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# Twin primes and the parity problem

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ABSTRACT

We formulate a conjecture regarding the equidistribution of the Möbius function over shifted primes in arithmetic progressions. Our main result is that such a conjecture for a fixed even integer $h$, in conjunction with the Elliott–Halberstam conjecture, can resolve the parity barrier and produce infinitely many primes $p$ such that $p+h$ is also prime.


## 1. Introduction

Let $\Lambda(n)$ denote the von Mangoldt function,

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p, \\ 0 & \text{otherwise.} \end{cases}$$

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Let $h$ be a fixed even integer. It is conjectured that

$$\sum_{n\leq x}\Lambda(n)\Lambda(n+h)\sim\mathfrak{S}(h)x, \tag{1.1}$$

where $\mathfrak{S}(h)$ is the singular series defined as

$$\mathfrak{S}(h)=\prod_{p|h}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid h}\left(1-\frac{1}{(p-1)^2}\right).$$

If $h=2$, this gives an asymptotic formula for the number of twin primes. It is believed that sieve methods cannot resolve this conjecture because of the parity problem. However, recently some breakthroughs have been made in proving the infinitude of bounded gaps between primes beginning with the work of Yitang Zhang [27] and later by Maynard [17]. The Polymath project [21] highlights the limitations of these new methods and amplifies the parity problem in resolving the twin prime conjecture.

The term "parity principle" was first coined by Atle Selberg [23], who came across this phenomenon in 1946, in his work on the ingenious sieve that bears his name today. He described this principle as follows: *"Sets of integers tend to be very evenly distributed with respect to the parity of their number of prime factors unless they have been particularly produced, constructed or selected in a way that has a built in bias."* The discussion by Selberg in [22] indicates that sieve methods are unable to distinguish whether an integer has an odd or an even number of prime factors. This is commonly referred to as the parity problem.

The parity problem can also be explained in the context of Bombieri's asymptotic sieve [1], which highlights that classical sieve methods are unable to sift out numbers having exactly $r$ prime factors, irrespective of the choice of $r$. There have been a number of attempts (cf. [13,5,8,10]) in various settings, to break the parity barrier by postulating additional analytic data into the sieve machinery. In this article, we follow this line of thought. A related result is due to J. Friedlander and H. Iwaniec [9] who assumed an estimate for certain bilinear forms, relying upon cancellations arising from sign changes of the Möbius function, in order to circumvent the parity problem and show the infinitude of primes of the form $a^2+b^4$.

A problem closely related to the parity principle is that of showing significant cancellation in the summatory function of the Möbius function:

$$M(x):=\sum_{n\leq x}\mu(n).$$

The assertions $M(x)=o(x)$ and $M(x)\ll_\epsilon x^{\frac{1}{2}+\epsilon}$ are equivalent to the prime number theorem and the Riemann hypothesis respectively. A higher rank version of this was conjectured by S. Chowla [4], in terms of the related Liouville function $\lambda(n)$, defined by $(-1)^{\Omega(n)}$ where $\Omega(n)$ is the total number of prime factors of $n$, counted with multiplicity.

**Conjecture** *(Chowla). Let $h_1 < \ldots < h_k$ be non-negative integers. Then, as $x \to \infty$, we have*

$$\sum_{n \le x} \lambda(n + h_1) \ldots \lambda(n + h_k) = o(x).$$

This conjecture is known only in the case $k = 1$, where again, the statement is equivalent to the prime number theorem. In general, one expects what is called the Möbius randomness law (cf. [11]), that is, for any "reasonable" sequence of complex numbers $a_n$, the sum of the Möbius function twisted by this sequence is relatively small.

A fundamental result of this type is Halász's mean value theorem (see [6], Theorem 6.2), which completely determines the asymptotic behaviour of the sum $\sum_{n \le x} x^{-1} g(n)$ for any multiplicative function $g$, with $|g(n)| \le 1$ for all $n \in \mathbb{N}$. A more general conjecture along these lines was formulated by P.D.T.A Elliott [7]. Recently, K. Matomäki, M. Radziwiłł and T. Tao [16,24] formulated a slightly modified, "corrected" form of Elliott's conjecture. They also succeeded in proving the averaged and logarithmically averaged (for $k = 2$) versions of the Chowla and Elliott conjectures. The Elliott conjecture essentially states that if $g_1, \ldots, g_k : \mathbb{N} \to \mathbb{C}$ are multiplicative functions absolutely bounded by 1, and are "reasonably distributed" (in a certain explicit sense, see Conjecture 1.5 of [16]), then for any $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{N}$, we have

$$\sum_{n \le x} g_1(a_1 n + b_1) \ldots g_k(a_k n + b_k) = o(x),$$

provided that $a_i b_j - a_j b_i \neq 0$ for any $1 \le i < j \le k$.

A natural generalization that suggests itself is the development of similar mean value estimates and conjectures for the behaviour of multiplicative functions over certain subsets of the natural numbers. One such subset of interest is the sequence of "shifted primes" $\{p + h\}$, for some fixed non-zero integer $h$, with $p$ running over all the primes. On such sequences, even the rank $k = 1$ analogue of Chowla's conjecture would suggest that

$$\sum_{n \le x} \Lambda(n) \mu(n + h) = o(x). \tag{1.2}$$

It will be convenient to use the notation $\mu_h$ to denote $\mu(n + h)$. One would expect this conjecture to be of the same level of difficulty as (1.1). Such estimates are inherently related to the parity problem in sieve theory, as we will see. It seems that Hildebrand [14] was the first to study such problems and derive a partial analogue of Halász's theorem for the subsequence of shifted primes. Later, these results have been improved by Timofeev [25] and Khripunova [15]. Some results in this direction have also been obtained by J. Pintz in [20].

In finding small gaps between primes, one of the main tools is the behaviour of primes in arithmetic progressions.

**Elliott–Halberstam Conjecture $\mathbf{EH}_{\boldsymbol{\Lambda}}(\boldsymbol{x^\theta})$.** *For any $A > 0$, we have*

$$\sum_{q \le x^\theta} \max_{y \le x} \max_{\substack{a \\ (a,q)=1}} \left| \sum_{\substack{n \le y \\ n \equiv a \pmod q}} \Lambda(n) - \frac{y}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A}. \tag{1.3}$$

This conjecture is true and is called the Bombieri–Vinogradov theorem when $\theta < 1/2$. This conjecture alone cannot resolve the parity problem and prove either (1.1) or (1.2), for instance cf. [21]. Let $h$ be a fixed non-zero integer. We postulate a conjecture regarding the equidistribution of the Möbius function on shifted primes $\{p + h\}$, in arithmetic progressions. We show that this more powerful conjecture does break the parity barrier.

**Shifted Möbius Elliott–Halberstam Conjecture $\mathbf{EH}_{\boldsymbol{\mu_h}}(\boldsymbol{x^\eta})$.** *For any $A > 0$, we have*

$$\sum_{q \le x^\eta} \max_{y \le x} \max_{\substack{a \\ (a,q)=1}} \left| \sum_{\substack{n \le y \\ n \equiv a \pmod q}} \Lambda(n)\mu(n+h) - \frac{1}{\phi(q)} \sum_{n \le y} \Lambda(n)\mu(n+h) \right| \ll_A \frac{x}{(\log x)^A}. \tag{1.4}$$

The conjecture (1.2) with a stronger error term would simplify the statement of this conjecture, but our main theorem below shows why we have not done this.

**Theorem 1.1.** *Let $h \neq 0$ be a fixed even integer. Suppose that the conjectures $EH_\Lambda(x^\theta(\log x)^C)$ and $EH_{\mu_h}(x^{1-\theta})$ are true for some fixed $\theta < 1$ and a suitably large fixed $C$. We then obtain the following:*

(a) *The assertions (1.1) and (1.2) are equivalent.*
(b) *We have*

$$\sum_{n \le x} \Lambda(n)\Lambda(n+h) \ge (1 - o(1))\mathfrak{S}(h)(1 - \mathcal{A}(h))x,$$

*where*

$$\mathcal{A}_h = \prod_{\substack{p \nmid h \\ p > 2}} \left(1 - \frac{1}{p(p-1)}\right). \tag{1.5}$$

In particular, part (b) of the theorem shows that the twin prime conjecture follows from $\mathrm{EH}_\Lambda(x^\theta(\log x)^C)$ and $\mathrm{EH}_{\mu_2}(x^{1-\theta})$. Note that if the Elliott–Halberstam conjecture holds, that is, we have $\mathrm{EH}_\Lambda(x^{1-\epsilon})$ for any small $\epsilon > 0$, then we only need $\mathrm{EH}_{\mu_2}(x^\epsilon)$ for any small positive $\epsilon$ in order to show the infinitude of twin primes!

We also remark that our proof goes through if we frame the equidistribution conjectures $\mathrm{EH}_\Lambda(x^\theta)$ and $\mathrm{EH}_{\mu_h}(x^\eta)$ for the fixed residue class $n \equiv -h \pmod q$ instead of taking the maximum over all residue classes co-prime to $q$. This may be of interest in the light of extensions of the Bombieri–Vinogradov theorem proved in [2,3].

## 2. Decomposition of $\Lambda$

Recall (cf. Ex 1.1.6, [18]) that

$$\Lambda(n) = \sum_{d|n} \mu(d) \log(1/d).$$

We can write

$$\Lambda(n) = \sum_{\substack{d|n \\ d \le y}} \mu(d) \log(1/d) + \sum_{\substack{d|n \\ d > y}} \mu(d) \log(1/d) := \Lambda_y(n) + \widetilde{\Lambda}_y(n).$$

We want to apply this decomposition only when the argument $n$ is square-free. Since $\Lambda(n) = \mu^2(n)\Lambda(n)$ except when $n = p^m$ with $m \ge 2$, and there are only $O(x^{1/2} \log x)$ such $n \le x$, we have

$$\begin{aligned}
\sum_{n \le x} \Lambda(n)\Lambda(n+h) &= \sum_{n \le x} \Lambda(n)\mu^2(n+h)\Lambda(n+h) + O(x^{1/2} \log x) \\
&= \sum_{n \le x} \Lambda(n)\mu^2(n+h)\Lambda_y(n+h) + \sum_{n \le x} \Lambda(n)\mu^2(n+h)\widetilde{\Lambda}_y(n+h) \\
&\quad + O(x^{1/2} \log x) \\
&:= S_1(y) + S_2(y) + O(x^{1/2} \log x). \qquad (2.1)
\end{aligned}$$

The condition that $n+h$ is square-free will be needed to evaluate $S_2$. Although this condition complicates the evaluation of $S_1$, we show in the next section that we get the same asymptotic value as we get for $S_1$ without this condition.

## 3. Evaluation of $S_1$ using $\mathrm{EH}_\Lambda(x^\theta(\log x)^C)$

We define the singular series by

$$\mathfrak{S}(h) = \begin{cases} 2C_2 \prod_{p>2, p|h} \left(\frac{p-1}{p-2}\right) & \text{if } h \text{ is even, } h \ne 0, \\ 0 & \text{if } h \text{ is odd,} \end{cases} \qquad (3.1)$$

where

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

In order to evaluate $S_1$, we will need the following special case of the Wiener–Ikehara Tauberian theorem due to D.J. Newman [19]. For a more general treatment of this, we refer the reader to the article [26].

**Theorem 3.1** *(Newman). Let $|a_n| \le 1$. We consider the series*

$$F(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

*which is absolutely convergent for* $\mathrm{Re}(s) > 1$*. If $F(s)$ can be analytically continued to* $\mathrm{Re}(s) \ge 1$*, then the series $\sum_{n=1}^{\infty} a_n/n^s$ converges for* $\mathrm{Re}(s) \ge 1$*.*

In particular, the proof of this theorem shows that for $\mathrm{Re}(s) \ge 1$, we have

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = F(s).$$

We need the following preliminary propositions.

**Proposition 3.2.** *For a fixed even integer $h$, let*

$$g_h(d) := \prod_{\substack{p|d \\ p\nmid h}} \left( \frac{(p-1)^2}{p(p-1)-1} \right). \tag{3.2}$$

*The series*

$$f(s) := \sum_{\substack{d=1 \\ (d,h)=1}}^{\infty} \frac{\mu(d) g_h(d)}{\phi(d) d^s}, \quad (\mathrm{Re}(s) > 0),$$

*can be analytically continued to* $\mathrm{Re}(s) \ge 0$*.*

**Proof.** We have the following Euler product for $f(s)$ for $\mathrm{Re}(s) > 0$,

$$f(s) = \prod_{p\nmid h} \left(1 - \frac{g_h(p)}{(p-1)p^s}\right) = \prod_{p\nmid h} \left(1 - \frac{(p-1)}{(p(p-1)-1)p^s}\right).$$

Multiplying and dividing by the factor $(1 - 1/p^{s+1})$, for each prime, we obtain

$$f(s) = \zeta(s+1)^{-1} G(s), \tag{3.3}$$

where $\zeta(s)$ is the Riemann-zeta function and $G(s)$ is the Euler product

$$G(s) = \prod_{p|h} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \prod_{p\nmid h} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \left(1 - \frac{(p-1)}{(p(p-1)-1)p^s}\right).$$

If we show that $G(s)$ is absolutely convergent for $\mathrm{Re}(s) \geq 0$, then the expression (3.3) gives the analytic continuation of $f(s)$ to $\mathrm{Re}(s) \geq 0$.

We can write $G(s)$ as

$$\begin{aligned} G(s) &= \prod_{p|h} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \prod_{p\nmid h} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \left(1 - \frac{1}{p^{s+1}\left(1 - \frac{1}{p(p-1)}\right)}\right) \\ &= \prod_{p|h} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \prod_{p\nmid h} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \left(1 - \frac{1 + O\left(\frac{1}{p(p-1)}\right)}{p^{s+1}}\right), \end{aligned}$$

since $(1 - x_p)^{-1} = 1 + O(x_p)$, where $x_p = 1/p(p-1)$. Continuing with this notation, we can simplify the above expression further as

$$\begin{aligned} G(s) &= \prod_{p|h} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \prod_{p\nmid h} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \left(1 - \frac{1}{p^{s+1}} + \frac{O(x_p)}{p^{s+1}}\right) \\ &= \prod_{p|h} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \prod_{p\nmid h} \left(1 + \frac{O(x_p)}{p^{s+1}} \left(1 - \frac{1}{p^{s+1}}\right)^{-1}\right). \end{aligned}$$

Since $(1 - p^{-s-1})^{-1} = 1 + O(1/p^{s+1})$, we obtain

$$G(s) = \prod_{p|h} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \prod_{p\nmid h} \left(1 + \frac{O(x_p)}{p^{s+1}} + \frac{O(x_p)}{p^{2s+2}}\right).$$

Keeping in mind that $x_p = 1/p(p-1)$, it is easy to see that $G(s)$ converges absolutely for $\mathrm{Re}(s) \geq -1$. □

**Proposition 3.3.** *We have*

$$\mathcal{A}_h \sum_{\substack{d=1 \\ (d,h)=1}}^{\infty} \frac{\mu(d) g_h(d) \log(1/d)}{\phi(d)} = \mathfrak{S}(h),$$

*where $\mathcal{A}_h$ and $\mathfrak{S}(h)$ are as defined by (1.5) and (3.1) respectively.*

**Proof.** From the definition of $f(s)$ above, we have

$$f'(s) = \sum_{\substack{d=1 \\ 2\nmid d}}^{\infty} \frac{\mu(d) g(d) \log(1/d)}{\phi(d) d^s}, \tag{3.4}$$

which is absolutely convergent for $\mathrm{Re}(s) > 0$. From the previous lemma we see that $f'(s)$ can be analytically continued to $\mathrm{Re}(s) \geq 0$. Indeed, we have

$$f'(s) = \left(\zeta(s+1)^{-1}G(s)\right)' = G'(s)\zeta(s+1)^{-1} + G(s)\left(-\frac{\zeta'(s+1)}{\zeta(s+1)}\zeta(s+1)^{-1}\right).$$

It can be checked that $G'(s)$ is absolutely convergent for $\mathrm{Re}(s) \geq 0$. Since $\zeta(s+1)^{-1}$ and $-\frac{\zeta'}{\zeta}(s+1)$ have a simple zero and a simple pole respectively at $s = 0$, with residue 1, at $s = 0$ we obtain:

$$f'(0) = G(0) = \prod_{p|h}\left(1-\frac{1}{p}\right)^{-1}\prod_{p\nmid h}\left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{(p-1)}{(p(p-1)-1)}\right).$$

We have obtained that the series

$$\sum_{\substack{d=1\\(d,h)=1}}^{\infty}\frac{\mu(d)g(d)\log(1/d)}{\phi(d)d^s}, \qquad (\mathrm{Re}(s) > 0),$$

can be analytically continued to $\mathrm{Re}(s) \geq 0$. Using the Tauberian Theorem 3.1 obtained by Newman, we see that the above series in fact converges for $\mathrm{Re}(s) \geq 0$ and is equal to $f'(0)$ at $s = 0$. This gives

$$\mathcal{A}_h\sum_{\substack{d=1\\(d,h)=1}}^{\infty}\frac{\mu(d)g(d)\log(1/d)}{\phi(d)} = \mathcal{A}_h f'(0) = \mathcal{A}_h G(0) = \mathfrak{S}(h),$$

where the last equality is easy to check. This completes the proof. □

We now prove the following lemma evaluating $S_1$.

**Lemma 3.4.** *Take $h$ to be a fixed positive even integer, $y = x^\theta$ for a fixed $\theta < 1$, and let $C$ be a fixed suitably chosen constant. Then assuming $EH_{\Lambda}(x^\theta(\log x)^C)$, we have*

$$S_1(y) \sim \mathfrak{S}(h)x.$$

It is instructive to observe how this lemma would be proved if we drop the factor $\mu^2(n+h)$ from $S_1$. Then,

$$\begin{aligned}\bar{S_1} &:= \sum_{n\leq x}\Lambda(n)\Lambda_y(n+h)\\ &= \sum_{n\leq x}\Lambda(n)\sum_{\substack{d|n+h\\d\leq y}}\mu(d)\log(1/d)\end{aligned}$$

$$= \sum_{d \le y} \mu(d) \log(1/d) \sum_{\substack{n \le x \\ n \equiv -h \pmod{d}}} \Lambda(n).$$

If the gcd $(d,h) > 1$, then this sum is bounded by $y \log y \sum_{n|h} \Lambda(n) \ll x^\theta \log x \log h$. We may thus assume $(d,h) = 1$. Then

$$\begin{aligned}
&\sum_{\substack{d \le y \\ (d,h)=1}} \mu(d) \log(1/d) \sum_{\substack{n \le x \\ n \equiv -h \pmod{d}}} \Lambda(n) \\
&= \sum_{\substack{d \le y \\ (d,h)=1}} \mu(d) \log(1/d) \left( \sum_{\substack{n \le x \\ n \equiv -h \pmod{d}}} \Lambda(n) - \frac{x}{\phi(d)} \right) \\
&+ x \sum_{\substack{d \le y \\ (d,h)=1}} \frac{\mu(d) \log(1/d)}{\phi(d)}.
\end{aligned}$$

By Lemma 2.1 of Goldston and Yıldırım [12], we have

$$\sum_{\substack{d \le y \\ (d,h)=1}} \frac{\mu(d) \log(1/d)}{\phi(d)} = \mathfrak{S}(h) + O(e^{-c\sqrt{\log y}}). \tag{3.5}$$

For the first term, we take $y = x^\theta$ and apply $\mathrm{EH}_\Lambda(x^\theta)$ to see that it is $\ll x/(\log x)^A$ for any $A > 0$. This proves the lemma for $\bar{S}_1$.

**Proof of Lemma 3.4.** Recall (cf. Ex 1.1.7, [18]) that $\mu^2(n) = \sum_{d^2|n} \mu(d)$. This gives

$$\begin{aligned}
S_1 &= \sum_{n \le x} \Lambda(n) \sum_{\substack{d|n+h \\ d \le y}} \mu(d) \log(1/d) \sum_{e^2|n+h} \mu(e) \\
&= \sum_{\substack{d \le y \\ e \le \sqrt{x+h} \\ [d,e^2] \le x+h}} \mu(d)\mu(e) \log(1/d) \sum_{\substack{n \le x \\ n \equiv -h \pmod{[d,e^2]}}} \Lambda(n)
\end{aligned}$$

Let $1 \le z \le x$ be chosen later. Then, when $e > z$, the terms in this sum are

$$\ll (\log x)^2 \sum_{\substack{[d,e^2] \le 2x \\ e > z}} \mu^2(d)\mu^2(e) \left( \frac{x}{[d,e^2]} + 1 \right).$$

Letting $\delta = (d,e)$, we can write $d = d'\delta$, $e = e'\delta$, so that $[d,e^2] = d'e'^2\delta^2$. Hence, when $e > z$, the contribution to the sum is

$$\ll (\log x)^2 \sum_{\substack{d' e'^2 \delta^2 \le 2x \\ e'\delta > z}} \left( \frac{x}{d' e'^2 \delta^2} + 1 \right)$$

$$\ll x(\log x)^2 \left( \sum_{d' \le 2x} \frac{1}{d'} \sum_{e' \le 2x} \frac{1}{e'^2} \sum_{\delta > z/e'} \frac{1}{\delta^2} \right) + (\log x)^2 \left( \sum_{\delta \le 2x} \sum_{e' > \frac{z}{\delta}} \sum_{d' \le 2x/e'^2\delta^2} 1 \right)$$

$$\ll \frac{x(\log x)^4}{z} + \frac{x(\log x)^3}{z} \ll \frac{x(\log x)^4}{z}.$$

Choosing $z = (\log x)^B$, $B = A + 4$, with $A$ sufficiently large, we obtain

$$S_1 = \sum_{\substack{d \le y \\ e \le z}} \mu(d)\mu(e) \log(1/d) \sum_{\substack{n \le x \\ n \equiv -h \pmod{[d,e^2]}}} \Lambda(n) + O\left( \frac{x}{(\log x)^A} \right).$$

Again if $(de, h) > 1$, we can show that the sum is bounded by $yz \log y \sum_{n|h} \Lambda(n) \ll x^\theta (\log x)^{B+2}$. We may thus assume $(de, h) = 1$, to get

$$S_1 = \sum_{\substack{d \le y \\ e \le z \\ (de,h)=1}} \mu(d)\mu(e) \log(1/d) \sum_{\substack{n \le x \\ n \equiv -h \pmod{[d,e^2]}}} \Lambda(n) + O\left( \frac{x}{(\log x)^A} \right)$$

$$= x \sum_{\substack{d \le y \\ e \le z \\ (de,h)=1}} \frac{\mu(d)\mu(e)\log(1/d)}{\phi([d,e^2])}$$

$$+ \sum_{\substack{d \le y \\ e \le z \\ (de,h)=1}} \mu(d)\mu(e) \log(1/d) \left( \sum_{\substack{n \le x \\ n \equiv -h \pmod{[d,e^2]}}} \Lambda(n) - \frac{x}{\phi([d,e^2])} \right)$$

$$+ O\left( \frac{x}{(\log x)^A} \right).$$

For the second term above, letting $r$ denote $[d, e^2]$, it is clear that $r \le x^\theta (\log x)^{2B}$ in the given range of $d, e$. Since the number of pairs of natural numbers $(n, m)$ such that $[n, m] = r$ is a multiplicative function of $r$, it is easy to see that this number can be bounded above by $\tau_3(r)$, where $\tau_3(r)$ is the number of ways of writing $r$ as a product of three positive integers. Then, by the Cauchy–Schwarz inequality and the trivial bound

$$\left| \sum_{\substack{n \le x \\ n \equiv -h \pmod{r}}} \Lambda(n) - \frac{x}{\phi(r)} \right| \ll \frac{x}{r} + 1,$$

the term under consideration is

$$\ll \log x \sum_{r \le x^\theta (\log x)^{2B}} \tau_3(r) \left( \sum_{\substack{n \le x \\ n \equiv -h \pmod r}} \Lambda(n) - \frac{x}{\phi(r)} \right)$$

$$\ll \log x \left( \sum_{r \le x^\theta (\log x)^{2B}} \tau_3(r)^2 \left( \frac{x}{r} + 1 \right) \right)^{1/2}$$

$$\times \left( \sum_{r \le x^\theta (\log x)^{2B}} \left| \sum_{\substack{n \le x \\ n \equiv -h \pmod r}} \Lambda(n) - \frac{x}{\phi(r)} \right| \right)^{1/2}.$$

Using elementary estimates for the first term in the above product and $\mathrm{EH}_\Lambda(x^\theta (\log x)^{2B})$ for the second, we finally obtain

$$S_1(y) = x \sum_{\substack{d \le y \\ e \le z \\ (de,h)=1}} \frac{\mu(d)\mu(e)\log(1/d)}{\phi([d,e^2])} + O\left( \frac{x}{(\log x)^A} \right).$$

To finish the proof of the lemma, we will show that as $y \to \infty$,

$$S_{11} := \sum_{\substack{d \le y \\ e \le z \\ (de,h)=1}} \frac{\mu(d)\mu(e)\log(1/d)}{\phi([d,e^2])} \sim \mathfrak{S}(h).$$

Since $d, e$ are square-free, we can write $[d, e^2]$ as the product of the co-prime integers $e^2$ and $d/(d,e)$, so that $\phi([d,e^2]) = e\phi(e)\phi(d)/\phi((d,e))$. Hence,

$$S_{11} = \sum_{\substack{d \le y \\ (d,h)=1}} \frac{\mu(d)\log(1/d)}{\phi(d)} \sum_{\substack{e \le z \\ (e,h)=1}} \frac{\mu(e)\phi((d,e))}{e\phi(e)}.$$

We can extend the last sum to infinity with an error

$$\ll \log y \sum_{d \le y} \frac{1}{\phi(d)} \sum_{e > z} \frac{\phi((d,e))}{e\phi(e)} \ll \log y \sum_{d \le y} \frac{1}{\phi(d)} \sum_{e > z} \frac{(d,e)}{e\phi(e)}.$$

On using $(d,e) = \sum_{r|(d,e)} \phi(r)$, and writing $d = rd'$ and $e = re'$ we find that this error is

$$\ll \log y \sum_{r \le y} \frac{1}{r\phi(r)} \sum_{d' \le y/r} \frac{1}{\phi(d')} \sum_{e' > z/r} \frac{1}{e'\phi(e')} \ll \frac{(\log y)^3}{z}.$$

This is $o(1)$ provided we take $B > 3$ while choosing $z = (\log x)^B$. We have now obtained

$$\begin{aligned} S_{11} &\sim \sum_{\substack{d\le y\\(d,h)=1}} \frac{\mu(d)\log(1/d)}{\phi(d)} \sum_{\substack{e=1\\(e,h)=1}}^{\infty} \frac{\mu(e)\phi((d,e))}{e\phi(e)} \\ &= \sum_{\substack{d\le y\\(d,h)=1}} \frac{\mu(d)\log(1/d)}{\phi(d)} \prod_{p\nmid h}\left(1-\frac{\phi(p,d)}{p(p-1)}\right) \\ &= \mathcal{A}_h \sum_{\substack{d\le y\\(d,h)=1}} \frac{\mu(d)\log(1/d)}{\phi(d)} g_h(d), \end{aligned}$$

where $\mathcal{A}_h$ and $g_h(d)$ are as in (1.5) and Proposition 3.2 respectively. Invoking Proposition 3.3 completes the proof. □

## 4. Evaluation of $S_2$ using $\mathrm{EH}_{\mu_h}(x^\eta)$

Recall that

$$S_2(y) = \sum_{n\le x} \Lambda(n)\mu^2(n+h)\widetilde{\Lambda}_y(n+h) = \sum_{n\le x}\Lambda(n)\mu^2(n+h)\sum_{\substack{d\mid n+h\\ d>y}}\mu(d)\log(1/d)$$

Writing $n+h = de$ and noting that $\mu^2(n+h)\mu(\frac{n+h}{e}) = \mu(n+h)\mu(e)$, we have

$$\begin{aligned} S_2(y) &= \sum_{n\le x}\Lambda(n)\mu(n+h)\sum_{\substack{e\mid n+h\\ e<\frac{n+h}{y}}}\mu(e)\log\left(\frac{e}{n+h}\right) \\ &= \sum_{e<\frac{x+h}{y}}\mu(e) \sum_{\substack{n\le x\\ n\equiv -h \pmod e}} \Lambda(n)\mu(n+h)\log\left(\frac{e}{n+h}\right). \end{aligned}$$

When $(e,h) > 1$, the contribution to the sum is

$$\ll \log x \sum_{e<\frac{x+h}{y}} \mu^2(e)\sum_{n\mid e}\Lambda(n) \ll \frac{x(\log x)^2}{y}.$$

Moreover, since there is at most one $e$ with $\frac{x}{y} < e \le \frac{x+h}{y} \le \frac{x}{y}+1$, for $y$ large, this term contributes

$$\ll \frac{x(\log x)^2}{e} \ll y(\log x)^2.$$

Hence, we have

$$S_2(y) = \sum_{\substack{e<\frac{x}{y} \\ (e,h)=1}} \mu(e) \sum_{\substack{n\le x \\ n\equiv -h \pmod e}} \Lambda(n)\mu(n+h)\log\left(\frac{e}{n+h}\right) + O\left(\left(\frac{x}{y}+y\right)(\log x)^2\right)$$

$$= S_3 - S_4 + O\left(\left(\frac{x}{y}+y\right)(\log x)^2\right),$$

where

$$S_3 := \sum_{\substack{e<\frac{x}{y} \\ (e,h)=1}} \mu(e)\log e \sum_{\substack{n\le x \\ n\equiv -h \pmod e}} \Lambda(n)\mu(n+h),$$

$$S_4 := \sum_{\substack{e<\frac{x}{y} \\ (e,h)=1}} \mu(e) \sum_{\substack{n\le x \\ n\equiv -h \pmod e}} \Lambda(n)\mu(n+h)\log(n+h).$$

We evaluate $S_3$ and $S_4$ in the following propositions.

**Proposition 4.1.** *Suppose $\frac{x}{y} \le x^\eta$, and $EH_{\mu_h}(x^\eta)$ holds. Then for any $A>0$, we have*

$$S_3 = (-\mathfrak{S}(h)+o(1))\left(\sum_{n\le x}\Lambda(n)\mu(n+h)\right) + O(x/(\log x)^A).$$

**Proof.** Since $x/y \le x^\eta$, we have by $\mathrm{EH}_{\mu_h}(x^\eta)$,

$$S_3 = \sum_{\substack{e<\frac{x}{y} \\ (e,h)=1}} \mu(e)\log e\left(\sum_{\substack{n\le x \\ n\equiv -h \pmod e}} \Lambda(n)\mu(n+h) - \frac{1}{\phi(e)}\sum_{n\le x}\Lambda(n)\mu(n+h)\right)$$

$$+ \sum_{\substack{e<\frac{x}{y} \\ (e,h)=1}} \frac{\mu(e)\log e}{\phi(e)}\sum_{n\le x}\Lambda(n)\mu(n+h)$$

$$= \sum_{\substack{e<\frac{x}{y} \\ (e,h)=1}} \frac{\mu(e)\log e}{\phi(e)}\sum_{n\le x}\Lambda(n)\mu(n+h) + O\left(\frac{x}{(\log x)^{A-1}}\right).$$

We complete the proof by using (3.5) to obtain that the first term is

$$(-\mathfrak{S}(h)+o(1))\left(\sum_{n\le x}\Lambda(n)\mu(n+h)\right). \qquad \square$$

Recall that conjecture $\mathrm{EH}_{\mu_h}(x^\eta)$ predicts that the function $\Lambda(n)\mu(n+h)$ is equidistributed in arithmetic progressions, within a certain range. In order to evaluate $S_4$, we need equidistribution in arithmetic progression for the function $\Lambda(n)\mu(n+h)\log(n+2)$. Indeed, this is an easy consequence of partial summation on $\mathrm{EH}_{\mu_h}(x^\eta)$, as illustrated below.

**Proposition 4.2.** *If $\mathrm{EH}_{\mu_h}(x^\eta)$ holds for some fixed $h \in \mathbb{Z}$, $h \neq 0$, then we have for any $A > 0$,*

$$\sum_{q<x^\eta} \max_{y \le x} \max_{(a,q)=1} \Bigg| \sum_{\substack{n \le y \\ n \equiv a \pmod q}} \Lambda(n)\mu(n+h)\log(n+h) - \frac{1}{\phi(q)} \sum_{n \le y} \Lambda(n)\mu(n+h)\log(n+h) \Bigg| \ll_A \frac{x}{(\log x)^A}.$$

**Proof.** For any $(a,q) = 1$, let us use the notation

$$\sum_{\substack{n \le x \\ n \equiv a \pmod q}} \Lambda(n)\mu(n+h) = M(x) + E_\mu(x,q,a), \tag{4.1}$$

where $M(x) = \frac{1}{\phi(q)} \sum_{n \le x} \Lambda(n)\mu(n+h)$. Then by partial summation,

$$\begin{aligned} &\sum_{\substack{n \le y \\ n \equiv a \pmod q}} \Lambda(n)\mu(n+h)\log(n+h) \\ &= \log(y+h) \sum_{\substack{n \le y \\ n \equiv a \pmod q}} \Lambda(n)\mu(n+h) \\ &\quad - \int_2^y \frac{1}{t+h} \sum_{\substack{n \le t \\ n \equiv a \pmod q}} \Lambda(n)\mu(n+h) dt \\ &= M(y)\log(y+h) - \int_2^y \frac{M(t)}{t+h} dt + O\left( \max_{t \le y} |E_\mu(t,q,a)| \log y \right). \end{aligned}$$

Again using partial summation, we see that the first two terms give us the desired main term

$$\frac{1}{\phi(q)} \sum_{n \le y} \Lambda(n)\mu(n+h)\log(n+h).$$

The term in parenthesis is the error term and can be checked to satisfy the required bound on average. $\square$

This allows us to bound $S_4$ as follows.

**Proposition 4.3.** *If $\frac{x}{y} \le x^\eta$, and $EH_{\mu_h}(x^\eta)$ holds, then for any $A > 0$, we have*

$$S_4 \ll_A \frac{x}{(\log x)^A}.$$

**Proof.** Applying Proposition 4.2, we obtain

$$S_4 = \sum_{\substack{e<\frac{x}{y}\\(e,h)=1}} \frac{\mu(e)}{\phi(e)} \sum_{n\le x} \Lambda(n)\mu(n+h)\log(n+h) + O\left(\frac{x}{(\log x)^{A-1}}\right).$$

By Lemma 2.1 of Goldston and Yıldırım [12], we have

$$\sum_{\substack{e<x\\(e,h)=1}} \frac{\mu(e)}{\phi(e)} \ll e^{-c\sqrt{\log x}},$$

which completes the proof. □

Putting together Propositions 4.1 and 4.3, we have proved the following result regarding the evaluation of $S_2(y)$.

**Lemma 4.4.** *For any $A > 0$, we have*

$$S_2(y) = (-\mathfrak{S}(h) + o(1)) \left(\sum_{n\le x} \Lambda(n)\mu(n+h)\right) + O\left(\frac{x}{(\log x)^A}\right) + O((x^\eta + y)(\log x)^2),$$

*provided $\frac{x}{y} \le x^\eta$ and the Shifted Möbius Elliott–Halberstam Conjecture $EH_{\mu_h}(x^\eta)$ holds.*

## 5. Proof of Theorem 1.1

We choose $y = x^\theta$ and $\eta = 1-\theta$, so that (2.1) and Lemmas 3.4, 4.4 give us

$$\sum_{n\le x} \Lambda(n)\Lambda(n+h) \sim (\mathfrak{S}(h) + o(1)) \left(x - \sum_{n\le x} \Lambda(n)\mu(n+h)\right),$$

provided $\mathrm{EH}_\Lambda(x^\theta(\log x)^C)$ and $\mathrm{EH}_{\mu_h}(x^{1-\theta})$ hold. Assuming these two conjectures, this shows that (1.1) and (1.2) are equivalent to each other.

The second part of the theorem follows from the trivial bound

$$\left|\sum_{n\le x} \Lambda(n)\mu(n+h)\right| \le \sum_{n\le x} \Lambda(n)\mu^2(n+h)$$

and the formula

$$\sum_{n\leq x}\Lambda(n)\mu^2(n+h)\sim\mathcal{A}_h x.$$

We prove this formula as follows. We have

$$\begin{aligned}\sum_{n\leq x}\Lambda(n)\mu^2(n+h)&=\sum_{n\leq x}\Lambda(n)\sum_{d^2|n+h}\mu(d)\\&=\sum_{d\leq\sqrt{x+h}}\mu(d)\sum_{\substack{n\leq x\\ n\equiv -h \pmod{d^2}}}\Lambda(n)\\&=\sum_{d\leq x^{1/4}}\mu(d)\sum_{\substack{n\leq x\\ n\equiv -h \pmod{d^2}}}\Lambda(n)\\&\quad+\sum_{x^{1/4}<d\leq\sqrt{x+h}}\mu(d)\sum_{\substack{n\leq x\\ n\equiv -h \pmod{d^2}}}\Lambda(n).\end{aligned}$$

We use the trivial bound on the inner sum to see that the second term is

$$\ll\sum_{x^{1/4}<d\leq\sqrt{x+h}}\log x\left(\frac{x}{d^2}+1\right)\ll x^{3/4}\log x.$$

For the first term, the contribution of $d$ such that $(d,h)>1$ is $\ll x^{1/4}\log x$. Using the Bombieri–Vinogradov theorem, we get

$$\begin{aligned}\sum_{\substack{d\leq x^{1/4}\\ (d,h)=1}}\mu(d)\sum_{\substack{n\leq x\\ n\equiv -h \pmod{d^2}}}\Lambda(n)&=x\sum_{\substack{d\leq x^{1/4}\\ (d,h)=1}}\frac{\mu(d)}{\phi(d^2)}+O\left(\frac{x}{(\log x)^A}\right)\\&=\mathcal{A}_h x+O(x^{3/4})+O\left(\frac{x}{(\log x)^A}\right),\end{aligned}$$

which proves the required formula and completes the proof.

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