) = 1$ of the Dirichlet L-series are given and that they do not vanish on that line.

The evolution of Tauberian theory during the twentieth century is a fascinating study (see for example [8]). Proofs of the Wiener–Ikehara theorem and other Tauberian theorems in the literature are usually found to be quite involved. The most expedient complex-analytic proof was discovered by Newman [13] and the reader can find a simple exposition in [18]. In this paper, we adopt a Fourier analytic approach to prove the Wiener–Ikehara Tauberian theorem. It is inspired by the method developed in [14,19] dealing with bounded gaps between consecutive primes and the higher rank sieve method. Our proof is not only simpler than classical proofs in the literature, but also allows us to obtain error terms. This method makes use of the Laurent series expansion of the Dirichlet series at $s = 1$, allowing us to further improve the error term if more information on the Laurent expansion of the Dirichlet series becomes available (see Remark 1). Our main result is the following.

Theorem 1.1. *Let*

$$G(s) = \sum_{t=1}^{\infty} \frac{b(t)}{t^s}$$

be a Dirichlet series with non-negative coefficients. Suppose

$$G(s) = \zeta^k(s) g(s),$$

where $k \in \mathbb{N}$, $g(s)$ is a Dirichlet series absolutely convergent in $\mathrm{Re}(s) \geq 1$, *with* $g(1) \neq 0$. *Then, as* $x \to \infty$,

$$\sum_{t \leq x} b(t) = \frac{R}{(k-1)!} x(\log x)^{k-1} + O\left(x(\log x)^{k-2}\right),$$

where R is the residue of $G(s)$ at $s = 1$.

Minor modifications in the proof of Theorem 1.1 lead to the following.

Theorem 1.2. *Let*

$$G(s) = \sum_{t=1}^{\infty} \frac{b(t)}{t^s}$$

be a Dirichlet series with non-negative coefficients and absolutely convergent for $\mathrm{Re}(s) > c > 0$. *Suppose $G(s)$ extends to a function in the region* $\mathrm{Re}(s) \geq c$, *which is analytic apart from a possible pole of order k at $s = c$. Then, we have as $x \to \infty$,*

$$\sum_{t \leq x} b(t) = \frac{R}{c(k-1)!} x^c(\log x)^{k-1} + O\left(x^c(\log x)^{k-2}\right),$$

where R is the residue of $G(s)$ at $s = c$.

The focus from $s = 1$ to $s = c$ is a simple translation of the argument. If $G(s)$ has a pole of order k at $s = 1$, then $G(s)$ can be written as $\zeta(s)^k g(s)$ for some function $g(s)$ analytic in the region $\mathrm{Re}(s) \geq 1$. These elementary observations are sufficient to derive Theorem 1.2 from the method of Theorem 1.1.

The non-negativity of coefficients can be dropped. More generally, we have the following result.

Theorem 1.3. *Let*

$$F(s) = \sum_{t=1}^{\infty} \frac{a(t)}{t^s}, \quad (a(t) \in \mathbb{C})$$

be absolutely convergent for $\mathrm{Re}(s) > c > 0$. *Suppose that $F(s)$ extends to a meromorphic function in the region* $\mathrm{Re}(s) \geq c$, *having a pole of order k at $s = c$ with residue r. Suppose that $|a(t)| \leq b(t)$ where $G(s) = \sum_{t=1}^{\infty} \frac{b(t)}{t^s}$ is absolutely convergent for* $\mathrm{Re}(s) > c > 0$ *and extends to a meromorphic function in the region* $\mathrm{Re}(s) \geq c$, *having a pole of order k at $s = c$. Then, we have*

$$\sum_{t \leq x} a(t) = \frac{r}{c(k-1)!} x^c(\log x)^{k-1} + O\left(x^c(\log x)^{k-2}\right),$$

as $x \to \infty$.

The case $k = 1$ yields the classical Wiener–Ikehara Tauberian theorem. If $G(s)$ has the Laurent series expansion

$$G(1+s) = \frac{c_k}{s^k} + \frac{c_{k-1}}{s^{k-1}} + \cdots + c_0 + O_m\big(|s|^m\big), \tag{1.3}$$

as $s \to 0^+$, for some $c_i \in \mathbb{C}$ and $m \in \mathbb{N}$, then we obtain a saving of $(\log x)^m$ in the error term. More precisely, we have the following.

Remark 1. Suppose $G(s)$ satisfies the conditions of Theorem 1.1 and has the Laurent series expansion (1.3) as $s \to 0^+$. Then, by following the proof of Theorem 1.1 one can obtain

$$\sum_{t \le x} \frac{b(t)}{t} = c_k \frac{(\log x)^k}{k!} + c_{k-1} \frac{(\log x)^{k-1}}{(k-1)!} + \cdots + c_0 + O_m\left(\frac{1}{(\log x)^m}\right). \tag{1.4}$$

By applying partial summation to (1.4), we have as $x \to \infty$,

$$\sum_{t \le x} b(t) = x \sum_{j=1}^{k} \frac{(\log x)^{k-j}}{(k-j)!} \lambda_{k-j} + O_m\left(\frac{x}{(\log x)^m}\right),$$

where

$$\lambda_{k-j} = \sum_{i=0}^{j-1} (-1)^{j-1-i} c_{k-i}.$$

A more general version of Theorem 1.1 where k is allowed to take complex values can be found in Theorem II.5.4 of [17]. We thank Professor Tenenbaum for bringing this to our attention.

Tauberian theorems have a rich history. Selberg [15] considered Dirichlet series of the form $f(s,z) = \sum_{n=1}^{\infty} a_z(n) n^{-s}$, $(z \in \mathbb{C}, \operatorname{Re}(s) > 1)$. Letting B be a positive constant depending on the particular function $f(s,z)$, if $\sum_{n=1}^{\infty} |a_z(n)| n^{-1} (\log 2n)^{B+3}$ is uniformly bounded for $|z| \le B$, then Selberg showed that the coefficients of $\zeta^z(s) f(s,z) = \sum_{n=1}^{\infty} b_z(n) n^{-s}$ satisfy

$$\sum_{n \le x} b_z(n) = \frac{f(1,z)}{\Gamma(z)} x (\log x)^{z-1} + O\left(x (\log x)^{z-2}\right),$$

uniformly for $|z| \le B$ and $x \ge 2$. This theory was further developed by Delange [4,5] giving birth to what is now widely referred to as the Selberg–Delange method. However, special cases of such series were handled by Landau [9] in 1908. A detailed treatment of these methods as well as more general and effective results can be found in Tenenbaum's book [17, Chapter II.5]. Results in [17] give remainder terms of the form $O(xe^{-c\sqrt{\log x}})$ for some $c > 0$ and rely either on analytic continuation in some region to the left of $\operatorname{Re}(s) = 1$ or on absolute convergence at $s = 1$ for a finite number of derivatives of the Dirichlet series. Weaker conditions such as

$$\sum_{p \le x} f(p) = (\alpha + o(1)) \frac{x}{\log x},$$

for some $\alpha \in \mathbb{C}$, where p denotes a prime, along with growth conditions on f have been considered by Wirsing [20,21] to obtain results on $\sum_{n \le x} f(n)$ with weaker error terms. Recently, Granville and Koukoulopoulos [6] considered the condition

$$\sum_{p \le x} f(p) \log p = \alpha x + O\left(\frac{x}{(\log x)^A}\right), \quad (x \ge 2) \tag{1.5}$$

for some $\alpha \in \mathbb{C}$ and some $A > 0$. In [2], Bretèche and Tenenbaum refined these results further. Among other aspects, they also considered a short interval version of (1.5).

The Wiener–Ikehara Tauberian theorem can be stated more generally in the following form. Let S be a non-negative, non-decreasing function. Suppose that

$$G(s) := \int_1^\infty S(t)t^{-s-1}dt$$

converges for $\mathrm{Re}(s) > 1$ and that there exists R such that $G(s) - \frac{R}{s-1}$ admits a continuous extension to $\mathrm{Re}(s) \geq 1$. Then we have

$$S(x) = Rx + o(x). \tag{1.6}$$

The question of strengthening the remainder term was investigated by Müger in [11]. He showed that if $G(s) - \frac{R}{s-1}$ can be analytically continued to $\mathrm{Re}(s) > \alpha$ for some $\alpha \in (0, 1)$, then

$$S(x) = Rx + O_\epsilon(x^{\gamma+\epsilon}),$$

for any $\epsilon > 0$, where $\gamma = \alpha$ if $R = 0$; and $\gamma = \frac{\alpha+1}{2}$ if $R > 0$ and $S(x) - Rx$ is of fixed sign for all sufficiently large x. Broucke, Debruyne and Vindas [1,3], demonstrated that in this general setup, analytic continuation of $G(s) - \frac{R}{s-1}$ to the left of $\mathrm{Re}(s) = 1$ cannot yield better remainders in (1.6). Our results can be understood in this context as follows. In the special case when $G(s)$ can be written as a Dirichlet series and $S(x)$ is the partial sum of the coefficients of $G(s)$, if we have analytic continuation of $G(s) - \frac{R}{s-1}$ to $\mathrm{Re}(s) \geq 1$, we obtain an error term of $x/\log x$ in (1.6). More constraints on the growth of $G(s) - \frac{R}{s-1}$ (such as in (1.3) with $k = 1$) lead us to stronger error terms of the form $x/(\log x)^m$.

2. Preliminary results

We begin with some basic Fourier analysis. Let $\mathcal{F}$ be a smooth compactly supported function on $\mathbb{R}$. Then by Fourier inversion for the function $\mathcal{F}(t)e^t$, we can write

$$\mathcal{F}(t)e^t = \int_{\mathbb{R}} \eta_{\mathcal{F}}(u)e^{-iut}du,$$

where $\eta_{\mathcal{F}}$ is the Fourier transform of $\mathcal{F}(t)e^t$. Thus, we have

$$\mathcal{F}(t) = \int_{\mathbb{R}} \eta_{\mathcal{F}}(u)e^{-(1+iu)t}du. \tag{2.1}$$

In particular,

$$\mathcal{F}\left(\frac{\log t}{\log x}\right) = \int_{\mathbb{R}} \eta_{\mathcal{F}}(u)t^{-\frac{1+iu}{\log x}}du. \tag{2.2}$$

Note that $\eta_{\mathcal{F}}$ is the Fourier transform of a smooth, compactly supported function and is hence rapidly decaying, satisfying the bound

$$|\eta_{\mathcal{F}}(u)| \ll_A \frac{1}{(1+|u|)^A}, \tag{2.3}$$

for any $A > 0$, as $|u| \to \infty$. For $r \in \mathbb{N}$, we define the notion of a function $f(x)$ being integrated r times with respect to x as follows:

$$\int^{(r)} f(x)dx := \int_{x_r=0}^\infty \int_{x_{r-1}\geq x_r} \cdots \int_{x_1 \geq x_2} f(x_1)dx_1 \cdots dx_r.$$

Proposition 2.1. *Let $r \in \mathbb{N}$ and $\mathcal{F}$ be a compactly supported smooth function satisfying (2.1). Then*

$$\int^{(r)} \mathcal{F}(t)dt = \int_{\mathbb{R}} \eta_{\mathcal{F}}(u)\frac{1}{(1+iu)^r}du.$$

Proof. Observe that for $\mathrm{Re}(s) > 0$,

$$\int^{(r)} e^{-sx}dx = \frac{1}{s^r}.$$

From (2.1),

$$\int^{(r)} \mathcal{F}(t)dt = \int^{(r)} \int_{\mathbb{R}} \eta_{\mathcal{F}}(u)e^{-(1+iu)t}du\,dt.$$

The rapid decay of the functions in the integrand ensures that we can apply Fubini's theorem repeatedly to obtain the required answer. □

We state the following lemma which will give a convenient means of reducing multiple integrals to single integrals.

Lemma 2.2. *Let $r \in \mathbb{N}$ and f be a function with compact support in $[0, \infty)$. Then*

$$\int^{(r)} f(x)dx = \frac{1}{(r-1)!}\int_0^\infty x^{r-1}f(x)dx.$$

Proof. Repeated use of Fubini's theorem gives us,

$$\int^{(r)} f(x)dx = \int_{x_1=0}^\infty \int_{0\le x_2\le x_1} \cdots \int_{0\le x_r\le x_{r-1}} f(x_1)dx_r \dots dx_1.$$

Now,

$$\int_{0\le x_r\le x_{r-1}} f(x_1)dx_r = x_{r-1}f(x_1),$$

$$\int_{0\le x_{r-1}\le x_{r-2}} x_{r-1}f(x_1)dx_{r-1} = \frac{x_{r-2}^2}{2!}f(x_1),$$

and so on. We continue this until we are left with only the integral with respect to x_1. □

In particular, from Proposition 2.1 and Lemma 2.2, we conclude that

$$\int_{\mathbb{R}} \eta_{\mathcal{F}}(u)\frac{1}{(1+iu)^r}du = \frac{1}{(r-1)!}\int_0^\infty t^{r-1}\mathcal{F}(t)dt. \tag{2.4}$$

We state a well-known partial summation formula below which is also known as Abel's identity.

Lemma 2.3. *For any arithmetical function $a(n)$, let*

$$A(x) = \sum_{n\le x} a(n),$$

where $A(x) = 0$ *if* $x < 1$. *Assume* f *has a continuous derivative on the interval* $[y, x]$, *where* $0 < y < x$. *Then we have*

$$\sum_{y<n\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt.$$

3. Proof of Theorem 1.1

The key idea of this proof is to twist the partial sum $\sum_{t\leq x} b(t)/t$ by a compactly supported smooth function $\mathcal{F}$, to be chosen later. This technique is often referred to as smoothing the required sum. Using (2.2), we have

$$\sum_{t=1}^{\infty} \frac{b(t)}{t} \mathcal{F}\left(\frac{\log t}{\log x}\right) = \int_{\mathbb{R}} \eta_{\mathcal{F}}(u) \sum_{t=1}^{\infty} \frac{b(t)}{t^{1+\frac{1+iu}{\log x}}} du, \tag{3.1}$$

where the interchange of the summation and integration is justified as follows. Note that the series on the right hand side above is absolutely convergent as $\mathrm{Re}(1 + \frac{1+iu}{\log x}) > 1$ and by using (2.3) we conclude that the integral above is absolutely convergent, which explains the interchange. For a fixed $0 < \epsilon < 1$, we partition the integral (3.1) into two cases to write,

$$\begin{aligned}\int_{\mathbb{R}} \eta_{\mathcal{F}}(u) \sum_{t=1}^{\infty} \frac{b(t)}{t^{1+\frac{1+iu}{\log x}}} du &= \left[\int_{|u|<(\log x)^{\epsilon}} + \int_{|u|\geq(\log x)^{\epsilon}}\right] \eta_{\mathcal{F}}(u) \sum_{t=1}^{\infty} \frac{b(t)}{t^{1+\frac{1+iu}{\log x}}} du \\ &= I_1(x) + I_2(x) \quad \text{(say)}.\end{aligned} \tag{3.2}$$

As $G(s)$ has a pole at $s = 1$ of order k with residue R, the Laurent series expansion of $G(s)$ yields

$$G(1+s) = \frac{R}{s^k} + \frac{c}{s^{k-1}} + O\left(\frac{1}{|s|^{k-2}}\right), \tag{3.3}$$

for some constant c, as $s \to 0^+$. Using (3.3), in the region $|u| < (\log x)^{\epsilon}$, we have

$$\sum_{t=1}^{\infty} \frac{b(t)}{t^{1+\frac{1+iu}{\log x}}} = R\frac{(\log x)^k}{(1+iu)^k} + c\frac{(\log x)^{k-1}}{(1+iu)^{k-1}} + O\left(\frac{(\log x)^{k-2}}{|1+iu|^{k-2}}\right), \tag{3.4}$$

since $(1 + iu)/\log x \to 0$, as $x \to \infty$. From (3.2) and (3.4), we obtain

$$\begin{aligned}I_1(x) = R(\log x)^k &\int_{|u|<(\log x)^{\epsilon}} \frac{\eta_{\mathcal{F}}(u)}{(1+iu)^k} du + c(\log x)^{k-1} \int_{|u|<(\log x)^{\epsilon}} \frac{\eta_{\mathcal{F}}(u)}{(1+iu)^{k-1}} du \\ &+ O\left((\log x)^{k-2} \int_{\mathbb{R}} \frac{|\eta_{\mathcal{F}}(u)|}{|1+iu|^{k-2}} du\right).\end{aligned} \tag{3.5}$$

By using (2.3), we see that the integral

$$\int_{\mathbb{R}} \frac{|\eta_{\mathcal{F}}(u)|}{|1+iu|^{k-2}} du$$

is absolutely convergent, hence the error term in (3.5) is $\ll (\log x)^{k-2}$. We now estimate the other integral $I_2(x)$. The rapid decay of $\eta_{\mathcal{F}}(u)$ gives

$$|\eta_{\mathcal{F}}(u)| \ll_A \frac{1}{(1+|u|)^{2A}},$$

as $|u| \to \infty$. In the region $|u| \geq (\log x)^{\epsilon}$, we obtain

$$|\eta_{\mathcal{F}}(u)| \ll_A (\log x)^{-\epsilon A} \frac{1}{(1+|u|)^A}, \tag{3.6}$$

for any $A > 0$, as $x \to \infty$. From (3.3), we deduce that

$$G(1+s) \ll \frac{1}{|s|^k},$$

as $s \to 0^+$, which then gives the bound

$$\left| \sum_{t=1}^{\infty} \frac{b(t)}{t^{1+\frac{1+iu}{\log x}}} \right| \leq \sum_{t=1}^{\infty} \frac{b(t)}{t^{1+\frac{1}{\log x}}} \ll (\log x)^k, \tag{3.7}$$

whenever $x \to \infty$. Using (3.6) and (3.7), we have the bound

$$|I_2(x)| \ll_A (\log x)^{k-\epsilon A} \int_{|u|\geq(\log x)^{\epsilon}} \frac{1}{(1+|u|)^A} du,$$

for any $A > 0$, as $x \to \infty$. Thus,

$$|I_2(x)| \ll_A (\log x)^{-A}, \tag{3.8}$$

for any $A > 0$. Combining (3.1), (3.2), (3.5), and (3.8), we obtain

$$\sum_{t=1}^{\infty} \frac{b(t)}{t} \mathcal{F}\left(\frac{\log t}{\log x}\right) = R(\log x)^k \int_{|u|<(\log x)^{\epsilon}} \frac{\eta_{\mathcal{F}}(u)}{(1+iu)^k} du + c(\log x)^{k-1} \int_{|u|<(\log x)^{\epsilon}} \frac{\eta_{\mathcal{F}}(u)}{(1+iu)^{k-1}} du + O\left((\log x)^{k-2}\right).$$

Further, one can get rid of the condition $|u| < (\log x)^{\epsilon}$ on the domain of the integrals above and revert to $\mathbb{R}$ as follows. Consider for $m \geq 0$,

$$\int_{|u|<(\log x)^{\epsilon}} \frac{\eta_{\mathcal{F}}(u)}{(1+iu)^m} du = \int_{\mathbb{R}} \frac{\eta_{\mathcal{F}}(u)}{(1+iu)^m} du - \int_{|u|\geq(\log x)^{\epsilon}} \frac{\eta_{\mathcal{F}}(u)}{(1+iu)^m} du.$$

Using (3.6), we see that

$$\int_{|u|\geq(\log x)^{\epsilon}} \frac{\eta_{\mathcal{F}}(u)}{(1+iu)^m} du \ll_A (\log x)^{-A} \int_{|u|\geq(\log x)^{\epsilon}} \frac{1}{(1+iu)^A} du \ll_A (\log x)^{-A},$$

for any $A > 0$. Thus,

$$\begin{aligned} \sum_{t=1}^{\infty} \frac{b(t)}{t} \mathcal{F}\left(\frac{\log t}{\log x}\right) &= R(\log x)^k \int_{\mathbb{R}} \frac{\eta_{\mathcal{F}}(u)}{(1+iu)^k} du + c(\log x)^{k-1} \int_{\mathbb{R}} \frac{\eta_{\mathcal{F}}(u)}{(1+iu)^{k-1}} du \\ &\quad + O\left((\log x)^{k-2}\right) \\ &= \frac{R(\log x)^k}{(k-1)!} \int_0^{\infty} \mathcal{F}(t) t^{k-1} dt + \frac{c(\log x)^{k-1}}{(k-2)!} \int_0^{\infty} \mathcal{F}(t) t^{k-2} dt \\ &\quad + O\left((\log x)^{k-2}\right), \end{aligned} \tag{3.9}$$

upon using (2.4).

In order to obtain the partial sum $\sum_{t\leq x} \frac{b(t)}{t}$, we will choose $\mathcal{F}(t)$ to be a good approximation to the indicator function of the interval $[1, x]$. More precisely, we take

$$0 < \delta < \log\left(\frac{\lfloor x\rfloor + 1}{x}\right)(\log x)^{-k}, \tag{3.10}$$

and define $\mathcal{F}(t)$ to be a smooth function bounded above by 1, supported on $[-\delta, 1+\delta]$ and identically equal to 1 on $[0, 1]$. Since $\delta < \log\left(\frac{\lfloor x\rfloor+1}{x}\right)(\log x)^{-1}$, we see that $x^{1+\delta} < \lfloor x\rfloor + 1$, so that this choice of $\mathcal{F}$ yields the required partial sum $\sum_{t\le x} \frac{b(t)}{t}$ on the left-hand side of (3.9). With this choice of $\mathcal{F}$, we have

$$\begin{aligned}\frac{R(\log x)^k}{(k-1)!}\int_0^\infty \mathcal{F}(t)t^{k-1}dt &= \frac{R(\log x)^k}{(k-1)!}\left[\int_0^1 t^{k-1}dt + \int_1^{1+\delta}\mathcal{F}(t)t^{k-1}dt\right]\\ &= \frac{R(\log x)^k}{k!} + O_k\big(\delta(\log x)^k\big),\end{aligned}$$

since the second integrand is bounded in $[1, 1+\delta]$. For δ as in (3.10), the above error term is

$$\ll \log\left(\frac{\lfloor x\rfloor + 1}{x}\right) \ll \frac{1}{x}.$$

Proceeding similarly for the second term in (3.9), we obtain

$$\sum_{t\le x}\frac{b(t)}{t} = \frac{R(\log x)^k}{k!} + \frac{c(\log x)^{k-1}}{(k-1)!} + O\left((\log x)^{k-2}\right).$$

Finally, we use partial summation to obtain

$$\begin{aligned}\sum_{t\le x} b(t) = \sum_{t\le x}\frac{b(t)}{t}t &= \frac{Rx(\log x)^k}{k!} + \frac{cx(\log x)^{k-1}}{(k-1)!} + O\left(x(\log x)^{k-2}\right)\\ &\quad - \int_1^x \left(\frac{R(\log t)^k}{k!} + \frac{c(\log t)^{k-1}}{(k-1)!}\right)dt.\end{aligned} \tag{3.11}$$

Performing repeated integration by parts, we have

$$\int_1^x \frac{R(\log t)^k}{k!}dt = \frac{Rx(\log x)^k}{k!} - \frac{Rx(\log x)^{k-1}}{(k-1)!} + O\left(x(\log x)^{k-2}\right),$$

and

$$\int_1^x \frac{c(\log t)^{k-1}}{(k-1)!}dt = \frac{cx(\log x)^{k-1}}{(k-1)!} + O\left(x(\log x)^{k-2}\right).$$

Substituting the above integrals into (3.11) and simplifying, we obtain

$$\sum_{t\le x} b(t) = \frac{R}{(k-1)!}x(\log x)^{k-1} + O\left(x(\log x)^{k-2}\right),$$

completing the proof. □

4. Proof of Theorem 1.3

The proof of Theorem 1.3 can be reduced to the case of Theorem 1.1 in the usual way (see for example page 276 of [12]). We will indicate very briefly how this reduction can be made. First, if the coefficients $a(t)$ are real, we can consider the Dirichlet series $F(s) - G(s)$ and apply Theorem 1.2. If the coefficients $a(t)$ are not real, we consider

$$F^*(s) = \sum_{t=1}^{\infty} \frac{\overline{a(t)}}{t^s},$$

and observe that

$$F(s) = \frac{F(s) + F^*(s)}{2} + i\left(\frac{F(s) - F^*(s)}{2i}\right).$$

One now applies the theorem to each of the terms on the right hand side to deduce the result. Since $F^*(s) = \overline{F(\overline{s})}$, it is clear that $F^*(s)$ is analytic in the same region as $F(s)$. This is evident from the Laurent expansion of $F(s)$ at any point in the domain of analyticity.

In conclusion, the Wiener–Ikehara Tauberian theorem is a versatile tool in analytic number theory in that it gives "instant" asymptotic formulas using the simple principle of analytic continuation of Dirichlet series.

CRediT authorship contribution statement

M. Ram Murty: Writing – original draft, Writing – review & editing. **Jagannath Sahoo:** Writing – original draft, Writing – review & editing. **Akshaa Vatwani:** Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] F. Broucke, G. Debruyne, J. Vindas, On the absence of remainders in the Wiener-Ikehara and Ingham-Karamata theorems: a constructive approach, Proc. Amer. Math. Soc. 149 (3) (2021) 1053–1060.
[2] R. De La Bretèche, G. Tenenbaum, Remarks on the Selberg-Delange method, Acta Arith. 200 (4) (2021) 349–369.
[3] G. Debruyne, J. Vindas, Note on the absence of remainders in the Wiener-Ikehara theorem, Proc. Amer. Math. Soc. 146 (12) (2018) 5097–5103.
[4] H. Delange, Sur des formules dues à Atle Selberg, Bull. Sci. Math. 83 (1959) 101–111.

[5] H. Delange, Sur des formules de Atle Selberg, Acta Arith. 19 (2) (1971) 105–146.

[6] A. Granville, D. Koukoulopoulos, Beyond the LSD method for the partial sums of multiplicative functions, Ramanujan J. 49 (2) (2019) 287–319.

[7] S. Ikehara, An extension of Landau's theorem in the analytical theory of numbers, J. Math. Phys. 10 (1931) 1–12.

[8] J. Korevaar, A century of complex Tauberian theory, Bull. Amer. Math. Soc. (N.S.) 39 (4) (2002) 475–531.

[9] E. Landau, On the partition of positive integers in four classes according to the minimal number of squares needed to their additive composition, Arch. der Math. u. Phys. (3) 13 (1908) 305–312.

[10] E. Landau, Über die Bedeutung einiger neuen Grenzwertsätze der Herren Hardy und Axer, Prace Matematyczno-Fizyczne 21 (1) (1910) 97–177.

[11] M. Müger, On Ikehara type Tauberian theorems with $O(x^{\gamma})$ remainders, Abh. Math. Semin. Univ. Hambg. 88 (1) (2018) 209–216.

[12] M.R. Murty, A Second Course in Analysis, in: HBA Lecture Notes in Mathematics, Springer, New Delhi, 2022, p. xiii+337, [2022] ©IMSc Lecture Notes in Mathematics.

[13] D.J. Newman, Simple analytic proof of the prime number theorem, Amer. Math. Monthly 87 (9) (1980) 693–696.

[14] D.H.J. Polymath, Variants of the Selberg sieve, and bounded intervals containing many primes, Res. Math. Sci. 1 (2014) 83, Art. 12.

[15] A. Selberg, Note on a paper by L. G. Sathe, J. Indian Math. Soc. (N.S.) 18 (1954) 83–87.

[16] A. Tauber, Ein satz aus der theorie der unendlichen reihen, Monatshefte für Math. Phys. 8 (1897) 273–277.

[17] G. Tenenbaum, Introduction to Analytic and Probabilistic Number Theory, third ed., in: Graduate Studies in Mathematics, vol. 163, American Mathematical Society, Providence, RI, 2015, p. xxiv+629.

[18] A. Vatwani, A simple proof of Wiener-Ikehara Tauberian theorem, Math. Student 84 (3–4) (2015) 127–134.

[19] A. Vatwani, A higher rank Selberg sieve and applications, Czechoslovak Math. J. 68 (143 (1)) (2018) 169–193.

[20] E. Wirsing, Das asymptotische Verhalten von Summen über multiplikative Funktionen, Math. Ann. 143 (1961) 75–102.

[21] E. Wirsing, Das asymptotische Verhalten von Summen über multiplikative Funktionen. II, Acta. Math. Acad. Sci. Hungar. 18 (1967) 411–467.