From: The Collected Papers of Sarvadaman Chowla, Volume, 1, pp. vii-xxviii.

The Work of Sarvadaman Chowla

M. Ram Murty, V. Kumar Murty and Kenneth S. Williams

Sarvadaman Chowla has been called a "poet of mathematics" and "an ambassador of number theory." Given the enthusiasm he had for number theory and the passion with which he did it, these descriptions are not exaggerations. Herman Weyl described his work in the following way. "Every one of his papers contains interesting observations, and most of them give their subject a new turn by introducing original ideas. Sometimes they are devoted to simplified proofs of difficult classical problems, but more often they give new results, not seldom of a quite surprising character." Indeed, the striking features of Chowla's papers are simplicity and beauty.

He was born into a mathematical family. His father was a professor of mathematics. His younger brother Inder Chowla (cited in [214]) received his Ph.D. under Heilbronn at Cambridge. Inder published twenty one papers in number theory and analysis from 1935 to 1943, until his untimely death. It is well-known that Chowla's daughter, Paromita, is also a number theorist and they have written twenty papers together. We will refer to some of them below. Chowla had many students who went on to become distinguished in their field. In particular, when Chowla taught at Government College in Lahore (which is in present day Pakistan) during the 1940's, two of his students were R.P. Bambah and Abdus Salaam. The former is a noted number theorist and the latter a renowned physicist and Nobel laureate.

It is not our purpose to give an exhaustive survey of Chowla's work. Nor is it possible to give an encyclopedic account of developments arising from Chowla's ideas. Our goal is only to give a brief overview to guide the reader through the numerous collected papers (which amount to more than 1400 printed pages) and indicate some of the recurring themes. His book [C] gives us a sampling of the nature of questions that Chowla was interested in. It is a friendly account, somewhat leisurely presented, of some central questions of number theory. It is sprinkled with enticing conjectures. For instance, Bambah and Chowla [166] proved the existence of a constant C such that between x and $x + Cx^{1/4}$ there is always a number which can be written as a sum of two squares. It is still an unsolved problem to show that between x and $x + o(x^{1/4})$ such a number exists [C, p. 92]. In Chapter 5 of [C], Chowla discusses the classical congruence for the binomial coefficient:

$$A := \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2a (\bmod p),$$

where p is a prime $\equiv 1 (\bmod 4)$ and $p = a^2 + b^2$ with $a \equiv 1 (\bmod 4)$. In joint work with Dwork and Evans, Chowla [347] obtained a refinement of this:

$$A \equiv \left(1 + \frac{2^{p-1} - 1}{2}\right)\left(2a - \frac{p}{2a}\right) (\bmod p^2),$$

establishing a conjecture of Beukers [Be].

A singular theme running through all of Chowla's papers is the study of class numbers, with an emphasis on quadratic fields and cyclotomic fields. This seems to form the background thought for many of his papers. Indeed, the study of class numbers takes one naturally into the domain of L-functions (see for example, [46], [171], [179], [186], [190]), modular forms (see e.g. [156], [157], [158]), Bernoulli numbers (see e.g., [S22], [2], [7], [34], [303]), Diophantine equations, Gauss sums and exponential sums (see e.g. [13], [22], [30], [33], [35], [38], [169], [228], [240]), elliptic integrals (see e.g. [8], [10], [167], [184], [276]), and even the combinatorics of block designs and difference sets (see for example [187] and [284] as well as [120], [121], [126], [131], [134], [135], [137], [141], [146], [147]).

As with many other Indian mathematicians, Chowla's earliest work was influenced by Ramanujan and his legacy. In [14], he proved several identities involving infinite series which had been stated by Ramanujan. For example, he showed that for any positive integer n,

$$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^{4n-1}}{\cosh\,((2k+1)\pi/2)} \;=\; 0.$$

Another early paper of Chowla dealt with the period of the continued fraction of $\sqrt{N}$. Since the time of Brahmagupta in the 6th century A.D. and Jayadeva in the 11th century A.D.(or perhaps even earlier), it was known that the continued fraction is periodic and that all the integral solutions of the equation

$$x^2 - Ny^2 = m, \quad |m| < \sqrt{N}$$

can be obtained from the knowledge of the continued fraction. (The first rigorous proof of this was given by Lagrange in the 18th century.) If we write in the usual notation,

$$\sqrt{N} = [a_0, \overline{a_1, \ldots, a_\ell}],$$

where $a_0 = [\sqrt{N}]$ and $a_\ell = 2a_0$, then T. Vijayaraghavan [V] proved in 1927 that the length ℓ of the period of the continued fraction satisfies $\ell = O(N^{1/2}\log N)$. He also showed that for any $\delta > 0$, there are infinitely many values of N such that $\ell > N^{1/2-\delta}$. In 1929, Chowla [20] showed that if N is squarefree, then for any $\delta > 0$,

$$\ell < \Big(\frac{6}{\pi^2} + \delta\Big) N^{1/2}\log N$$

for all $N > N_0(\delta)$. In 1931, Pillai and Chowla [36] proved that

$$\ell = O(N^{1/2}\log\log N)$$

under the generalized Riemann hypothesis. They also showed that there are positive constants c_1 and c_2 such that $c_1 N^{1/2} < \ell < c_2 N^{1/2}$ for infinitely many N and that ℓ is "on the average" of order $N^{1/2}$.

In joint work with his daughter Paromita, Chowla returned to this topic in two later papers [306] and [307] when they were inspired by a theorem of Hirzebruch [Hi] that related the class number $h(d)$ of $\mathbf{Q}(\sqrt{d})$ to the terms in the continued fraction expansion of $\sqrt{|d|}$. More precisely, define for each non-square integer N, the function $s(N) = a_\ell - a_{\ell-1} + a_{\ell-2} - \cdots \pm a_1$. Hirzebruch proved that if p is a prime $\equiv 3(\bmod 4)$ with $p > 3$ and $h(p) = 1$, then $s(p) = 3h(-p)$. This result suggested various conjectures to them, most notable of which is that if p is as above and $h(2p) = 1$, then $s(2p) = 6h(-p)$. (See also the related work of Zagier [Z].)

In 1930, Chowla [34] considered a conjecture of Ramanujan that the numerator of B_{2n}/n, where B_n denotes the n-th Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!},$$

is always a prime number. However,

$$\frac{B_{32}}{16} = -\frac{7709321041217}{8160}$$

is divisible by 37 giving a counterexample to Ramanujan's conjecture. Chowla showed that in fact there are infinitely many counterexamples by proving that the numerator of

$$\frac{B_{690k+12}}{345k+6}$$

is divisible by 691. In [303], Chowla together with his student Hartung, derived an exact formula for B_{2n}. In [351], together with Paromita Chowla, he asks whether Ramanujan's conjecture can be modified to ask if the numerator is always squarefree.

In one of his earliest papers, Chowla [2] gave a new proof of the von Staudt and Clausen theorem which gives the fractional part of the Bernoulli number B_{2n}, namely for every natural number n,

$$B_{2n} \equiv -\sum_{p-1|2n} \frac{1}{p} (\bmod 1),$$

where the summation is over primes. This theorem has attracted the attention of many mathematicians over the years. Dilcher, Skula, and Slavutskii [DSS] list over 35 papers dealing with this theorem or its generalizations in their bibliography of Bernoulli numbers covering the period from 1713 to 1990. Indeed, the theorem has a p-adic interpretation. It is the key ingredient in Kummer's celebrated theorem that a prime p is regular (that is, does not divide the class number of the p-th cyclotomic field) if and only if p does not divide the numerators of the Bernoulli numbers $B_2, B_4, \ldots, B_{p-3}$. (See Rosen [R] for an elegant exposition of this.)

Chowla returned to the subject of Bernoulli numbers on many occasions during his life. In [S22], for example, in answer to a question of Hemraj, Chowla showed that

$$\sum_{k=1}^{(p-3)/2} \frac{B_{2k}}{a^{2k}} \equiv \frac{1}{2a} - 1 - \sum_{j=1}^{a} \frac{1}{j} (\bmod p)$$

for any integer a not divisible by the prime p. In [348], P. and S. Chowla use Bernoulli numbers to derive a criterion for the class number of a real quadratic field to be 1.

In the early 1930's, Chowla was in Cambridge studying with Littlewood. There, in 1930, he met a young student who had just arrived from India to study physics and who had already attracted much attention with his work on Compton scattering. Just the year before, C.V. Raman had won the Nobel Prize in Physics for his work in spectroscopy. One can therefore imagine the excitement that was attached to the study of physics in the early 1930's among students from India. This young man, Subrahmanyan Chandrasekhar, was in fact Raman's nephew and he was soon to make a tremendous impact in the newly emerging field of astrophysics, eventually winning the Nobel Prize himself in 1983.

Chowla and Chandrasekhar developed a friendship as both were studious and hard working. In 1931, Chandrasekhar wrote to his father about Chowla [W]: "He sometimes comes to my room and we study together. $\cdots$ But we don't talk much except during walks. For instance, once we studied nearly four hours without a single word passing between us. Yet, the feeling that somebody is with us makes it easier." In another letter, written in 1935, he described Chowla as "... not only my personal friend but one whose work and abilities I greatly admire."

The period 1930-1935, including the years in Cambridge, were amongst the most productive for Chowla. During this period, he published about 65 papers. In surveying these papers, the year 1934 stands out as one of tremendous excitement. For it was in that year that Siegel proved his famous theorem about class numbers of quadratic fields. This was the climax of a series of parallel and simultaneous discoveries by a number of mathematicians, with Chowla playing a serious role.

One can get a sense of the excitement if one goes to the library to look at say *Acta Arithmetica* or the *Journal of the London Mathematical Society* of that period. In article 303 of *Disquisitiones*, Gauss conjectured that the class number $h(-d)$ of $\mathbb{Q}(\sqrt{-d})$ for $d > 0$ goes to infinity. A century went by without mathematicians having any idea of how to resolve the conjecture of Gauss as so little was known about class numbers. Even proving the weaker assertion that $h(-d) \geq 2$ for d sufficiently large, which was also stated by Gauss, was beyond the scope of 19th century mathematics. The remarkable achievements of that period were the class number formulas of Dirichlet which certainly set the stage for Hecke's work of 1918. In that year, Hecke showed assuming the generalized Riemann hypothesis that $h(-d) \gg \sqrt{d}/\log d$. In 1933, Deuring proved the unexpected result that the *falsity* of the Riemann hypothesis for $\zeta(s)$ implies $h(-d) \geq 2$ if d is sufficiently large, and shortly after, Mordell proved that this assumption also implies $h(-d) \to \infty$. Their work was based on the study of the Epstein zeta function

$$Z(s) = \sum_{m,n} Q(m,n)^{-s},$$

where $Q(m,n) = am^2 + bmn + cn^2$ is a binary quadratic form and m, n range through all pairs of integers $\neq (0,0)$ and as is well-known, this function plays a dominant role in the study of class numbers. Its study is a recurring theme in much of Chowla's work.

Assuming that there are infinitely many imaginary quadratic fields of class number one, Chowla shows in [55] that the Epstein zeta function has a real zero close to $s = 1$. A few years later, Davenport and Heilbronn [DH] showed that this zeta function has infinitely many zeros in the half plane $Re(s) > 1$. More than one hundred papers and 13 years later, Chowla returned to this problem and in [170] he states that the Epstein zeta function corresponding to the form $x^2 + dy^2$ of discriminant d has a real zero $s(d)$ with

$$s(d) \sim 1 - 3/(\pi d^{1/2})$$

as $d \longrightarrow \infty$. More recently, Bombieri and Hejhal [BH] showed (assuming the Riemann Hypothesis for certain Hecke L-functions and a weak form of the pair correlation conjecture) that almost all (in the sense of density) of the zeros of the Epstein zeta function are on the critical line.

What happened shortly after Mordell's work amazed the number theory community. In 1934, Heilbronn showed that the *falsity* of the generalized Riemann hypothesis implies that $h(-d) \to \infty$. Combined with Hecke's result, this gave an unconditional proof that $h(-d) \to \infty$ as $d \to \infty$. Heilbronn's result stimulated Chowla to give [62] another proof of Heilbronn's theorem which avoided the use of the theory of ideals. In [65], he gave a further simplification which avoided the appeal to a theorem of Hecke and, in [61], [63], [67], he was able to prove the stronger result

$$h(-d)/2^{\nu(d)} \to \infty, \quad \text{as} \quad d \to \infty,$$

where $\nu(d)$ is the number of distinct prime factors of d. This result has applications to the problem of determining all *idoneal* numbers. These numbers D have the property that if $D = ab$, every number represented by $f = ax^2 + by^2$ (with ax coprime to by) in a single way is a prime, the square of a prime, the double of a prime or a power of 2. If an odd number > 1 is represented by f in a single way, it is a prime. (See Dickson [Di, p. 89]. Our definition of *idoneal* numbers also corrects inaccuracies on page 89 of [Di].) Heilbronn's work naturally suggests the problem of giving a lower bound for $h(-d)$ in terms of $|d|$. Chowla considered this in [70] linking the question to primes in arithmetic progressions. Let $\pi(x, k, a)$ denote the number of primes $\equiv a(\bmod k)$ not exceeding x. Let $0 < \delta < 1/2$. In [70], he proved that if there is a positive constant c such that

$$\pi(x, k, a) > \frac{cx}{\phi(k) \log x}$$

for $x \geq e^{k^\delta}$, $\quad k \geq k_0(\delta)$, $\quad (a, k) = 1$, then

$$h(-d) > |d|^{1/2-\delta-\epsilon}$$

for every $\epsilon > 0$ and all d with $|d| > d_0(\delta, \epsilon)$. Later, in [188], Ankeny and Chowla linked the constant in the Brun-Titchmarsh theorem (a central theorem in sieve theory) to a lower bound for $h(-d)$. This relationship was subsequently made more precise by Motohashi [Mo].

The number-theoretic highlights of 1934 are definitely striking to say the least. First an important result is proved assuming the generalised Riemann hypothesis, and then assuming the falsity of the same hypothesis. It is certainly a curious sequence of ideas. Of course, Siegel's breakthrough theorem of 1934 that proved $h(-d) \gg d^{1/2-\epsilon}$ was certainly a highlight. Of course, this as well as the earlier results were all ineffective. Since then, much has been learned through the work of Goldfeld, Gross and Zagier as well as the refinement of Oesterlé, namely the effective bound

$$h(-d) > \frac{1}{7000} \prod_{p|d} \left(1 - \frac{2\sqrt{p}}{p+1}\right) \log |d|.$$

It is interesting that in Goldfeld's reminiscences, he mentions that Chowla had an influence in getting him to think about the class number problem. "Whenever I had a question," writes Goldfeld, "he would dig out an old original reprint of Siegel, Heilbronn or refer me to even older works of Dedekind or Legendre, and slowly I got to know the classical literature. The collected papers of Hecke were always easy to find in his apartment and he pushed me to study it from cover to cover. I continued to work on the class number problem for several years and when it was finally solved ... it gave Chowla an enormous satisfaction to know that it had been partially done by his old student in the theory of quadratic fields."

It was also in 1934 that Chowla made an important contribution to the Oppenheim conjecture. Given any $\epsilon > 0$ and an indefinite quadratic form $Q(\mathbf{x}) = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2$ with real coefficients λ_i and at least one irrational ratio λ_i/λ_j, when does the inequality $|Q(\mathbf{x})| < \epsilon$ have a non-trivial integral solution? Oppenheim conjectured that for $n \geq 3$ there is always an integral solution. In 1934, Chowla [50] proved this conjecture for $n \geq 9$. Davenport and Heilbronn established in 1946 the conjecture for $n \geq 5$. Finally in 1986, the conjecture was settled by Margulis [Mar] using ergodic theory.

The excitement created in 1934 by Siegel's theorem led Chowla to study its proof in the subsequent years and in 1950, Chowla [186] gave a very simple proof of it. In subsequent papers, it is clear that Epstein zeta functions became the focus of Chowla's attention. They were the object of study of the celebrated collaboration with Atle Selberg. Though the research for this paper was done in 1949 and an announcement had already been published by them, the actual paper came out much later, in 1967. The paper begins with Deuring's formula for $Z(s)$ which really was the catalyst behind the developments described above but it takes Deuring's work much further. They start with the representation

$$Z(s) = 2\zeta(2s)a^{-s} + 2^{2s}a^{s-1}\pi^{1/2}(\Gamma(s))^{-1}\Delta^{1/2-s}\zeta(2s-1)\Gamma(s-1/2) + Q(s)$$

where $\Delta = 4ac - b^2 > 0$ and

$$Q(s) = 4\pi^s 2^{s-1/2}a^{-1/2}\Gamma(s)^{-1}\Delta^{1/4-s/2} \sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)\cos(n\pi b/a)I_n,$$

with

$$I_n = \int_0^\infty y^{s-3/2} e^{-n(y+y^{-1})\Delta^{1/2}\pi/2a} dy,$$

from which many consequences can be deduced. For instance, if $\Delta = p > 7$ is prime and $h(-p) = 1$, then $Z(s) = 2\zeta(s)L_p(s)$ and the above formula can be used to show that $L_p(1/2) \neq 0$. By similar techniques, they study

$$H_d(s) = \sum{}' (x^2 + y^2 + dz^2)^{-s},$$

where the sum is over integer triples $(x, y, z) \neq (0,0,0)$. They derive a new meromorphic continuation of $H_d(s)$ and show that there exists a real zero $\rho_d \neq 0$ with $\rho_d \to 0$. Finally, they derive a remarkable formula in the theory of elliptic functions. As usual, let us write

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{-1/2} d\phi$$

and

$$K' = \int_0^{\pi/2} (1 - k'^2 \sin^2 \phi)^{-1/2} d\phi,$$

where $k^2 + k'^2 = 1$. If $K'/K = n^{1/2}$, and $h(-n)$ is small, it is shown that K may be computed in finite terms involving the Γ-function. For example, if $n = p$ is prime and $h(-p) = 1$, one has

$$2K = (2kk')^{-1/6}(\pi/p)^{1/2}\Big(\prod_{(a/p)=1} \Gamma(a/p) \Big/ \prod_{(b/p)=-1} \Gamma(b/p)\Big)^{w/4}$$

where $w = 2$ for $p > 3$ and $w = 6$ for $p = 3$. The more general formula can be stated as follows. Let $\eta(z)$ be the Dedekind eta function and let $[a, b, c]$ run through the equivalence classes of positive definite, primitive, integral binary quadratic forms $ax^2 + bxy + cy^2$ of discriminant d. Then, the Chowla-Selberg formula is

$$\prod_{[a,b,c]} a^{-1/4} \Big|\eta\big((b + \sqrt{d})/2a\big)\Big| = (2\pi|d|)^{-h(d)/4}\Big(\prod_{m=1}^{|d|} \Gamma(m/|d|)^{(d/m)}\Big)^{w(d)/8},$$

where (d/m) in the exponent is the Kronecker symbol and $w(d)$ is the number of roots of unity of $\mathbf{Q}(\sqrt{d})$. This work is the culmination of earlier work of Chowla, namely [8] and [10], where he investigated the problem of finding infinite series for $1/K$ and $1/K^2$. The evaluation of complete elliptic integrals of the first kind is still very much a subject of current interest. It is now recognized that a version of the Chowla-Selberg formula was given by Lerch in the late 19th century.

As Emil Grosswald states in his review of the Chowla-Selberg paper, "it is inevitable that during this long interval [between the announcement of the results and the publication of the paper] others would work on the same and related topics." Indeed, one can see papers by Rosser, Anferteva, Bateman, Grosswald, Ramachandra, Rankin, Low, and Stark, to name a few, that either proved directly the formulas announced in 1947 or developed a related motif. The Chowla-Selberg formula has recently been extended by Huard, Kaplan and Williams [HKW] and by van der Poorten and Williams [PW]. By interpreting the left hand side as the period of an elliptic curve with complex multiplication, Gross [Gro] and Deligne [D] obtained a generalization of the Chowla-Selberg formula for abelian varieties with multiplication by an imaginary quadratic field by expressing their periods (up to a rational number) as a product of Γ-function values at rational numbers.

In the 1940's there was considerable interest in the Ramanujan τ function. Most notably, Weil formulated his famous conjectures concerning the number of solutions mod p for general algebraic varieties. It was slowly being suspected, strongly propelled by early work of Sato and Shimura, that perhaps the Ramanujan conjecture predicting $|\tau(p)| \leq 2p^{11/2}$ for all primes p may in fact be a special case of the Weil conjectures. This indeed turned out to be the case though this was not known until Deligne's work related to the Weil conjectures appeared in 1972. Chowla wrote numerous papers on the τ function. For example, in collaboration with Bambah, Gupta and Lahiri, he proved many congruences satisfied by $\tau(n)$. Bambah and Chowla [156, 157] showed that

$$\tau(n) \equiv \sigma(n)(\bmod 3), \qquad (n,3) = 1,$$

and

$$\tau(n) \equiv \sigma_{11}(n)(\bmod 256), \qquad (n,2) = 1,$$

where $\sigma_k(n)$ is the sum of the k-th powers of the divisors of n. He [158] also obtained a number of results of the type

$$\tau(n) \equiv 0(\bmod 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 23 \cdot 691)$$

for almost all n. We now understand this in the light of the theory of the ℓ-adic representations attached to Ramanujan's cusp form (see for example Serre [Se]).

In 1952, Ankeny, Artin and Chowla [194, 199] derived a beautiful congruence relating the class number $h(d)$ of a real quadratic field of discriminant d to its fundamental unit. More precisely, if p is a prime divisor of d with $3 < p < d$, they proved that

$$-2\frac{u}{t}h(d) \equiv \sum_{0<a<d} \frac{p}{da}\left(\frac{d}{a}\right)\left[\frac{a}{p}\right](\bmod p),$$

where (d/a) is the Kronecker symbol for discriminant d, $[x]$ denotes the greatest integer function and $(t+u\sqrt{d})/2$ is the fundamental unit > 1 of the quadratic field. In [348], P. and S. Chowla use this congruence to give a nice criterion to determine when the class number of $\mathbf{Q}(\sqrt{p})$ is 1. In fact, their result allows for the explicit determination of the class number of $\mathbf{Q}(\sqrt{p})$ via the congruence

$$hu/t \equiv B_{(p-1)/4}(\bmod p),$$

where $(t + u\sqrt{p})/2$ is the fundamental unit of $\mathbf{Q}(\sqrt{p})$ and B_k denotes the k-th Bernoulli number. In case $u \not\equiv 0(\bmod p)$, we obtain a congruence for h mod p. Since $h < p$, this determines h explicitly from a knowledge of the fundamental unit of $\mathbf{Q}(\sqrt{p})$.

Later, in 1955, Ankeny and Chowla showed [203] that given any natural number g, there are infinitely many imaginary quadratic fields whose class number is divisible by g. Some results for real quadratic fields were also obtained. This work has recently been extended by R. Murty [RM] who obtained quantitative estimates for the number of such quadratic fields. There is also some related work of Kohnen and Ono [KO] who obtained quantitative estimates for the number of imaginary quadratic fields whose class number is coprime to g, improving earlier work of Chowla's student, Hartung. These theorems, however, fall short of the Cohen-Lenstra conjectures [CL] that predict that a positive density of such fields should exist in either case.

Another focus of attention for Chowla was Kummer's conjecture. Let h_p be the class number of the p-th cyclotomic field, with p prime. It is well-known that h_p can be factored as $h_p^+ h_p^-$ where h_p^+ is the class number of the maximal real subfield of index 2. Kummer conjectured that for p prime,

$$h_p^- \sim \frac{p^{(p+3)/4}}{2^{(p-3)/2}\pi^{(p-1)/2}}$$

as $p \to \infty$. Actually Kummer never conjectured this. Rather, he claimed he had a proof but never published it. In 1851, he wrote, "Je me réserve la démonstration et les développements ultérieurs à une autre occasion." Unfortunately, this "autre occasion" never arose. If we call $G(p)$ the expression on the right hand side of the conjectured asymptotic formula, Ankeny and Chowla proved in [193] that for p prime,

$$\frac{\log(h_p^-/G(p))}{\log p} \to 0$$

as $p \to \infty$. In a recent paper, Granville [Gra] shows that Kummer's conjecture is incompatible with the Elliot-Halberstam conjecture and the Hardy-Littlewood conjecture predicting that there are at least $\gg x/\log^2 x$ primes $p \le x$ such that $2p+1$ is also prime.

Around the time that [193] was written, there was an interesting correspondence between Chowla and A. Weil regarding what is referred to in the literature as Maillet's determinant. For a positive integer m and any integer x, denote by $R(x)$ the least positive integer which is congruent to $x(\bmod m)$. If $(x,m) = 1$, denote by x' any integer such that $xx' \equiv 1(\bmod m)$. Define the set

$$S = \{ 1 \le a < m/2, \quad (a,m) = 1 \}$$

and consider the determinant of the $\frac{1}{2}\phi(m) \times \frac{1}{2}\phi(m)$ matrix $A_m = (R(ab'))$ where a and b range over elements of S. In a letter written March 6, 1952, Weil asked Chowla if he could evaluate this determinant. Both of them had successfully considered the case when m is a prime p. In this case,

$$\det(A) = \pm p^{(p-3)/2} h_p^-,$$

where h_p^- is as above. In fact, they also had a similar formula for prime powers. There was a plan to write a joint paper about this and also to consider the general case. Unfortunately, this plan never materialised. The prime case was rediscovered two years later by Carlitz and Olson [CO].

Chowla returned to this question in [293]. There, he posed the general question of determining the non-trivial $\mathbb{Q}$-linear relations amongst the roots of an irreducible polynomial over $\mathbb{Q}$. He proved that for a prime $p \equiv 3(\bmod 4)$, the only such relations amongst the numbers $\{\cot(\frac{\pi r}{p}) \; : \; 1 \le r < p\}$ are those which can be deduced from the relations

$$\cot(\frac{\pi r}{p}) \; + \; \cot(\frac{\pi(p-r)}{p}) \; = \; 0.$$

Chowla deduced this from the non-vanishing of the Dirichlet L-functions $L(s,\chi)$ at $s = 1$. This pretty result attracted enough attention that three new proofs were given, by Ayoub [Ay1], Hasse [Has] and Iwasawa (see [Ay2]). Kai Wang [Wa] extended this result of Chowla and later used it to evaluate Maillet's determinant for any odd m as

$$\det A \; = \; -2^{1-m} \prod_{\chi(-1)=-1} \left(\sum_{(a,m)=1} a\chi(a) \right).$$

From Weil's letter, one infers that this formula was probably known to him and Chowla. Maillet's determinant itself seems to come up in other areas. For example, it arises in the study of torsion on elliptic curves [FZ] and in the computation of the Hodge group of an abelian variety with complex multiplication by the cyclotomic field [Haz].

In [208], Ankeny, Brauer and Chowla study lower bounds of class numbers for more general algebraic number fields. They prove that given two non-negative integers r_1 and r_2 such that $r_1 + 2r_2 = n \ge 2$ and $\epsilon > 0$, there exist infinitely many fields F which have exactly r_1 real and $2r_2$ non-real embeddings such that

$$h_F > |d_F|^{1/2-\epsilon},$$

where h_F is the class number of F and d_F is the discriminant of F. This is a kind of analogue of Siegel's lower bound for the class number of an imaginary quadratic field. By a classical result of Minkowski and Landau, it is known that

$$h_F \ll |d_F|^{1/2}(\log |d_F|)^{n-1},$$

where the implied constant depends on n alone. Thus, the result of [208] is best possible.

Naturally, class numbers are intimately connected with special values of Dirichlet L-functions, namely at $s = 1$. If χ is a real non-principal primitive character mod k, the Dirichlet L-series is

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

which converges for $Re(s) > 0$. In [179], Chowla proved that there are infinitely many k such that

$$L(1,\chi) < (1+o(1))\pi^2/6e^\gamma \log\log k,$$

where γ is Euler's constant. In [46], he showed that there is a positive constant D such that for infinitely many k we have

$$L(1,\chi) > D \log\log k.$$

Both of these results were previously proved by Littlewood under the assumption of the generalised Riemann hypothesis. Chowla gave some further improvements of these results in [171] and in [190] which he wrote with Bateman and Erdös. It is interesting to note the connection of these results to some earlier work of Chowla, namely [72] written in 1935. There, he considers characters sums for a real non-trivial character χ mod k,

$$S_1(x) = \sum_{n \le x} \chi(n)$$

and more generally

$$S_m(x) = \sum_{n \le x} S_{m-1}(n)$$

defined recursively. He then proves that if for some m, $S_m(x) \geq 0$ for all $x \geq 1$, then $L(s,\chi) > 0$ for $0 < s < 1$. Write $m(\chi)$ to be the least such m if it exists. There he writes, "I have been unable to find a real non-principal character χ for which $m(\chi)$ does not exist." His results on upper bounds for $L(1,\chi)$ allow him to deduce that

$$m(\chi) = \Omega(\log\log k).$$

We think it is safe to assume that Chowla was conjecturing that $m(\chi)$ always exists. In 1937, Heilbronn [H] disproved this and showed that $m(\chi)$ does not always exist. The disproof is exceedingly simple and makes use of the Liouville function $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the total number of prime factors of n. It is significant that Heilbronn's proof is based on the fact that by choosing an appropriate quadratic character, $\chi(n)$, one can make it coincide with $\lambda(n)$, up to $n \leq N$ for any preassigned value of N. Such a technique was used by Bateman and Chowla in [201], written in 1953, where they study the equivalence of two conjectures, namely

$$\sum_{n=1}^{N} \frac{\chi(n)}{n} > 0$$

for all N if and only if

$$\sum_{n=1}^{N} \frac{\lambda(n)}{n} > 0.$$

The latter conjecture was disproved by Haselgrove [Ha] and hence the former one is also false.

In *The Collected Papers of Hans Arnold Heilbronn*, Chowla makes comments on [H]. Since Heilbronn's counterexample produced quadratic characters χ whose conductor is composite, Chowla asks if there are infinitely many primes $p \equiv 3(\bmod 4)$ such that

$$\sum_{n \leq x} \left(\frac{n}{p}\right) \geq 0$$

for all $x \geq 1$. All of these questions revolve around the theme of non-vanishing of Dirichlet L-functions on the real line segment $0 < s < 1$. Chowla continued to think about this question and he wrote various papers with his students, notably, [309], [313], [317], [318], [321] to cite a few. In fact, in [312] and [313], the authors formulate variations of the original conjecture and ask the question above. Several notable developments have arisen from attempts to answer this question. For instance, there is the paper of Jutila [J] that shows for the first time, infinitely many primes p for which the $L_p(1/2) \neq 0$. R. Balasubramanian and Kumar Murty [BM] considered the L-functions corresponding to all the characters mod p and proved that a positive proportion of them do not vanish at $s = 1/2$. An analogous result for quadratic characters was established in the Ph.D. thesis of Soundararajan [S]. Iwaniec and Sarnak [IS] relate a proportion larger than 1/2 of normalized eigenforms of fixed weight and level having the property that their L-functions do not vanish at the central critical point to the non-existence of Siegel zeros. There is the paper of P. Chowla [PC] which shows non-vanishing at $s = 1/2$ of the zeta function of a particular biquadratic field. This is of interest since it is contained in a field k of degree 8 for which the zeta function of k does have a zero, and in fact arising from a non-abelian Artin L-function attached to a 2 dimensional representation. This example, due to Serre, is explained in paper [328].

In his paper Milnor [Mi] points out that P. and S. Chowla [342] suggested the following conjecture: let $a_1, a_2, \ldots,$ be a sequence of integers which is periodic with period p, a prime. Then

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \neq 0,$$

except in the special case $a_1 = a_2 = \cdots = a_{p-1} = a_p/(1-p^2)$. If $\zeta(x,s)$ is the Hurwitz zeta function

$$\zeta(x,s) = x^{-s} + (x+1)^{-s} + \cdots$$

then Milnor proves that Chowla's conjecture is true if and only if

$$\zeta(2,1/p), \zeta(2,2/p), \ldots, \zeta(2,(p-1)/p)$$

are linearly independent over $\mathbf{Q}$.

Alan Baker [B] describes a related conjecture of Chowla. In the 1968 Stony Brook conference, Chowla asked whether there exists a rational valued function $f(n)$, periodic with prime period p, such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0.$$

In [BBW], Baker, Birch and Wirsing show that no such function exists. As a consequence, they prove that for any modulus q, satisfying $(q, \phi(q)) = 1$, the values $L(1, \chi)$, with χ ranging over the characters mod q, are linearly independent over the rationals. They ask if the theorem is still true in the case $(q, \phi(q)) > 1$.

In [191], Erdös and Chowla study the distribution of values of Dirichlet L-functions attached to quadratic characters. Consider the set of $D \equiv 0, 1 (\bmod 4)$ and which are not squares. Denote by χ_D the quadratic character associated to D. Fix a real $\sigma > 3/4$. Let $g(z, x)$ denote the set of such $D \leq x$ for which $L(\sigma, \chi_D) < z$. Then, they show that $g(z, x)/(x/2)$ tends to a continuous, strictly increasing function $\phi(z)$ with $\phi(0) = 0$ and $\phi(\infty) = 1$. In particular, applying this with $z = 0$ shows that for a fixed σ, the value $L(\sigma, \chi_D)$ is positive for a set of D of density 1.

Elliott [E] generalized this in several ways by considering quadratic characters to prime modulus and any argument $\sigma > 1/2$. One can ask whether σ can be replaced by a complex argument s which is allowed to vary as a function of D. This problem is more complicated, but as shown in [E], it is still possible to get a distribution result provided one restricts the proximity of the real part of s to the critical line and bounds the imaginary part in terms of x. Distribution of the L-values on the critical line seems to be an open question.

In his Amalfi paper, Selberg [Sel] showed for quite a general class of functions that one can get such a distribution result if one assumes the analogue of the Riemann Hypothesis (or a weaker hypothesis about the number of zeros off the critical line).

In [176], Chowla and Vijayaraghavan considered the important function of determining the number of integers $\leq x$ all of whose prime factors are $\leq y$. This ubiquitous function is of fundamental importance in sieve theory and it makes its appearance in various questions of number theory. For example, it is a key tool in the work of Ankeny, Brauer and Chowla [208] cited above. In [204], Chowla and Briggs return to the study of this function.

These investigations naturally lead one to the study of error terms for the summatory functions of various classical arithmetical functions to which Chowla turned his attention. The Dirichlet divisor problem is one of the more well-known problems. If $d(n)$ denotes the number of divisors of n, and γ denotes Euler's constant, then it is a famous conjecture that

$$\Delta(x) := \sum_{n \leq x} d(n) - \Big(x \log x + (2\gamma - 1)x\Big) = O(x^{1/4+\epsilon})$$

for any $\epsilon > 0$. If $\psi(x) = x - [x] - 1/2$, then the conjecture would follow from

$$\sum_{n \leq \sqrt{x}} \psi(x/n) = O(x^{1/4+\epsilon}).$$

Chowla and Walum [247] formulated in 1963, the more general conjecture

$$\sum_{n\leq\sqrt{x}} n^a B_r(\{x/n\}) = O(x^{a/2+1/4+\epsilon})$$

for every non-negative integer a and with B_r denoting the r-th Bernoulli polynomial. They proved the result for $a = 1, r = 2$. In 1985, Kanemitsu and Sitaramachandrarao [KS] proved this conjecture for $a \geq 1/2$ and all integers $r \geq 2$. A related problem concerns the error term in a classical result of Mertens. If $\phi(n)$ denotes Euler's function, Mertens proved that

$$E(x) = \sum_{n\leq x} \phi(n) - \frac{3}{\pi^2}x^2$$

satisfies $E(x) = O(x \log x)$. In a joint paper with Pillai, Chowla [25] showed that

$$\sum_{n\leq x} E(n) \sim \frac{3}{2\pi^2}x^2$$

and that $E(x) \neq o(x \log\log\log x)$. In [39], he showed that

$$\int_1^x E^2(t)dt = \frac{1}{6\pi^2}x^3 + O(x^3/\log^4 x).$$

Our knowledge of Gauss sums is very far from being complete. Chowla returned to this topic throughout his life. In an early paper [13], Chowla derives the classical results of Gauss by evaluating certain contour integrals and comparing them with evaluations of Ramanujan. He returned to this theme again in [22], [33], and [169]. In [240], Chowla proved that $\tau(\chi)/\sqrt{p}$ is a root of unity only when $\chi^2 = \chi_0$. In [322], Berndt and Chowla consider the explicit determination of quartic and octic Gauss sums.

Before closing, there are two more themes of number theory that originated in early papers of Chowla. One is the problem of estimating the least prime in an arithmetic progression. Given relatively prime integers k and ℓ, Chowla [48] conjectured that there is a prime $P(k,\ell) \equiv \ell (\bmod k)$ satisfying

$$P(k,\ell) \ll k^{1+\epsilon}.$$

The analog of the Riemann Hypothesis for Dirichlet L-functions implies only the weaker estimate

$$P(k,\ell) \ll k^{2+\epsilon}.$$

Linnik [L] proved unconditionally that there is an absolute constant C such that

$$P(k,\ell) \ll k^C$$

and many authors have sought to determine C explicitly. The latest result is due to Heath-Brown [H-B] who shows that $C \leq 5.5$

The problem of the least prime in an arithmetic progression has a generalization to number fields. Let F/K be a Galois extension of number fields with group G and let C be a conjugacy class in G. The analogue of Dirichlet's theorem on the infinitude of primes in an arithmetic progression is the Chebotarev density theorem, one version of which states that

$$\pi_C(x, F/K) \sim \frac{|C|}{|G|} \operatorname{li} x,$$

where $\pi_C(x, F/K)$ denotes the number of prime ideals of K which are unramified in F and whose Frobenius conjugacy class is C and

$$\operatorname{li} x = \int_2^x \frac{dt}{\log t}.$$

If $K = \mathbf{Q}$ and F is the cyclotomic field of k-th roots of unity, then we recover the problem of primes in an arithmetic progression modulo k. Amongst all primes $\mathcal{P}$ of K whose Frobenius conjugacy class in F is C, consider the one of least norm and denote this norm by $P(C, F/K)$. The analogue of Chowla's conjecture on the least prime in an arithmetic progression is

$$P(C, F/K) \ll \frac{(\log d_F)^{2+\epsilon}}{n_F},$$

where d_F denotes the absolute value of the discriminant of $F/\mathbf{Q}$ and $n_F = [F : \mathbf{Q}]$. (See [KM1] for this and other results and conjectures on the least prime in a conjugacy class. Some evidence is found in the paper [KM2]). Assuming the Riemann Hypothesis for all Dedekind zeta functions, Lagarias and Odlyzko [L-O] showed that

$$P(C, F/K) \ll (\log d_F)^2.$$

If in addition to the GRH, one also assumes Artin's conjecture on the holomorphy of non-Abelian L-functions, then it follows from the results of Murty, Murty and Saradha [MMS] that

$$P(C, F/K) \ll \frac{(\log d_F)^2}{|C|}.$$

The second theme is determining an upper bound for the number of integral points on a given curve. In [41] he addresses this question and in fact disproves a conjecture of Siegel. Later in 1983, Silverman [Si] showed how Chowla's work can be related to ranks of elliptic curves. This work of Chowla is related to his earlier work on Waring's problem. If $R_k(n)$ denotes the number of representations of n as the sum of k k-th powers, then Hardy and Littlewood conjectured in 1925 that $R_k(n) = O(n^\epsilon)$ for every $\epsilon > 0$. Chowla proved in [93] that $R_k(n) \neq O(1)$. Later, in [95] he showed with Pillai that there is a positive constant c depending only on k such that $R_k(n) > c \log\log n$ for infinitely many positive integers n, and that

$$R_3(n) > \frac{c \log n}{\log\log n}$$

for infinitely many n.

It is difficult to survey in one sweeping instance the mathematical work of Chowla. No doubt, we have left out many areas which he touched and influenced. We have not mentioned many of his conjectures, his contributions towards them, and their later developments. There are many mathematicians whose work has been influenced by Chowla either directly or indirectly whom we have not mentioned in the brief compass of this survey. The reader may find additional material in [Ay3] and [AHW]. We hope, however, that this short overview conveys some of the beauty, charm and excitement that colours the mathematical personality of Sarvadaman Chowla.

Acknowledgements. We would like to thank Kalyan Chakraborty, John Friedlander, Yiannis Petridis and Harold Stark for their comments on an earlier version of this manuscript.

REFERENCES

[AHW] R.G. Ayoub, J.G. Huard, and K.S. Williams, Sarvadaman Chowla (1907-1995), *Notices Amer. Math. Soc.* **45** (1998), 594-598.

[Ay1] R. Ayoub, On a theorem of S. Chowla, *J. Number Theory* **7** (1975), 105-107.

[Ay2] R. Ayoub, On a theorem of Iwasawa, *J. Number Theory* **7** (1975), 108-120.

[Ay3] R. Ayoub, Sarvadaman Chowla, *J. Number Theory* **11** (1979), 286-301.

[B] A. Baker, Effective methods in Diophantine Problems, II, in Fields Medallists Lectures, edited by M. Atiyah and D. Iagolnitzer, World Scientific Press, Singapore, 1997, 171-189.

[BBW] A. Baker, B. Birch and E. Wirsing, On a problem of Chowla, *J. Number Theory* **5** (1973), 224-236.

[BM] R. Balasubramanian and V. Kumar Murty, Zeros of Dirichlet L-functions, *Ann. Sci. École Norm. Sup. (4)* **25** (1992), 567-615.

[Be] F. Beukers, Arithmetical properties of Picard-Fuchs equations, *Séminaire de théorie des nombres*, Paris 82-83, Birkhäuser Boston, 1984, 33-38.

[BH] E. Bombieri and D. Hejhal, On the zeros of Epstein zeta functions, *C.R. Acad. Sci. Paris Ser. I Math.* **304** (1987), 213-217.

[CO] L. Carlitz and F.R. Olson, Maillet's determinant, *Proc. Amer. Math. Soc.* **6** (1955), 265-269.

[PC] P. Chowla, On the nonvanishing of a certain L-series at $s = 1/2$, *J. Number Theory* **6** (1974), 158-159.

[C] S. Chowla, The Riemann hypothesis and Hilbert's tenth problem, Gordon and Breach, London, 1965.

[CL] H. Cohen and H.W. Lenstra, Heuristics on class groups of number fields, in Number Theory, Noordwijkerhout 1983, *Lecture Notes in Mathematics*, **1068** (1984), Springer-Verlag, 33-62.

[DH] H. Davenport and H. Heilbronn, On zeros of certain Dirichlet series I, *J. London Math. Soc.* **11** (1936), 181-185.

[D] P. Deligne, Valeurs de fonctions L et periodes d'integrales, *Proc. Symp. Pure Math.* **33** (1979), 313-346.

[Di] L.E. Dickson, Introduction to the Theory of Numbers, University of Chicago Press, 1929.

[DSS] K. Dilcher, L. Skula, and I. Slavutskii, Bernoulli numbers, *Queen's Papers in Pure and Applied Mathematics*, **87** (1991), 175pp., Queen's University, Kingston, Ontario, Canada.

[E] P.D.T.A. Elliott, On the distribution of the values of quadratic L-series in the half plane $\sigma > 1/2$, *Invent. Math.* **21** (1973), 319-338.

[FZ] H. Folz and H. Zimmer, What is the rank of the Demjanenko matrix, *J. Symbolic Comput.* **4** (1987), 53-67.

[Gra] A. Granville, On the size of the first factor of the class number of a cyclotomic field, *Invent. Math.* **100** (1990), 321-338.

[Gro] B.H. Gross, On the periods of abelian integrals and a formula of Chowla and Selberg, *Invent. Math.* **45** (1978), 193-211.

[Ha] C.B. Haselgrove, A disproof of a conjecture of Pólya, *Mathematika* **5** (1958), 141-145.

[Has] H. Hasse, On a question of S. Chowla, *Acta Arith.* **18** (1971), 275-280.

[Haz] F. Hazama, Demjanenko matrix, class number and Hodge group, *J. Number Theory* **34** (1990), 174-177.

[H-B] R. Heath-Brown, Zero-free regions for Dirichlet L-functions and the least prime in an arithmetic progression, *Proc. London Math. Soc.* **64** (1992), 265-338.

[H] H. Heilbronn, On real characters, *Acta Arith.* **2** (1937), 212-213.

[Hi] F. Hirzebruch, Hilbert modular surfaces, *Enseign. Math.* **19** (1973), 183-281.

[HKW] J. Huard, P. Kaplan, and K.S. Williams, The Chowla-Selberg formula for genera, *Acta Arith.* **73** (1995), 273-301.

[IS] H. Iwaniec and P. Sarnak, The non-vanishing of central values of automorphic L-functions and Siegel's zeros, in preparation.

[J] M. Jutila, On the mean value of $L(1/2, \chi)$ for real characters, *Analysis* **1** (1981), 149-161.

[KS] S. Kanemitsu and R. Sitaramachandrarao, On a conjecture of S. Chowla and H. Walum, I, *J. Number Theory* **20** (1985), 255-261.

[KO] W. Kohnen and K. Ono, Indivisibility of class numbers of imaginary quadratic fields and orders of Tate-Shafarevich groups of elliptic curves with complex multiplication, *Invent. Math.* **135** (1999), 387-398.

[L] Yu. Linnik, On the least prime in an arithmetic progression, I and II, *Mat. Sbornik (New Series)* **15** (1944), 139-178 and 347-368.

[L-O] J. Lagarias and A. Odlyzko, Effective versions of the Chebotarev Density Theorem, in: Algebraic Number Fields: L-functions and Galois properties, (ed. A. Fröhlich), Proc. Symp., Univ. Durham, Durham (1975), 409, 465.

[Mar] G.A. Margulis, Formes quadratiques indéfinies et flots unipotents sur les espaces homogènes, *C.R. Acad. Sci. Paris Sér. I Math.* **304** (1987), 249-253.

[Mi] J. Milnor, On polylogarithms, Hurwitz zeta functions and Kubert identities, *Enseign. Math.* **29** (1983), 281- 322.

[Mo] Y. Motohashi, A note on Siegel's zeros, *Proc. Japan Acad.* **55** (1979), 190-192.

[RM] M. Ram Murty, Exponents of class groups of quadratic fields, in *Topics in Number Theory*, (edited by S.D. Ahlgren, G.E. Andrews, and K. Ono), Kluwer Academic Publishers, 1999.

[MMS] M. Ram Murty, V. Kumar Murty and N. Saradha, Modular forms and the Chebotarev Density Theorem, *Amer. J. Math.* **110** (1988), 253-281.

[KM1] V. Kumar Murty, The least prime in a conjugacy class, in preparation.

[KM2] V. Kumar Murty, The least prime that does not split completely, *Forum Math.* **6** (1994), 555-565.

[PW] A. van der Poorten and K.S. Williams, Values of the Dedekind eta function at quadratic irrationalities, *Canad. J. Math.* **51** (1999), 176-224.

[R] M. Rosen, The History of Fermat's Last Theorem, in *Modular Forms and Fermat's Last Theorem* (edited by G. Cornell, J. Silverman and G. Stevens), Springer-Verlag, 1997.

[Sel] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, see Collected Papers, Volume 2, pp. 47-63, Springer Verlag, Heidelberg, 1991.

[Se] J.-P. Serre, Divisibilité de certaines fonctions arithmétiques, *Enseign. Math.* **22** (1976), 227-260.

[Si] J. Silverman, Integer points on curves of genus 1, *J. London Math. Soc. (2)* **28** (1983), 1-7.

[S] K. Soundararajan, Quadratic twists of Dirichlet L-functions at $s = 1/2$, Ph.D. thesis, Princeton University, 1998.

[V] T. Vijayaraghavan, Periodic simple continued fractions, *Proc. London Math. Soc.* **26** (1927), 403-414.

[W] K. C. Wali, Chandra – A biography of S. Chandrasekhar, University of Chicago Press, 1991.

[Wa] K. Wang, On Maillet's determinant, *J. Number Theory* **18** (1984), 306-312.

[Z] D. Zagier, Nombres de classes et fractions continues, *Journées Arithmétiques de Bordeaux, Astérisque* **24-25** (1975), 81-97.